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Full Length Research Paper

Non-perturbative solutions of a nonlinear heat conduction model of the human head

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Temperature profiles within the human head are highly influenced by the geometry and the rate of metabolic heat production. In this paper, we provide a very accurate, non-perturbative, semi-analytical solution to a nonlinear singular boundary value problem modelling the distribution of heat sources in the human head and their dependence on environmental temperature. Our analysis is based on Adomian decomposition method. Graphical results are presented and discussed quantitatively. The model serves as a tool for extrapolating human head temperature profiles.

Key words: Human head, nonlinear model, temperature distribution, decomposition method.

INTRODUCTION

The human body, when healthy, maintains its core temperature of about 37°C within small margins. The core consists of the brain and the internal organs in the trunk. Temperatures in the remaining parts of the body - the periphery: surface tissue and the limbs - are much less constant.

Variations in temperature can have significant consequences for the behaviour of individual cells and the body as a whole. The temperature dependence of biological processes can be used as clinical effect. Examples are hyperthermia treatment against cancer, cooling of the head to prevent hair loss as a side effect of chemotherapy and cooling of patients during major surgery to protect the brain. Intervention can also be necessary to maintain or restore normal temperature, e.g. during or after exposure to harsh environmental conditions. For instance, when environmental conditions vary, core temperature is maintained partly by behaviour (e.g. clothing) and partly by the body's thermoregulatory system. Meanwhile, thermogenesis is an obligatory consequence of human cellular metabolism; it enhances the maintenance of constant body temperature in spite of variation in the environmental temperature, (Ricquier, 2006; Janssen et al., 2005).

Moreover, mathematical models of temperature distribution in human head have a role both in treatment and diagnosis. They can aid in predicting the time course of temperatures or in giving information on the temperature where (invasive) thermometry is lacking. Colin et al. (1966) presented experimental data on the distribution of temperature in the human head. These data have been summarized by Flesch (1975) who interpreted them successfully on a theoretical basis. His conclusions were that the generation of heat was greater in the periphery of the human head than in the centre and that it was sensitive to ambient temperature changes at the periphery. Gray (1980) investigated a simplified linear model for heat sources distribution in the human head and showed that in response to a drop in ambient temperature the peripheral heat generation increases more than the central heat generation. Anderson and Arthurs (1981) derived the complementary variational principles and the associated approximate-mate solutions to a nonlinear model of heat conduction in the human head. Duggan and Goodman (1986) applied the theory of maximum principle to a non-linear heat conduction model of the human head to obtain accurate analytical upper and lower bounding curves. Çelik and Gokmen (2003) presented a finite difference numerical solution for nonlinear model for the distribution of the temperature in the human head with variable thermal conductivity. Janssen et al. (2005) presented a computer model that describes heat transfer in the human head during scalp cooling.

In the present study, we constructed a non-perturbative semi-analytical solution of a steady state nonlinear model for heat sources distribution in the human head under the influence of environmental temperature variation. Our work stems mainly from Adomian decomposition method

(Adomian, 1994; Makinde, 2009). The chief merit of the method is that it is capable of greatly reducing the size of computation work while still maintaining high accuracy of the numerical solution. In the following sections, we established the mathematical analysis of the problem and apply some rudiments of Adomian decomposition technique in order to obtain the desired solution. Both numerical and graphical results are presented and discussed quantitatively with respect to various parameters embedded in the problem.

MATHEMATICAL MODEL

We consider a steady state one-dimensional spherically symmetrical heat conduction equation modeling a simplified human head. Following Anderson and Arthurs (1981), Duggan and Goodman (1986), the governing energy balance equation together with its corresponding boundary conditions then becomes:

$$\frac{d^2T}{d\bar{r}^2}+\frac{2}{\bar{r}}\frac{dT}{d\bar{r}}+\frac{\delta}{k}e^{-\alpha T}=0\,, \tag{1}$$

$$\frac{dT}{d\bar{r}}=0\,, \qquad \text{on } \bar{r}=0, \tag{2}$$

$$-k\frac{dT}{d\bar{r}}=\beta(T-T_a)\,, \qquad \text{on } \bar{r}=R, \tag{3}$$

where k is the thermal conductivity (average) inside the head, T is the absolute temperature, $0 \le \bar{r} \le R$ is the radial distance measured from the centre to the periphery of the head, β is a heat exchange coefficient from the head to the surrounding medium, T_a is the ambient temperature, α and δ are the metabolic thermogenesis slope parameter and thermogenesis heat production parameter respectively. We introduce the following dimensionless variables in equations (1 - 3):

$$\theta=\frac{T}{T_a},\ r=\frac{\bar{r}}{R},\ \lambda=\frac{\delta R^2}{kT_a},\ m=\alpha T_a,\ Bi=\frac{\beta R}{k}, \tag{4}$$

and obtain the dimensionless governing equations together with the corresponding boundary conditions as

$$\frac{d^2\theta}{dr^2}+\frac{2}{r}\frac{d\theta}{dr}+\lambda e^{-m\theta}=0\,, \tag{5}$$

$$\frac{d\theta}{dr}=0\,, \qquad \text{on } r=0, \tag{6}$$

$$\frac{d\theta}{dr}=Bi(1-\theta)\,, \qquad \text{on } r=1, \tag{7}$$

where Bi, $m,\lambda,$ represent the Biot number, metabolic thermogenesis slope parameter and thermogenesis heat production parameter respectively.

COMPUTATIONAL METHOD

In this section, we constructed an approximate non-perturbative solution for the nonlinear differential equations (5 - 7) using Adomian decomposition technique (Adomian, 1994; Makinde, 2009). The advantage of this method is that it provides a direct scheme for solving the problem, that is, without the need for linearization, perturbation, massive computation and any transformation. We rewrite equation (5) in the form

$$L_r\theta=-\lambda e^{-m\theta}\,, \tag{8}$$

where the subscript r represents derivative with respect to r and the differential operator employs the first two derivatives in the form

$$L_r=\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right), \tag{9}$$

in order to overcome the singularity behaviour at the point $r = 0$. In view of equation (9), the inverse operator L_r^{-1} is considered a twofold integral operator defined by

$$L_r^{-1}=\int_0^r r^{-2}\int_0^r r^2(\cdot)drdr\,. \tag{10}$$

Applying L_r^{-1} to both sides of equations (8) using the condition in equation (6), we obtain

$$\theta(r)=\theta(0)-\lambda L_r^{-1}(e^{-m\theta})\,. \tag{11}$$

As usual in Adomian decomposition method the solution of equation (11) is approximated as an infinite series

$$\theta(r)=\sum_{n=0}^{\infty}\theta_n\,, \tag{12}$$

and the nonlinear terms are decomposed as follows:

$$e^{-m\theta}=\sum_{n=0}^{\infty}A_n\,, \tag{13}$$

where A_n are polynomials (called Adomian polynomials) given by

$$A_n=\frac{1}{n!}\frac{d^n}{dw^n}\left[e^{-m\sum_{i=0}^{\infty}\theta_i w^i}\right]_{w=0}, \tag{14}$$

Table 1. Computations showing the convergence of the procedure (λ=1, m=1, Bi=1).

ψ_N	$\theta(0)$
ψ_2	1.1607192415883329
ψ_6	1.1608198195214159
ψ_8	1.1608198195900541
ψ_{10}	1.1608198195900541

Table 2. Computations showing the core temperature at various parameter values.

Bi	*λ*	*m*	*θ(0)*
0.1	1.0	1.0	1.670707772962304
0.5	1.0	1.0	1.246067802886287
1.0	1.0	1.0	1.160819819590054
1.0	3.0	1.0	1.395994649701994
1.0	5.0	1.0	1.569878890257090
1.0	1.0	0.5	1.270842849149902
1.0	1.0	0.1	1.436256582344328

Thus, we can identify

$$\theta_0 = \theta(0),$$

$$\theta_{n+1} = -\lambda L_r^{-1}(A_n), \text{ for } n \geq 0, \qquad (15)$$

where $\theta(0)$ is a constant to be determined from the boundary condition in equation (7). Using equation (13), we compute some of the Adomian polynomials as follows:

$$A_0 = e^{-m\theta_0}, \; A_1 = -m\theta_1 e^{-m\theta_0},$$

$$A_2 = (\frac{m^2}{2}\theta_1^2 - m\theta_2)e^{-m\theta_0},$$

$$A_3 = (m^2\theta_2\theta_1 - m\theta_3 - \frac{m^3}{6}\theta_1^3)e^{-m\theta_0},$$

$$A_4 = (m^2\theta_3\theta_1 - m\theta_4 + \frac{m^2}{2}\theta_2^2 - \frac{m^3}{2}\theta_2\theta_1^2 + \frac{m^4}{24}\theta_1^4)e^{-m\theta_0},$$

$$\ldots \qquad (16)$$

Substituting equation (15) into equation (12) and using MAPLE we obtained a few terms approximation to the solution as

$$\psi_N = \sum_{n=0}^{N}\theta_n, \qquad (17)$$

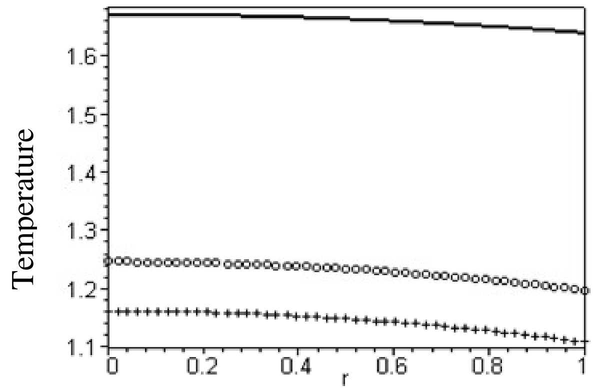

Figure 1. Human head temperature profile_____Bi=0.1; ooooooBi=0.5; ++++++Bi=1.0.

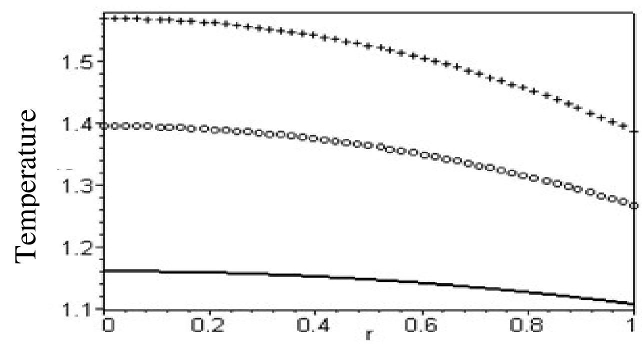

Figure 2 Human head temperature profile _____λ=1; oooooλ=3; +++++λ=5.

where $\theta(r) = \lim_{N\to\infty}(\psi_N)$. Usually, the decomposition method yields rapidly convergent series solutions by using a few iterations for the nonlinear deterministic equations Abboui and Cherruault (1995).

NUMERICAL RESULTS AND DISCUSSION

In this section, we present both the numerical and graphical results for the human head temperature distribution based on the decomposition method. For illustration purposes the parameter values in the Tables 1 - 2 are used.

In Table 1, we observed that the convergence of our computational procedure in section (3) improves with gradual increase in the number of decomposition series coefficients utilized. Figures 1 - 3 illustrate the temperature profiles in a human head as demonstrated by the computational result in Table 2. Generally, it is very interesting to note that human head temperature is higher at the core and decreases transversely with minimum value at the periphery. An increase in the Biot number due to an increase in the rate of heat-transfer at the interface

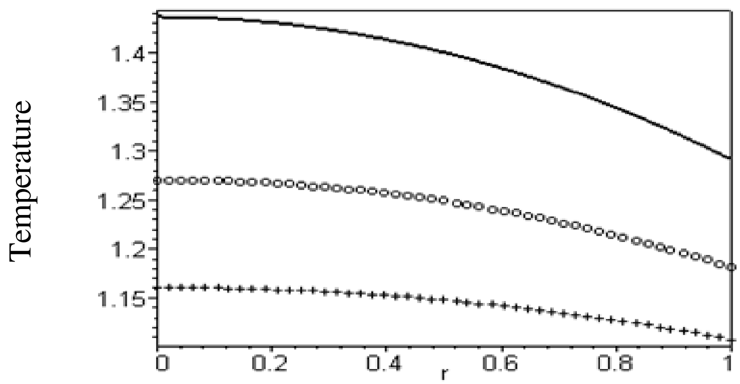

Figure 3. Human head temperature profile _____*m*=0.1; ooooo*m*=0.5; +++++*m*=1.0.

of human head with the surrounding environment causes a general cooling effect and a reduction in human head temperature as shown in Figure 1. This is in agreement with the earlier results obtained by Çelik and Gokmen (2003). In Figure 2, we observed that an increase in the rate of thermogenesis heat production in a human head resulting from cellular metabolism (as normally experienced during feverish condition) may lead to an elevation in human head temperature; however, an increase in the magnitude of metabolic thermogenesis slope parameter may cause a slight decrease in human head temperature as demonstrated in Figure 3.

Conclusion

In this paper, Adomian decomposition method is employed to obtain a non-perturbative semi-analytical solution of a nonlinear model of heat conduction in the human head. The procedure reveals accurately the human head temperature profiles and confirms earlier results reported in the literature. The distribution of the temperature in the human head is very sensitive to environment temperature changes at the periphery due to convective heat exchange with the ambient. Also it can be said the temperature at the center of the head is not affected by the environment temperature as well.

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