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On the Fekete-Szegö like inequality for meromorphic functions with fixed Residue d

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In the present paper, we will consider the class of meromorphic starlike functions with fixed residue d. Silverman et al. (2008) has obtained sharp upper bounds for Fekete-Szegö like functional $\left|a_1 - \mu a_0^2\right|$ for certain subclasses of meromorphic functions. In this paper, we will find sharp upper bounds for $\left|a_1 - \mu a_0^2\right|$ for the class meromorphic starlike functions with fixed residue d. The aim of the present paper, is to completely solve Fekete Szegö problem for a certain subclass of meromorphic starlike functions with fixed residue d.

Key words: Fekete-Szegö inequality, starlike function, analytic function, subordination, meromorphic function.

INTRODUCTION

Let $H(U)$ be the set of functions which are regular in the unit disc $U=\{z:|z|<1\}$, $A=\{f\in H(U): f(0)=f'(0)-1=0\}$ and S denote the class of functions of the form

$$f(z)=z+\sum_{n=2}^{\infty} a_n z^n \tag{1}$$

that are analytic and univalent in unit disc $U=\{z:|z|<1\}$. Let t be a fixed point in U and $A_t=\{f\in H(U): f(t)=f'(t)-1=0\}$. It is easy to see that a function $f\in A_t$ has the series expansion:

$$f(z)=(z-t)+a_2(z-t)^2+\dots .$$

Kanas and Ronning (1999) introduced the following classes:

$$S(t)=\{f\in A(t): f \text{ is univalent in } U\}$$

$$ST(t)=\left\{f\in S(w): \operatorname{Re}\left(\frac{(z-t)f'(z)}{f(z)}\right)>0,\ z\in U\right\}$$

$$CV(t)=\left\{f\in S(t): 1+\operatorname{Re}\left(\frac{(z-t)f''(z)}{f'(z)}\right)>0,\ z\in U\right\}.$$

The class $ST(t)$ is defined by the geometric property that the image of any circular arc centered at t is starlike with respect to $f(t)$ and the corresponding class $CV(t)$ is defined by the property that the image of any circular arc centered t is convex.

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Let $\sum$ denote the class of the functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \tag{2}$$

that are regular and univalent in $U^* = \{z : 0 < |z| < 1\} = U - \{0\}$ with a simple pole at the origin with Residue 1.

For $0 \le t < 1$, let $\sum_t$ denote the class of functions f which are meromorphic and univalent in the unit disc U with the normalization $\lim_{z \to t} f(z) = \infty$.

Let A_t denote the set of function analytic in $U_t^* = \{z \in C : t < |z - t| < 1\} = U - \{t\}$ with the topology given by uniform convergence on compact subsets of $U - \{t\}$. Then A_t is locally convex linear topological space and $\sum_t$ is a compact subset of A_t (Schober, 1975).

In the punctured open unit disk U_t^*, every function f in $\sum_t$ has an expansion of the form $f(z) = \frac{d}{z-t} + \sum_{k=0}^{\infty} a_k (z-t)^k$; $d \in C - \{0\}$.

A function $f \in \sum_t$ is said to be meromorphic starlike functions with fixed Residue d if

$$-\mathrm{Re}\frac{(z-t) f'(z)}{f(z)} > 0 \quad ; \quad z \in U^* \tag{3}$$

and the class of all such meromorphic starlike functions in U_t^* is denoted by $\sum_t^*$.

Let $\phi(z)$ be an analytic funtions with positive real part on U with $\phi(0) = 1$, $\phi'(0) > 1$, which maps the unit disk U onto a region starlike with respect to 1 and is symmetric with respect to the real axis. Let $\sum^*(\phi)$ be the class of functions $f \in \sum$ for which

$$-\frac{zf'(z)}{f(z)} \prec \phi(z) \tag{4}$$

where $\prec$ denotes subordination between analytic functions. This class was studied by Silverman et al. (2008). They have obtained Fekete Szegö like inequality for functions in the class $\sum^*(\phi)$.

Let denote with $\wp(t)$ the class of all functions $p(z) = 1 + \sum_{n=1}^{\infty} c_n (z-t)^n$ that are regular in U and satisfy $p(t) = 1$ and $\mathrm{Re}(p(z)) > 0$ for $z \in U$.

Definition 1

Let $\phi(z)$ be an analytic funtions with positive real part on U with $\phi(0) = 1$, $\phi'(0) > 1$, which maps the unit disk U onto a region starlike with respect to 1 and is symmetric with respect to the real axis. Let $\sum_t^*(\phi)$ be the class of functions $f \in \sum_t$ for which

$$-\frac{(z-t) f'(z)}{f(z)} \prec \phi(z) \tag{5}$$

where $\prec$ denotes subordination between analytic functions and $\sum_t^*(\phi)$ is the meromorphic analogue of the class $S_t^*(\phi)$ which is introduced and studied by Kanas and Ronning (1999).

Fekete and Szegö (1933) obtained sharp bounds for $|a_3 - \mu a_2^2|$ for $f \in S$ and μ real. For different subclasses of S, Fekete-Szegö problem has been investigated by many authors including Darus and Akbalary (2004), Orhan and Raducanu (2009), Orhan et al. (2010), Ravichadran et al. (2005) and Srivastava et al. (2001). Recently, Silverman et al. (2008) has obtained sharp upper bounds for Fekete-Szegö like functional $|a_1 - \mu a_0^2|$ for certain subclasses of $\sum$. In this paper, we will find sharp upper bounds for $|a_1 - \mu a_0^2|$ for the class $\sum_t^*$.

To prove our result, we need the following lemmas. First lemma was obtained by Keogh and Merkes (1969).

Lemma 1

If $p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \ldots$ is a function with positive real part in U, then for any complex number μ,

$$|c_2 - \mu c_1^2| \le 2 \max\{1, |1 - 2\mu|\}. \tag{6}$$

Wald (1978) gives the sharp bounds for the coefficients c_n of the function $p \in \wp(t)$ as follows.

Lemma 2

If $p(z) \in \wp(t)$, $p(z) = 1 + \sum_{n=1}^{\infty} c_n (z-t)^n$ then

$$|c_n| \le \frac{2}{(1+e)(1-e)^n} \quad ; \; where \; e = |t|, \; n \ge 1. \tag{7}$$

Coefficient bounds

By making use of Lemmas, we prove the following bounds for the classes $\sum_t^*$.

Theorem 1

Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \ldots$ if $f(z)$ given by Equation (1) belongs to $\sum_t^*$, then for any complex number μ,

(i) $$|a_1-\mu a_0^2| \le \frac{|B_1|}{2|d|(1+t)^2(1-t)^2}\max\left\{1, \left| t+\frac{B_2}{B_1}-\left(\frac{1}{d}-2\mu d^3\right)B_1\right|\right\} \;;\; B_1 \ne 0 \tag{8}$$

(ii) $$|a_1-\mu a_0^2| \le \frac{1}{|d|(1+e)^2(1-e)^2} \;;\; B_1 = 0 . \tag{9}$$

The bounds are sharp.

Proof

If $f(z) \in \sum_t^*(\phi)$, then there is a Schwarz function $w(z) = (z-t) + A_2(z-t)^2 + A_3(z-t)^3 + \ldots$, analytic in U_t^* with $w(t) = 0$ and $|w(z)| < 1$ in U_t^* such that

$$-\frac{(z-t)f'(z)}{f(z)} = \phi(w(z)) \tag{10}$$

Define the function $p(z)$ by

$$p(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1(z-t) + c_2(z-t)^2 + \ldots \tag{11}$$

Since $w(z)$ is Schwarz function, we see that $R(p(z)) > 0$ and $p(t) = 1$. Therefore

$$\begin{aligned}\phi(w(z)) &= \phi\left(\frac{p(z)-1}{p(z)+1}\right)\\ &= \phi\left(\frac{1}{2}\left[c_1(z-t)+\left(c_2-\frac{c_1^2}{2}\right)(z-t)^2+\left(c_3+\frac{c_1^3}{4}-c_1c_2\right)(z-t)^3+\ldots\right]\right)\\ &= 1+\frac{1}{2}B_1c_1(z-t)+\left(\frac{1}{2}B_1\left(c_2-\frac{1}{2}c_1^2\right)+\frac{1}{4}B_2c_1^2\right)(z-t)^2+\ldots\end{aligned} \tag{12}$$

Now adding Equation (12) in (10), we have

$$-\frac{(z-t)f'(z)}{f(z)} = 1+\frac{1}{2}B_1c_1(z-t)+\left(\frac{1}{2}B_1\left(c_2-\frac{1}{2}c_1^2\right)+\frac{1}{4}B_2c_1^2\right)(z-t)^2+\ldots \tag{13}$$

From this Equation (1), we obtain

$$\frac{a_0}{d} + \frac{B_1c_1}{2} = 0 \tag{14}$$

$$-\frac{a_1}{d} = \frac{a_1}{d} + \frac{a_0B_1c_1}{2d^2} + \frac{B_1c_2}{2} - \frac{B_1c_1^2}{4} + \frac{B_2c_1^2}{4}$$

or equivalently

$$\begin{aligned} a_0 &= -\frac{B_1c_1d}{2}\\ a_1 &= -\frac{1}{2}\left[\frac{1}{2}\frac{B_1c_2}{d} + \frac{1}{4}\left(\frac{B_2}{d} - \frac{B_1}{d} - \frac{B_1^2}{d^2}\right)c_1^2\right]\end{aligned} \tag{15}$$

Therefore,

$$a_1 - \mu a_0^2 = -\frac{B_1}{4d}\left\{c_2 - vc_1^2\right\} \tag{16}$$

Where

$$v = \frac{1}{2}\left[1 - \frac{B_2}{B_1} + \left(\frac{1}{d} - 2\mu d^3\right)B_1\right] \tag{17}$$

The result of Equation (8) follows by an application of Lemma 1.

If $B_1 = 0$, then from Equation (15) $a_0 = 0$ and $a_1 = -\frac{B_2c_1^2}{8d}$. Since $p(z)$ has positive real part, $|c_1| \le \frac{2}{(1+e)(1-e)}$, so that $|a_1 - \mu a_0^2| \le \left|\frac{B_2}{2d(1+e)^2(1-e)^2}\right|$.

Since $\phi(z)$ also has positive real part, $|B_2| \le 2$. Thus, $|a_1 - \mu a_0^2| \le \frac{1}{|d|(1+e)^2(1-e)^2}$, proving Equation (9). The bounds are sharp for the functions $F_1(z)$ and $F_2(z)$ defined by,

$$-\frac{(z-t)F_1'(z)}{F_1(z)} = \phi\left((z-t)^2\right) \quad \text{where}$$

$$F_1(z) = \frac{1+z^2-t(2z-t)}{(z-t)\left(1-z^2+t(2z-t)\right)} \tag{18}$$

$$-\frac{(z-t)F_2'(z)}{F_2(z)} = \phi\left((z-t)\right) \quad \text{where}$$

$$F_2(z) = \frac{1+z-t}{(z-t)(1-z+t)}$$

Clearly, the functions $F_1(z), F_2(z) \in \sum_t$.

Example 1

By taking $\phi(z) = \dfrac{1+z}{1-z}$, we obtain the sharp inequality,

$$\left|a_1 - \mu a_0^2\right| < \frac{1}{|d|} \max\left\{1, \left|\frac{B_2}{B_1} + \left(\frac{1}{d} - 2\mu d^3\right) B_1\right|\right\} \quad ; \quad B_1 \neq 0 \ .$$

Putting $d = 1$ and $t \to 0$ in Theorem 1, we get the following result obtained by Silverman et al. (2008).

Corollary 1

Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \ldots$ if $f(z)$ given by Equation (1) belongs to $\sum_0^*(\phi) = \sum^*(\phi)$, then for any complex number μ,

(i) $\left|a_1 - \mu a_0^2\right| \leq \dfrac{|B_1|}{2} \max\left\{1, \left| t + \dfrac{B_2}{B_1} - (1 - 2\mu) B_1\right|\right\} \quad ; \quad B_1 \neq 0$

(ii) $\left|a_1 - \mu a_0^2\right| \leq 1 \quad ; \quad B_1 = 0$.

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