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The further generalization on the inequalities for *Hadamard* products of any number of invertible Hermitian matrices

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Abstract
Without 'positive definiteness' demanded in the present papers, the forward and reverse inequalities for *Hadamard* products of any number of invertible Hermitian matrices are obtained, and the sufficient and necessary conditions for the equations in these inequalities are given. As Hermitian positive matrices naturally satisfy the added constraints, these results generalize and improve the corresponding results in the present papers. Beyond that, with no demand of 'positive definiteness', these forward and backward inequalities are not determined mutually any longer.

Keywords: Hermitian matrix; *Hadamard* product; matrix inequality; equation condition

1 Introduction

Throughout the paper, we assume $C^{m\times n}$ is the set of $m \times n$ complex matrices, I is a identity matrix, E_{ii} is a diagonal matrix with 1 at its (i,i)th position and 0 elsewhere, and $Z_n = [E_{11}, E_{22}, \dots, E_{nn}]^* \in C^{n^2\times n}$ is a selection matrix. A^* stands for the conjugate transpose of $A \in C^{m\times n}$. The matrix A is Hermitian if $A^* = A$, denoted by $A \in H(n)$. Furthermore, we denote by $H_0^+(n)$ and $H^+(n)$ the sets of Hermitian semi-positive matrices and Hermitian positive matrices, respectively. Recall that A, B are said to have the inequality $A \geq B$ or $B \leq A$, if $A - B \in H_0^+(n)$. In particular, $A \in H_0^+(n)$ (resp. $A \in H^+(n)$), denoted by $A \geq 0$ (resp. $A > 0$), and denoted by $A^{\frac{1}{2}}$ the square root of A. For a positive integer k, we have $\alpha(k) \subseteq \langle n^k\rangle = \{1, 2, \dots, n^k\}$, $\alpha'(k) = \langle n^k\rangle - \alpha(k)$. Especially, $\alpha = \alpha(1) \in \langle n\rangle$. For $A \in C^{n\times n}$, $A(\alpha, \beta)$ denotes the submatrix of A lying in rows indexed by α and the columns indexed by β and $A(\alpha, \alpha) = A(\alpha)$. The *Hadamard* and *Kronecker* products of $A = (a_{ij})$, $B = (B_{ij})$ are defined as $A \circ B = (a_{ij}b_{ij})$ and $A \otimes B = (a_{ij}B)$, respectively. If $AXA = X$, $XAX = X$, $(AX)^* = AX$, and $(XA)^* = XA$, then X is said to be a *Moore-Penrose* generalized inverse of A, denoted by $X = A^+$.

Recall that if every diagonal element of $R \in H_0^+(n)$ is 1, then R is said to be a correlation matrix, in symbol $R \in CH_0^+(n)$ (see [1]). The invertible matrix $A \in H_0^+(n)$ implies $A \in H^+(n)$, thus the set of invertible matrices $CH^+(n) \subset H^+(n)$.

By multivariate analysis, Styan obtained the inequalities as follows in 1973 (see [1, Theorem 4.1, Corollary 4.2(4.21) and Corollary 4.3]):

$$R \circ R - 2\left(R^{-1} \circ R + I\right)^{-1} \geq 0, \quad R \in CH^+(n); \tag{1.1}$$

$$R^{-1} \circ R + I - 2(R \circ R)^{-1} \geq 0, \quad R \in CH^+(n); \tag{1.2}$$

$$A \circ A - 2(A \circ I)\bigl(A^{-1} \circ A + I\bigr)^{-1}(A \circ I) \geq 0, \quad A \in H^+(n); \tag{1.3}$$

$$A^{-1} \circ A + I - 2(A \circ I)(A \circ A)^{-1}(A \circ I) \geq 0, \quad A \in H^+(n). \tag{1.4}$$

Meanwhile, Styan pointed out 'A matrix-theoretic proof of Theorem 4.1 would be of interest.' (see [1]).

Many papers [2–10] focus on the generalization of inequalities (1.1)-(1.4) to Hermitian (semi-)positive matrices by matrix methods. As a generalization of the usual *Hadamard* product, the *Khatri-Rao* product [5, 6, 8] has many similar properties to the *Hadamard* product (see [5]), thus here we only focus on the *Hadamard* product, a basic product. Referring to [2] and [6], we denote

$$\prod_{l=1}^{k} \circ A_l = A_1 \circ A_2 \circ \cdots \circ A_k, \qquad \prod_{l=1}^{k} \otimes A_l = A_1 \otimes A_2 \otimes \cdots \otimes A_k, \quad A_l \in C^{n\times n}, l = 1, 2, \ldots, k.$$

In 1979, Ando [2] obtained the following.

Proposition 1.1 (see [2, Theorem 20]) *Let $A \in H^+(n)$, for any positive integer k (≥ 2), one has*

$$\prod_{l=1}^{k} \circ A \geq k(A \circ I)^{k-1}\left[A^{-1} \circ \left(\prod_{l=1}^{k-1} \circ A\right) + (k-1)(A \circ I)^{k-2}\right]^{-1}(A \circ I)^{k-1}. \tag{1.5}$$

When $k = 2$, the inequality (1.5) implies (1.3). Ando also made clear, by applying the method of Proposition 1.1, that one has the following.

Proposition 1.2 (see [2, p.239]) *Let $A, B \in H^+(n)$. Then*

$$A \circ B \geq (A \circ I + B \circ I)\bigl(A \circ B^{-1} + A^{-1} \circ B + 2I\bigr)^{-1}(A \circ I + B \circ I). \tag{1.6}$$

In 2000, Zhang showed the following.

Proposition 1.3 (see [3, Application 4]) *Let $A, B \in H^+(n)$. Then*

$$A \circ B^{-1} + A^{-1} \circ B + 2I \geq (A \circ I + B \circ I)(A \circ B)^{-1}(A \circ I + B \circ I). \tag{1.7}$$

As applications, (1.2) and (1.4) were also given by Visick in 2000 (see [4, Theorem 20]).

Moreover, Al Zhour and Kilicman obtained a matrix inequality as follows in 2006 (see [6, Theorem 4.4]):

$$\begin{aligned}&\left(A_1 \circ \prod_{l=2}^{k} \circ A_l^{\frac{1}{2}} A_l^{+\frac{1}{2}} + A_1^{\frac{1}{2}} A_1^{+\frac{1}{2}} \circ \prod_{l=2}^{k} \circ A_l\right)\left(\prod_{l=1}^{k} \circ A_l\right)^{+} \\ &\quad \times \left(A_1 \circ \prod_{l=2}^{k} \circ A_l^{\frac{1}{2}} A_l^{+\frac{1}{2}} + A_1^{\frac{1}{2}} A_1^{+\frac{1}{2}} \circ \prod_{l=2}^{k} \circ A_l\right)\end{aligned}$$

$$\leq 2\left(\prod_{l=1}^{k} \circ A_l^{\frac{1}{2}} A_l^{+\frac{1}{2}}\right) + \left(A_1 \circ \prod_{l=2}^{k} \circ A_l^{+}\right) + \left(A_1^{+} \circ \prod_{l=2}^{k} \circ A_l\right),$$

$$A_l \in H_0^{+}(n), l = 1, 2, \ldots, k; \tag{1.8}$$

$$\left(A \circ B^{\frac{1}{2}} B^{+\frac{1}{2}} + A^{\frac{1}{2}} A^{+\frac{1}{2}} \circ B\right)(A \circ B)^{+}\left(A \circ B^{\frac{1}{2}} B^{+\frac{1}{2}} + A^{\frac{1}{2}} A^{+\frac{1}{2}} \circ B\right)$$

$$\leq A \circ B^{+} + A^{+} \circ B + 2A^{\frac{1}{2}} A^{+\frac{1}{2}} \circ B^{\frac{1}{2}} B^{+\frac{1}{2}}, \quad A, B \in H_0^{+}(n). \tag{1.9}$$

For $A \in H_0^{+}(n)$, by [11], one has $A^{\frac{1}{2}} A^{+\frac{1}{2}} = AA^{+}$. Because of the commutativity of the *Hadamard* product, the inequality (1.9) is equivalent to [7, Proposition 1], [8, Corollary 1(5)], it could be viewed as a generalization of (1.4) over Hermitian semi-positive matrices.

For $A, B \in H^{+}(n)$, (1.6) and (1.7) are the forward and backward inequalities to each other, and determined mutually as well (Theorem 2.6), [8] has ever called them as companion inequalities. Of course, (1.1) and (1.2), (1.3) and (1.4) are also the companion inequalities determined by each other (Lemma 2.4). In fact, [9, Theorem 1] shows us the backward inequality which is companied to (1.9).

Papers [1–10] all discuss on Hermitian (semi-)positive matrices. However, the following Example 1.4 illustrates that the condition 'positive definiteness' is not necessary for the inequality (1.7) to hold.

Example 1.4 Let $A = \frac{1}{2}\left[\begin{smallmatrix} -1 & 1 \\ 1 & 1 \end{smallmatrix}\right], B = \frac{1}{3}\left[\begin{smallmatrix} 1 & 1 \\ 1 & -2 \end{smallmatrix}\right] \in H(2)$. Then the matrix inequality (1.7) holds, this is because

$$\left(A \circ B^{-1} + A^{-1} \circ B + 2I\right) - \left((A + B) \circ I\right)(A \circ B)^{-1}\left((A + B) \circ I\right) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \in H_0^{+}(2).$$

Example 1.5 Let $A = \left[\begin{smallmatrix} -1 & 1 \\ 1 & 1 \end{smallmatrix}\right], B = \left[\begin{smallmatrix} -1 & 1 \\ 1 & 2 \end{smallmatrix}\right] \in H(2)$. Then the inequality (1.7) does not hold anymore, this is because

$$\left(A \circ B^{-1} + A^{-1} \circ B + 2I\right) - \left((A + B) \circ I\right)(A \circ B)^{-1}\left((A + B) \circ I\right) = \frac{1}{6}\begin{bmatrix} -29 & -31 \\ -31 & -34 \end{bmatrix}.$$

Example 1.5 indicates that, in general, the matrix inequality (1.7) does not hold for all invertible Hermitian matrices. Hence we will add some constraint conditions in our discussion.

In this paper, without the 'positive definiteness' demanded in the inequality (1.5), we will show inequalities and their companion forms for any number of invertible Hermitian matrices and get sufficient and necessary conditions for the equations in the related inequalities to hold. In view of [12, 13] *etc.*, we see the discussion of the equation conditions for the inequalities is significant. Further, Hermitian positive matrices satisfy the added constraints naturally, thus these results generalize and improve the corresponding results in the present literature. However, with no demand of 'positive definiteness', the new forward and backward companion inequalities are not determined mutually.

2 Preliminaries

For matrices of appropriate sizes, in view of [14, Proposition 4.2.14] and [15, Propositions 7.13-7.17], by induction to k, it follows that

$$(A \otimes B) \otimes C = A \otimes (B \otimes C); \tag{2.1}$$

$$(A + B) \otimes C = A \otimes C + B \otimes C, \qquad C \otimes (A + B) = C \otimes A + C \otimes B; \tag{2.2}$$

$$\left(\prod_{l=1}^{k} \otimes A_l\right)^* = \prod_{l=1}^{k} \otimes A_l^*, \quad A_l \in C^{m\times n}, l = 1, 2, \dots, k; \tag{2.3}$$

$$\left(\prod_{l=1}^{k} \otimes A_l\right)\left(\prod_{l=1}^{k} \otimes B_l\right) = \prod_{l=1}^{k} \otimes A_l B_l, \qquad \prod_{l=1}^{k} (A_l \otimes B_l) = \left(\prod_{l=1}^{k} A_l\right) \otimes \left(\prod_{l=1}^{k} B_l\right); \tag{2.4}$$

$$\left(\prod_{l=1}^{k} \otimes A_l\right)^{-1} = \prod_{l=1}^{k} \otimes A_l^{-1}, \quad \text{any of } A_l \in C^{n\times n} \text{ is invertible}, l = 1, 2, \dots, k; \tag{2.5}$$

$$\begin{aligned} &A_l \in H(n)\big(H_0^+(n), H^+(n)\big), \qquad l = 1, 2, \dots, k \\ &\quad \Rightarrow \quad \prod_{l=1}^{k} \otimes A_l \in H\big(n^k\big)\big(H_0^+\big(n^k\big), H^+\big(n^k\big)\big). \end{aligned} \tag{2.6}$$

Lemma 2.1 *Let* $A_l \in C^{n\times n}$, $l = 1, 2, \dots, k$. *Then*

$$\prod_{l=1}^{k} \circ A_l = \left(\prod_{l=1}^{k} \otimes A_l\right)\big(\alpha(k)\big) \in C^{n\times n}, \quad \alpha(k) = \big(\alpha_k^{(1)}, \alpha_k^{(2)}, \dots, \alpha_k^{(n)}\big) \subset \langle n^k \rangle; \tag{2.7}$$

$$\alpha_k^{(l)} = \big[(l-1)\big(n^k - 1\big)/(n-1)\big] + 1, \quad 1 \le l \le n. \tag{2.8}$$

Proof In view of [4, Theorem 1 and Corollary 2] or [15, Proposition 7.3.1] yields

$$A_1 \circ A_2 = P(n,2)^*(A_1 \otimes A_2)P(n,2), \quad P(n,2) = Z_n \in C^{n^2\times n}, A_1, A_2 \in C^{n\times n},$$

that is,

$$A_1 \circ A_2 = (A_1 \otimes A_2)\big(\alpha(2)\big), \quad \alpha(2) = \big(1, n+2, 2n+3, \dots, n^2\big) \subset \langle n^2 \rangle, \tag{2.9}$$

which indicates (2.7) and (2.8) hold for $k = 2$.

By (2.9), (2.1), (2.3), and (2.4), similar to [10, Lemma 2.1], it follows that

$$\begin{aligned} A_1 \circ A_2 \circ A_3 &= A_1 \circ (A_2 \circ A_3) \\ &= P(n,2)^*\big[IA_1 I \otimes P(n,2)^*(A_2 \otimes A_3)P(n,2)\big]P(n,2) \\ &= P(n,2)^*\big(I \otimes P(n,2)\big)^*(A_1 \otimes A_2 \otimes A_3)\big(I \otimes P(n,2)\big)P(n,2), \end{aligned}$$

that is,

$$A_1 \circ A_2 \circ A_3 = P(n,3)^*(A_1 \otimes A_2 \otimes A_3)P(n,3) \in C^{n\times n}, \quad P(n,3) = \big(I \otimes P(n,2)\big)Z_n. \tag{2.10}$$

As $E_{ii}E_{jj} = E_{ii}$ $(i = j)$, $E_{ii}E_{jj} = 0$ $(i \neq j)$, and $E_{ii} \in H(n)$, by (2.9) and (2.10), we see

$$\begin{aligned} P(n,3) &= \operatorname{diag}\bigl(P(n,2), P(n,2), \ldots, P(n,2)\bigr)Z_n \\ &= (E_{11}, 0, \ldots, 0; 0, E_{22}, 0, \ldots, 0; \ldots; 0, \ldots, 0, E_{nn}) \in C^{n^3 \times n}, \end{aligned}$$

which shows the $\alpha(3)$ determined by $P(n,3)$ satisfies (2.8), then (2.7) holds.

Just as the proof of [10, Lemma 2.1], similarly, one has

$$\prod_{l=1}^{k} \circ A_l = P(n,k)^* \left(\prod_{l=1}^{k} \otimes A_l \right) P(n,k), \quad P(n,k) = \bigl(I \otimes P(n,k-1)\bigr)Z_n \in C^{n^k \times n}, \tag{2.11}$$

thus the $\alpha(k)$ determined by $P(n,k)$ satisfies (2.8), then (2.7) follows by (2.11). □

The proof method of Lemma 2.1 plays a great role in discussing the matrix inequalities for *Khatri-Rao* products of any finite number of positive matrices (see [16, Lemma 2.1], [6, Lemmas 2.1 and 2.2]). Recently, [17, Theorem 3] has also discussed a similar problem to Lemma 2.1, but our results (2.7) and (2.8) should be more convenient in applications.

When $A \in C^{n\times n}$, if $\alpha = \alpha(1) \in \langle n \rangle$ and $A(\alpha)$ is invertible, we call

$$A/\alpha = A\bigl(\alpha'\bigr) - A\bigl(\alpha', \alpha\bigr)A(\alpha)^{-1}A\bigl(\alpha, \alpha'\bigr), \quad \alpha' = \langle n \rangle - \alpha, A(\alpha)^{-1} = \bigl(A(\alpha)\bigr)^{-1}$$

the Schur complement of $A(\alpha)$ in A (see [8, 18, 19]).

Lemma 2.2 *Let $A \in H(n)$ and $A(\alpha')$ be both invertible. Then both of A/α' and $A^{-1}(\alpha)$ are invertible as well, and $(A/\alpha')^{-1} = A^{-1}(\alpha)$.*

Proof By assumption, there exists a permutation matrix U such that

$$U^*AU = \begin{bmatrix} A(\alpha) & A(\alpha,\alpha') \\ A(\alpha',\alpha) & A(\alpha') \end{bmatrix}, U^*A^{-1}U = \begin{bmatrix} A^{-1}(\alpha) & A^{-1}(\alpha,\alpha') \\ A^{-1}(\alpha',\alpha) & A^{-1}(\alpha') \end{bmatrix} \in H(n). \tag{2.12}$$

Since both of A and $A(\alpha')$ are invertible, by (2.12), one has a matrix $V = \left[\begin{smallmatrix} I & 0 \\ -A(\alpha')^{-1}A(\alpha',\alpha) & I \end{smallmatrix}\right]$ such that

$$(UV)^*A(UV) = \operatorname{diag}\bigl(A/\alpha', A\bigl(\alpha'\bigr)\bigr) = A/\alpha' \oplus A\bigl(\alpha'\bigr) \in H(n),$$

thus A/α' is invertible. By (2.12),

$$U^*A^{-1}U = \bigl(U^*AU\bigr)^{-1} = V\bigl(\bigl(A/\alpha'\bigr)^{-1} \oplus A\bigl(\alpha'\bigr)^{-1}\bigr)V^* = \begin{bmatrix} (A/\alpha')^{-1} & * \\ * & * \end{bmatrix},$$

by comparing with (2.12), it follows that $A^{-1}(\alpha)$ is invertible and $(A/\alpha')^{-1} = A^{-1}(\alpha)$. □

When $A \in H^+(n)$, it is natural that $A(\alpha')$ is invertible, hence we could obtain [18, formula (4)] again by Lemma 2.2.

Lemma 2.3 *Let $C \in C^{n\times n}$ and $A \in H(n)$ be invertible, $\alpha \subset \langle n\rangle$, $\alpha' = \langle n\rangle - \alpha$, if $A^{-1}(\alpha') > 0$, then $A(\alpha)$ is invertible and*

$$C(\alpha)^* A(\alpha)^{-1} C(\alpha) \le \big(C^* A^{-1} C\big)(\alpha), \quad \textit{where } C(\alpha)^* = \big(C(\alpha)\big)^*. \tag{2.13}$$

Moreover, the equation in (2.13) *holds if and only if* $A^{-1}(\alpha',\alpha)C(\alpha) + A^{-1}(\alpha')C(\alpha',\alpha) = 0$.

Proof In this case, there is a permutation matrix U such that (2.12) holds and $U^*CU = \left[\begin{smallmatrix} C(\alpha) & C(\alpha,\alpha') \\ C(\alpha',\alpha) & C(\alpha') \end{smallmatrix}\right]$, then

$$U^*\big(C^*A^{-1}C\big)U = \begin{bmatrix} (C^*A^{-1}C)(\alpha) & (C^*A^{-1}C)(\alpha,\alpha') \\ (C^*A^{-1}C)(\alpha',\alpha) & (C^*A^{-1}C)(\alpha') \end{bmatrix} \in H(n). \tag{2.14}$$

By assumption, both of $A^{-1} \in H(n)$ and $A^{-1}(\alpha')$ are Hermitian and invertible, then by applying Lemma 2.2, $A(\alpha)$ is invertible and

$$A^{-1}/\alpha' = A^{-1}(\alpha) - A^{-1}\big(\alpha,\alpha'\big)A^{-1}\big(\alpha'\big)^{-1}A^{-1}\big(\alpha',\alpha\big)^{-1} = \big(A^{-1}\big)^{-1}(\alpha)^{-1} = A(\alpha)^{-1};$$

combining with (2.12), there exists $W = \left[\begin{smallmatrix} I & 0 \\ -A^{-1}(\alpha')^{-1}A^{-1}(\alpha',\alpha) & I \end{smallmatrix}\right]$ such that

$$W^*U^*A^{-1}UW = \operatorname{diag}\big(A^{-1}/\alpha', A^{-1}\big(\alpha'\big)\big) = A(\alpha)^{-1} \oplus A^{-1}\big(\alpha'\big), \tag{2.15}$$

$$W^{-1}U^*CU = \begin{bmatrix} C(\alpha) & C(\alpha,\alpha') \\ X & Y \end{bmatrix}, \quad X = A^{-1}\big(\alpha'\big)^{-1}A^{-1}\big(\alpha',\alpha\big)C(\alpha) + C\big(\alpha',\alpha\big). \tag{2.16}$$

By (2.14)-(2.16), as $A^{-1}(\alpha') > 0$, then $(C^*A^{-1}C)(\alpha) - C(\alpha)^*A(\alpha)^{-1}C(\alpha) = X^*A^{-1}(\alpha')X \ge 0$, that is, (2.13) follows. Meanwhile, the equation in (2.13) holds; therefore $X = 0$, it is equivalent to

$$A^{-1}\big(\alpha'\big)X = A^{-1}\big(\alpha',\alpha\big)C(\alpha) + A^{-1}\big(\alpha'\big)C\big(\alpha',\alpha\big) = 0. \qquad \square$$

For a Hermitian positive matrix A, one has $A^{-1}(\alpha') > 0$; then we get [3, Theorem 1(7)] again by Lemma 2.3.

Lemma 2.4 *Let $F, G \in H^+(n)$. Then*

$$F \ge TG^{-1}T^* \quad \Leftrightarrow \quad G \ge T^*F^{-1}T \quad \textit{and} \quad F = TG^{-1}T^* \quad \Leftrightarrow \quad G = T^*F^{-1}T.$$

Proof In this case, $M = \left[\begin{smallmatrix} F & T \\ T^* & G \end{smallmatrix}\right] \in H(2n)$, and there exist invertible matrices $P = \left[\begin{smallmatrix} I & 0 \\ -G^{-1}T^* & I \end{smallmatrix}\right]$ and $Q = \left[\begin{smallmatrix} I & -F^{-1}T \\ 0 & I \end{smallmatrix}\right]$ such that

$$P^*MP = \big(F - TG^{-1}T^*\big) \oplus G, \qquad Q^*MQ = F \oplus \big(G - T^*F^{-1}T\big),$$

which indicates

$$\begin{aligned} F \ge TG^{-1}T^* \quad &\Leftrightarrow \quad P^*MP \in H_0^+(2n) \\ &\Leftrightarrow \quad Q^*MQ \in H_0^+(2n) \quad \Leftrightarrow \quad G \ge T^*F^{-1}T, \end{aligned}$$

$$F = TG^{-1}T^* \quad \Leftrightarrow \quad \operatorname{rank} P^*MP = \operatorname{rank} G = n$$
$$\Leftrightarrow \quad \operatorname{rank} Q^*MQ = \operatorname{rank} F = n \quad \Leftrightarrow \quad G = T^*F^{-1}T. \qquad \square$$

Lemmas 2.2-2.4 will play an important role in the discussion.

Theorem 2.5 *Let $A \in H^+(n)$, for any positive integer k (≥ 2). Then*

$$A^{-1} \circ \left(\prod_{l=1}^{k-1} \circ A\right) + (k-1)(A \circ I)^{k-2} \geq k(A \circ I)^{k-1}\left(\prod_{l=1}^{k} \circ A\right)^{-1}(A \circ I)^{k-1}. \tag{2.17}$$

Moreover, the equation in (1.5) *holds if and only if the one in* (2.17) *holds.*

Proof As $A \in H^+(n)$, by (2.6), $\prod_{l=1}^{k} \circ A, A^{-1} \circ (\prod_{l=1}^{k-1} \circ A) + (k-1)(A \circ I)^{k-2} \in H^+(n)$. Taking $T^* = T = \sqrt{k}(A \circ I)^{k-1} \in H(n)$, in view of Proposition 1.1, yields the inequality (1.5), that is, $\prod_{l=1}^{k} \circ A \geq T^*(A^{-1} \circ (\prod_{l=1}^{k-1} \circ A) + (k-1)(A \circ I)^{k-2})^{-1}T$, then by Lemma 2.4, $A^{-1} \circ (\prod_{l=1}^{k-1} \circ A) + (k-1)(A \circ I)^{k-2} \geq T^*(\prod_{l=1}^{k} \circ A)^{-1}T$, which shows (2.17) holds, meanwhile, the equation in (1.5) holds; therefore the one in (2.17) holds. $\square$

Theorem 2.5 not only leads to the backward inequality (2.17) of (1.5) (in this case, the inequalities (1.5) and (2.17) are mutually determined), but it also shows us that the inequalities (1.1) and (1.2), (1.3) and (1.4) given by Styan are companied and determined by each other (the case of $k = 2$ in Theorem 2.5).

By applying Lemma 2.4, Propositions 1.2 and 1.3, with a similar discussion as Theorem 2.5, we have the following.

Theorem 2.6 *Let $A \in H^+(n)$. Then the inequalities* (1.6) *and* (1.7) *are companied and determined by each other, and the equation in* (1.6) *holds*; *therefore the one in* (1.7) *holds.*

3 Main results

For Hermitian matrices A_l $(l = 1, 2, \ldots, k)$, unless otherwise specified, we always assume

$$C = \sum_{t=1}^{k}\left[\left(\prod_{l=1}^{t-1} \otimes A_l\right) \otimes I \otimes \left(\prod_{l=t+1}^{k} \otimes A_l\right)\right], \qquad N = \sum_{t=1}^{k}\left(\left(\prod_{l\neq t} \circ A_l\right) \circ I\right).$$

Theorem 3.1 *Let $A_l \in H(n)$ be invertible, $l = 1, 2, \ldots, k$, and $\alpha(k)$ be as in* (2.7) *and* (2.8), *if* $(\prod_{l=1}^{k} \otimes A_l^{-1})(\alpha'(k)) > 0$, *then $\prod_{l=1}^{k} \circ A_l$ is also invertible and*

$$\sum_{t=1}^{k}\left[A_t^{-1} \circ \left(\prod_{l\neq t} \circ A_l\right)\right] + 2\sum_{1\leq t<s\leq k}\left[\left(\prod_{l\neq t,s} \circ A_l\right) \circ I\right] \geq N\left(\prod_{l=1}^{k} \circ A_l\right)^{-1} N. \tag{3.1}$$

Moreover, the equation in (3.1) *holds if and only if*

$$\left(\prod_{l=1}^{k} \otimes A_l^{-1}\right)\big(\alpha'(k), \alpha(k)\big)N + \left(\prod_{l=1}^{k} \otimes A_l^{-1}\right)\big(\alpha'(k)\big)C\big(\alpha'(k), \alpha(k)\big) = 0. \tag{3.2}$$

Proof By (2.6), we see $C = \sum_{t=1}^{k}[(\prod_{l=1}^{t-1} \otimes A_l) \otimes I \otimes (\prod_{l=t+1}^{k} \otimes A_l)] \in H(n^k)$, and by (2.5), $\prod_{l=1}^{k} \otimes A_l^{-1} = (\prod_{l=1}^{k} \otimes A_l)^{-1}$ is invertible. From our assumption, $(\prod_{l=1}^{k} \otimes A_l^{-1})(\alpha'(k)) > 0$, then by applying (2.7) and Lemma 2.3, $\prod_{l=1}^{k} \circ A_l = (\prod_{l=1}^{k} \otimes A_l)(\alpha(k))$ is invertible. By the commutativity of *Hadamard* products, combining with (2.1)-(2.6) yields

$$\begin{aligned}
C^*\left(\prod_{l=1}^{k} \otimes A_l\right)^{-1} C = C\left(\prod_{l=1}^{k} \otimes A_l^{-1}\right)C \\
&= \left[\sum_{t=1}^{k}\left(\left(\prod_{l=1}^{t-1} \otimes A_l\right) \otimes I \otimes \left(\prod_{l=t+1}^{k} \otimes A_l\right)\right)\right]\left(\prod_{l=1}^{k} \otimes A_l^{-1}\right) \\
&\quad \times \left[\sum_{t=1}^{k}\left(\left(\prod_{l=1}^{t-1} \otimes A_l\right) \otimes I \otimes \left(\prod_{l=t+1}^{k} \otimes A_l\right)\right)\right] \\
&= \left[\sum_{t=1}^{k}\left(\left(\prod_{l=1}^{t-1} \otimes A_l A_l^{-1}\right) \otimes I A_t^{-1} \otimes \left(\prod_{l=t+1}^{k} \otimes A_l A_l^{-1}\right)\right)\right] \\
&\quad \times \left[\sum_{t=1}^{k}\left(\left(\prod_{l=1}^{t-1} \otimes A_l\right) \otimes I \otimes \left(\prod_{l=t+1}^{k} \otimes A_l\right)\right)\right] \\
&= \left[\sum_{t=1}^{k}\left(\left(\prod_{l=1}^{t-1} \otimes I\right) \otimes A_t^{-1} \otimes \left(\prod_{l=t+1}^{k} \otimes I\right)\right)\right] \\
&\quad \times \left[\sum_{t=1}^{k}\left(\left(\prod_{l=1}^{t-1} \otimes A_l\right) \otimes I \otimes \left(\prod_{l=t+1}^{k} \otimes A_l\right)\right)\right] \in H(n^k),
\end{aligned}$$

that is,

$$\begin{aligned}
& C^*\left(\prod_{l=1}^{k} \otimes A_l\right)^{-1} C \\
&= \sum_{t=1}^{k}\left(A_1 \otimes \cdots \otimes A_{t-1} \otimes A_t^{-1} \otimes A_{t+1} \otimes \cdots \otimes A_k\right) \\
&\quad + 2\sum_{1\le t<s\le k}\left(A_1 \otimes \cdots \otimes A_{t-1} \otimes I \otimes A_{t+1} \otimes \cdots \otimes A_{s-1} \otimes I \otimes A_{s+1} \otimes \cdots \otimes A_k\right). \qquad (3.3)
\end{aligned}$$

By (2.7), (2.13), and (3.3), we have $C(\alpha(k)) = \sum_{t=1}^{k}[(\prod_{l\neq t} \circ A_l) \circ I] = N \in H(n)$, and

$$\begin{aligned}
& \left(C^*\left(\prod_{l=1}^{k} \otimes A_l\right)^{-1} C\right)(\alpha(k)) \\
&= \sum_{t=1}^{k}\left(\left(\prod_{l=1}^{t-1} \otimes A_l\right) \otimes A_t^{-1} \otimes \left(\prod_{l=t+1}^{k} \otimes A_l\right)\right)(\alpha(k)) \\
&\quad + 2\sum_{1\le t<s\le k}\left(\left(\prod_{l=1}^{t-1} \otimes A_l\right) \otimes I \otimes \left(\prod_{l=t+1}^{s-1} \otimes A_l\right) \otimes I \otimes \left(\prod_{l=s+1}^{k} \otimes A_l\right)\right)(\alpha(k)) \\
&= \sum_{t=1}^{k}\left(A_1 \circ \cdots \circ A_{t-1} \circ A_t^{-1} \circ A_{t+1} \circ \cdots \circ A_k\right)
\end{aligned}$$

$$
\begin{aligned}
&+ 2\sum_{1\le t<s\le k}(A_1 \circ \cdots \circ A_{t-1} \circ I \circ A_{t+1} \circ \cdots \circ A_{s-1} \circ I \circ A_{s+1} \circ \cdots \circ A_k) \\
&= \sum_{t=1}^{k}\left(A_t^{-1} \circ \left(\prod_{l\neq t} \circ A_l\right)\right) + 2\sum_{1\le t<s\le k}\left(\left(\prod_{l\neq t,s} \circ A_l\right) \circ I\right) \\
&\ge C\big(\alpha(k)\big)^*\left(\prod_{l=1}^{k} \otimes A_l\right)\big(\alpha(k)\big)^{-1} C\big(\alpha(k)\big) \\
&= \left[\sum_{t=1}^{k}\left(\left(\prod_{l\neq t} \circ A_l\right) \circ I\right)\right]\left(\prod_{l=1}^{k} \circ A_l\right)^{-1}\left[\sum_{t=1}^{k}\left(\left(\prod_{l\neq t} \circ A_l\right) \circ I\right)\right] \\
&= N\left(\prod_{l=1}^{k} \circ A_l\right)^{-1} N \in H(n),
\end{aligned}
$$

so (3.1) holds.

From Lemma 2.3, the equation in (3.1) holds; therefore

$$
\begin{aligned}
&\left(\prod_{l=1}^{k} \otimes A_l^{-1}\right)\big(\alpha'(k),\alpha(k)\big)N + \left(\prod_{l=1}^{k} \otimes A_l^{-1}\right)\big(\alpha'(k)\big)C\big(\alpha'(k),\alpha(k)\big) \\
&= \left(\prod_{l=1}^{k} \otimes A_l\right)^{-1}\big(\alpha'(k),\alpha(k)\big)C\big(\alpha(k)\big) + \left(\prod_{l=1}^{k} \otimes A_l\right)^{-1}\big(\alpha'(k)\big)C\big(\alpha'(k),\alpha(k)\big) = 0. \qquad \square
\end{aligned}
$$

From Theorems 2.5 and 2.6, we see the inequalities (1.1) and (1.2), (1.3) and (1.4), and the general ones (1.5) and (2.17), (1.6) and (1.7) are companied and determined by each other, hence one of the companion inequalities could be obtained from the other one immediately. However, the following example indicates that the matrix inequality (3.1) is no longer equivalent to its backward inequality, without 'positive definiteness'.

Example 3.2 Let A, B just as the one in Example 1.4, as $(A^{-1} \otimes B^{-1})(\alpha'(2)) = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right] > 0$, then (1.7) follows by Theorem 3.1, but the inequality (1.6) does not hold; this is because

$$
(A \circ B) - (A \circ I + B \circ I)\big(A^{-1} \circ B + B^{-1} \circ A + 2I\big)^{-1}(A \circ I + B \circ I) = \frac{1}{30}\begin{bmatrix} -10 & 10 \\ 10 & -16 \end{bmatrix}.
$$

Theorem 3.3 *Let $A_l \in H(n)$ $(l = 1,2,\ldots,k)$ be invertible, and $\alpha(k)$ be as in* (2.7) *and* (2.8), *if C is invertible and $(C^{-1}(\prod_{l=1}^{k} \otimes A_l)C^{-1})(\alpha'(k)) > 0$, then $\sum_{t=1}^{k}(A_t^{-1} \circ (\prod_{l\neq t} \circ A_l)) + 2\sum_{1\le t<s\le k}((\prod_{l\neq t,s} \circ A_l) \circ I) \in H(n)$ is invertible as well and*

$$
N\left[\sum_{t=1}^{k}\left(A_t^{-1} \circ \left(\prod_{l\neq t} \circ A_l\right)\right) + 2\sum_{1\le t<s\le k}\left(\left(\prod_{l\neq t,s} \circ A_l\right) \circ I\right)\right]^{-1} N \le \prod_{l=1}^{k} \circ A_l. \tag{3.4}
$$

Moreover, the equation in (3.4) *holds if and only if*

$$
\begin{aligned}
&\left(C^{-1}\left(\prod_{l=1}^{k} \otimes A_l\right)C^{-1}\right)\big(\alpha'(k),\alpha(k)\big)N + \left(C^{-1}\left(\prod_{l=1}^{k} \otimes A_l\right)C^{-1}\right)\big(\alpha'(k)\big)C\big(\alpha'(k),\alpha(k)\big) \\
&= 0.
\end{aligned} \tag{3.5}
$$

Proof By (2.6), $C \in H(n^k)$ is invertible, then $\prod_{l=1}^k \otimes A_l = C^*(C^*(\prod_{l=1}^k \otimes A_l)^{-1}C)^{-1}C \in H(n^k)$ is also invertible. As

$$\left(C^{-1}\left(\prod_{l=1}^{k} \otimes A_l\right)C^{-1}\right)\bigl(\alpha'(k)\bigr) = \left(C^*\left(\prod_{l=1}^{k} \otimes A_l\right)^{-1} C\right)^{-1}\bigl(\alpha'(k)\bigr) > 0,$$

in view of (3.3), (2.7), and Lemma 2.3 we find that

$$\begin{aligned}&\left(C^*\left(\prod_{l=1}^{k} \otimes A_l\right)^{-1} C\right)\bigl(\alpha(k)\bigr)\\ &\quad= \sum_{t=1}^{k}\left(A_t^{-1} \circ \left(\prod_{l\neq t} \circ A_l\right)\right) + 2\sum_{1\leq t<s\leq k}\left(\left(\prod_{l\neq t,s} \circ A_l\right) \circ I\right) \in H(n)\end{aligned}$$

is invertible. Then combining with (2.7), (2.13), and Lemma 2.3, it follows that

$$\begin{aligned}\prod_{l=1}^{k} \circ A_l &= \left(\prod_{l=1}^{k} \otimes A_l\right)\bigl(\alpha(k)\bigr)\\ &= \left(C^*\left(C^*\left(\prod_{l=1}^{k} \otimes A_l\right)^{-1} C\right)^{-1} C\right)\bigl(\alpha(k)\bigr)\\ &\geq C\bigl(\alpha(k)\bigr)^*\left(C^*\left(\prod_{l=1}^{k} \otimes A_l\right)^{-1} C\right)\bigl(\alpha(k)\bigr)^{-1}C\bigl(\alpha(k)\bigr)\\ &= C\bigl(\alpha(k)\bigr)\left(C\left(\prod_{l=1}^{k} \otimes A_l\right)^{-1} C\right)\bigl(\alpha(k)\bigr)^{-1}C\bigl(\alpha(k)\bigr),\end{aligned}$$

where $C(\alpha(k)) = \sum_{t=1}^k((\prod_{l\neq t} \circ A_l) \circ I) = N$, meanwhile

$$\left(C\left(\prod_{l=1}^{k} \otimes A_l\right)^{-1} C\right)\bigl(\alpha(k)\bigr) = \sum_{t=1}^{k}\left(A_t^{-1} \circ \left(\prod_{l\neq t} \circ A_l\right)\right) + 2\sum_{1\leq t<s\leq k}\left(\left(\prod_{l\neq t,s} \circ A_l\right) \circ I\right) \in H(n),$$

thus the inequality (3.4) follows by (3.3).

From Lemma 2.3 and the proof course as above, with a similar discussion as Theorem 3.1, we see the condition for the equation in (3.4) is determined by (3.5). □

From above, the case discussed here is without 'positive definiteness', which is different from [1–10], and in form, the inequalities (3.1) and (3.4) are the reverses to each other; however, Theorems 3.1 and 3.3, and Example 3.2 indicate that their constraints are different, so (3.1) and (3.4) are not determined by each other any longer.

When $k = 2$, we could obtain [8, Corollaries 2 and 3] from Theorems 3.1 and 3.3 immediately.

Corollary 3.4 *Let $A \in H(n)$ be invertible, and $\alpha(k)$ be as in* (2.7) *and* (2.8),

$$C = \left(\sum_{t=1}^{k}\left(\prod_{l=1}^{t-1} \otimes A\right) \otimes I \otimes \left(\prod_{l=t+1}^{k} \otimes A\right)\right),$$

(i) *if* C *is invertible and* $(C^{-1}(\prod_{l=1}^{k}\otimes A)C^{-1})(\alpha'(k)) > 0$, *then* $A^{-1}\circ(\prod_{l=1}^{k-1}\circ A) + (k-1)(A\circ I)^{k-2}$ *is also invertible and the inequality* (1.5) *holds. Meanwhile, we have the equation in* (1.5) *if and only if*

$$k\left(C^{-1}\left(\prod_{l=1}^{k}\otimes A\right)C^{-1}\right)\bigl(\alpha'(k),\alpha(k)\bigr)(A\circ I)^{k-1} + \left(C^{-1}\left(\prod_{l=1}^{k}\otimes A\right)C^{-1}\right)\bigl(\alpha'(k)\bigr)C\bigl(\alpha'(k),\alpha(k)\bigr) = 0; \tag{3.6}$$

(ii) *if* $(\prod_{l=1}^{k}\otimes A^{-1})(\alpha'(k)) > 0$, *then* $\prod_{l=1}^{k}\circ A \in H(n)$ *is invertible and the inequality* (2.17) *holds. Meanwhile, the equation in* (2.17) *holds if and only if*

$$k\left(\prod_{l=1}^{k}\otimes A^{-1}\right)\bigl(\alpha'(k),\alpha(k)\bigr)(A\circ I)^{k-1} + \left(\prod_{l=1}^{k}\otimes A^{-1}\right)\bigl(\alpha'(k)\bigr)C\bigl(\alpha'(k),\alpha(k)\bigr) = 0. \tag{3.7}$$

Proof In this case, $\sum_{t=1}^{k}((\prod_{l=1}^{k-1}\circ A)\circ I) = k(A\circ I)^{k-1} \in H(n)$, by (3.3),

$$\left(C^{*}\left(\prod_{l=1}^{k}\otimes A\right)^{-1}C\right)\bigl(\alpha(k)\bigr) = k\left[A^{-1}\circ\left(\prod_{l=1}^{k-1}\circ A\right) + (k-1)(A\circ I)^{k-2}\right] \in H(n),$$

then we could obtain the results by taking $A_l = A$ $(l = 1,2,\ldots,k)$ in Theorems 3.1 and 3.3. □

Corollary 3.4 indicates that, without 'positive definiteness', not only the inequality (1.5) still holds under some constraints, but also its reverse inequality (2.17) still holds as well. Clearly their constraints are different.

Corollary 3.5 *Let* $A_l \in H(n)$ $(l = 1,2,\ldots,k)$ *be invertible with all diagonal elements* 1 *and* $\alpha(k)$ *as in* (2.7) *and* (2.8),

(i) *if* $(\prod_{l=1}^{k}\otimes A_l^{-1})(\alpha'(k)) > 0$, *then* $\prod_{l=1}^{k}\circ A_l$ *is invertible and*

$$k(k-1)I + \sum_{t=1}^{k}\left(A_t^{-1}\circ\left(\prod_{l\neq t}\circ A_l\right)\right) \geq k^2\left(\prod_{l=1}^{k}\circ A_l\right)^{-1}; \tag{3.8}$$

the equation in (3.8) *holds if and only if*

$$k\left(\prod_{l=1}^{k}\otimes A_l^{-1}\right)\bigl(\alpha'(k),\alpha(k)\bigr) + \left(\prod_{l=1}^{k}\otimes A_l^{-1}\right)\bigl(\alpha'(k)\bigr)C\bigl(\alpha'(k),\alpha(k)\bigr) = 0; \tag{3.9}$$

(ii) *if* C *is invertible and* $(C^{-1}(\prod_{l=1}^{k}\otimes A_l)C^{-1})(\alpha'(k)) > 0$, *then* $k(k-1)I + \sum_{t=1}^{k}(A_t^{-1}\circ(\prod_{l\neq t}\circ A_l))$ *is also invertible and*

$$\prod_{l=1}^{k}\circ A_l \geq k^2\left[k(k-1)I + \sum_{t=1}^{k}\left(A_t^{-1}\circ\left(\prod_{l\neq t}\circ A_l\right)\right)\right]^{-1}; \tag{3.10}$$

the equation in (3.10) *holds if and only if*

$$k\left(C^{-1}\left(\prod_{l=1}^{k}\otimes A_l\right)C^{-1}\right)\left(\alpha'(k),\alpha(k)\right)$$
$$+\left(C^{-1}\left(\prod_{l=1}^{k}\otimes A_l\right)C^{-1}\right)\left(\alpha'(k)\right)C\left(\alpha'(k),\alpha(k)\right)=0. \tag{3.11}$$

Proof By the assumption, $A_l \in H(n)$ $(l = 1,2,\ldots,k)$ is invertible with all diagonal elements 1, so $N = \sum_{t=1}^{k}((\prod_{l\neq t}\circ A_l)\circ I) = kI$, $2\sum_{1\leq t<s\leq k}((\prod_{l\neq t,s}\circ A_l)\circ I) = k(k-1)I$, then in view of (3.1), (3.2), (3.4), and (3.5) we have the conclusions. □

For $A, B \in H^+(n)$, by (2.6), they satisfy the constraints demanded in Theorems 3.1 and 3.3 naturally. Hence similar to Theorem 2.6, by Lemma 2.4, we have the following.

Theorem 3.6 *Let $A_l \in H^+(n)$ $(l = 1,2,\ldots,k)$ and $\alpha(k)$ be the one as in* (2.7) *and* (2.8), *then both of inequalities* (3.1) *and* (3.4) *hold, and the equation in* (3.1) *holds iff the one in* (3.4) *holds iff* (3.2) *holds iff* (3.5) *holds.*

When $A = A_l \in H^+(n)$ $(l = 1,2,\ldots,k)$, by (1.8), one has

$$\left(A\circ I+I\circ\prod_{l=1}^{k-1}\circ A\right)\left(\prod_{l=1}^{k}\circ A\right)^{-1}\left(A\circ I+I\circ\prod_{l=1}^{k-1}\circ A\right)$$
$$\leq 2I+\left(A\circ\prod_{l=1}^{k-1}\circ A^{-1}\right)+\left(A^{-1}\circ\prod_{l=1}^{k-1}\circ A\right). \tag{3.12}$$

Now in view of Theorem 2.5 and (2.17), we see the inequality (3.12) obtained from (1.8) is different from the one in (1.5). When $A = A_l \in H^+(n)$ $(l = 1,2,\ldots,k)$, by Theorem 3.6, we have the following.

Corollary 3.7 *Let $A \in H^+(n)$ and $\alpha(k)$ be the one as in* (2.7) *and* (2.8). *Then both of inequalities* (1.5) *and* (2.17) *hold, and the equation in* (1.5) *holds iff the one in* (2.17) *holds iff* (3.6) *holds iff* (3.7) *holds.*

By applying Theorem 3.6 and Corollaries 3.4, 3.5, we are led to the following conclusion.

Corollary 3.8 *Let $A_l \in CH^+(n)$ $(l = 1,2,\ldots,k)$, $\alpha(k)$ be the one as in* (2.7) *and* (2.8). *Then both of inequalities* (3.8) *and* (3.10) *hold, moreover, the equation in* (3.8) *holds if and only if the one in* (3.10) *holds*; *thus* (3.9) *and* (3.11) *hold.*

When $k = 2$, the companion inequalities (1.1)-(1.4), (1.6), and (1.7), and their equation conditions are obtained.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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