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On a certain new subclass of meromorphic close-to-convex functions

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Abstract
In this paper, we introduce and investigate a new subclass $MK^{(k)}(\beta,\gamma)$ of meromorphic close-to-convex functions. For functions belonging to the class $MK^{(k)}(\beta,\gamma)$, we obtain some coefficient inequalities and a distortion theorem. The results presented here would unify and extend some recent work of Wang *et al.* (Appl. Math. Lett. 25:454-460, 2012).
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1 Introduction

Let Σ be the class of functions f of the form:

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the punctured open unit disk $U^* = \{z \in C : 0 < |z| < 1\} = U \setminus \{0\}$.

Let P denote the class of functions p given by

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \quad (z \in U), \tag{1.2}$$

which are analytic and convex in U and satisfy the condition $\Re(p(z)) > 0$ $(z \in U)$.

A function $f \in \Sigma$ is said to be in the class $MS^*(\alpha)$ of meromorphic starlike functions of order α if it satisfies the inequality

$$\Re\left(\frac{zf'(z)}{f(z)}\right) < -\alpha \quad \bigl(z \in U^*; 0 \le \alpha < 1\bigr).$$

In addition, a function $f \in \Sigma$ is said to be in the class MC of meromorphic close-to-convex functions if it satisfies the inequality

$$\Re\left(\frac{zf'(z)}{g(z)}\right) < 0 \quad \bigl(z \in U^*; g \in MS^*(0) = MS^*\bigr).$$

Recently, Srivastava *et al.* [1] (see also [2, 3]) introduced and studied the class MS_s^* of meromorphic starlike functions with respect to symmetric points, which satisfies the condition

$$\Re\left(\frac{zf'(z)}{f(z)-f(-z)}\right) < 0 \quad \left(z \in U^*\right).$$

More recently, Wang *et al.* [4] discussed a class MK of meromorphic close-to-convex functions, that is, a function $f \in \Sigma$ is said to be in the class MK if it satisfies the inequality

$$\Re\left(\frac{f'(z)}{g(z)g(-z)}\right) > 0 \quad \left(z \in U^*\right),$$

where $g \in MS^*(\frac{1}{2})$.

Let $f(z) = z + a_2 z^2 + \cdots$ be analytic in U. If there exists a function $g \in S^*(\frac{1}{2})$, such that

$$\left|\frac{z^2 f'(z)}{g(z)g(-z)} + 1\right| < \left|\frac{z^2 f'(z)}{g(z)g(-z)} - 1 + 2\gamma\right| \quad (z \in U),$$

then we say that $f \in K_s(\gamma)$, $0 \le \gamma < 1$, where $S^*(\frac{1}{2})$ denotes the usual class of starlike functions of order 1/2. The function class $K_s(\gamma)$ was introduced and studied recently by Kowalczyk and Les-Bomba [5] (see also [6–9]).

For two functions f and g analytic in U, we say that the function $f(z)$ is subordinate to $g(z)$ in U, and we write $f(z) \prec g(z)$ $(z \in U)$ if there exists a Schwarz function $w(z)$, analytic in U with $w(0) = 0$ and $|w(z)| \le 1$, such that $f(z) = g(w(z))$ $(z \in U)$. In particular, if the function g is univalent in U, then we have $f(0) = g(0)$ and $f(U) \subset g(U)$ (see, for example, [10]).

Motivated essentially by the above mentioned function classes MK and $K_s(\gamma)$, we now introduce a new class $MK^{(k)}(\beta,\gamma)$ of meromorphic functions.

Definition 1 Let $MK^{(k)}(\beta,\gamma)$ denote the class of functions in Σ satisfying the inequality

$$\left|\frac{z^{2-k} f'(z)}{g_k(z)} + 1\right| < \beta\left|\frac{z^{2-k} f'(z)}{g_k(z)} + 2\gamma - 1\right| \quad \left(z \in U^*; 0 < \beta \le 1; 0 \le \gamma < 1\right), \tag{1.3}$$

where $g \in MS^*(\frac{k-1}{k})$, $k \ge 1$, is a fixed positive integer and $g_k(z)$ is defined by the following equality:

$$g_k(z) = \prod_{\nu=0}^{k-1} \varepsilon^{-\nu} g\left(\varepsilon^{\nu} z\right) \quad \left(\varepsilon^k = 1\right). \tag{1.4}$$

We note that $MK^{(2)}(1,0) = MK$ (see [4]), so the class $MK^{(k)}(\beta,\gamma)$ is a generation of the class MK.

In this paper, we prove that the class $MK^{(k)}(\beta,\gamma)$ is a subclass of meromorphic close-to-convex functions. Moreover, we provide some coefficient inequalities and a distortion theorem for functions in the class $MK^{(k)}(\beta,\gamma)$. Our results unify and extend the corresponding results obtained by Wang *et al.* [4].

2 Main results

First of all, we give two meaningful conclusions about the class $MK^{(k)}(\beta,\gamma)$. The proof of Theorem 1 below is much akin to that of Theorem 1 in [11], so we choose to omit the details involved.

Theorem 1 *A function $f \in MK^{(k)}(\beta,\gamma)$ if and only if there exists $g \in MS^*(\frac{k-1}{k})$ such that*

$$-\frac{z^{2-k}f'(z)}{g_k(z)} \prec \frac{1+(1-2\gamma)\beta z}{1-\beta z} \quad (z \in U^*). \tag{2.1}$$

Remark 1 From Theorem 1, we know that

$$\Re\left(\frac{z^{2-k}f'(z)}{g_k(z)}\right) < 0 \quad (z \in U^*), \tag{2.2}$$

because of $\Re(\frac{1+(1-2\gamma)\beta z}{1-\beta z}) > 0$ $(z \in U^*)$.

Lemma 1 *Let $\varphi_i \in MS^*(\alpha_i)$, where $0 \le \alpha_i < 1$ $(i = 0,1,\dots,k-1)$. Then for $k-1 \le \sum_{i=0}^{k-1}\alpha_i < k$, we have*

$$z^{k-1}\prod_{i=0}^{k-1}\varphi_i(z) \in MS^*\left(\sum_{i=0}^{k-1}\alpha_i - (k-1)\right).$$

Proof Since $\varphi_i \in MS^*(\alpha_i)$ $(i = 0,1,\dots,k-1)$, by the definition of meromorphic starlike functions, we have

$$\Re\left(\frac{z\varphi_0'(z)}{\varphi_0(z)}\right) < -\alpha_0, \qquad \Re\left(\frac{z\varphi_1'(z)}{\varphi_1(z)}\right) < -\alpha_1, \qquad \dots, \qquad \Re\left(\frac{z\varphi_{k-1}'(z)}{\varphi_{k-1}(z)}\right) < -\alpha_{k-1}. \tag{2.3}$$

We now let

$$F(z) = z^{k-1}\varphi_0(z)\varphi_1(z)\cdots\varphi_{k-1}(z). \tag{2.4}$$

Differentiating (2.4) with respect to z logarithmically, we easily get

$$\frac{zF'(z)}{F(z)} = \frac{z\varphi_0'(z)}{\varphi_0(z)} + \frac{z\varphi_1'(z)}{\varphi_1(z)} + \cdots + \frac{z\varphi_{k-1}'(z)}{\varphi_{k-1}(z)} + (k-1). \tag{2.5}$$

From (2.5) together with (2.3), we obtain

$$\begin{aligned}\Re\left(\frac{zF'(z)}{F(z)}\right) &= \Re\left(\frac{z\varphi_0'(z)}{\varphi_0(z)}\right) + \Re\left(\frac{z\varphi_1'(z)}{\varphi_1(z)}\right) + \cdots + \Re\left(\frac{z\varphi_{k-1}'(z)}{\varphi_{k-1}(z)}\right) + (k-1)\\ &< -\sum_{i=0}^{k-1}\alpha_i + (k-1) = -\left(\sum_{i=0}^{k-1}\alpha_i - (k-1)\right)\end{aligned}$$

by noting that $0 \le \sum_{i=0}^{k-1}\alpha_i - (k-1) < 1$, which implies that

$$F(z) = z^{k-1}\prod_{i=0}^{k-1}\varphi_i(z) \in MS^*\left(\sum_{i=0}^{k-1}\alpha_i - (k-1)\right).$$

The proof of Lemma 1 is thus completed. □

Theorem 2 *Let* $g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n \in MS^*(\frac{k-1}{k})$, *then* $z^{k-1} g_k(z) \in MS^*$.

Proof From (1.4), we know

$$\begin{aligned} z^{k-1} g_k(z) &= z^{k-1} \prod_{\nu=0}^{k-1} \varepsilon^{-\nu} g(\varepsilon^{\nu} z) = z^{k-1} \prod_{\nu=0}^{k-1} \varepsilon^{-\nu} \left[\frac{1}{\varepsilon^{\nu} z} + \sum_{n=1}^{\infty} b_n (\varepsilon^{\nu} z)^n \right] \\ &= z^{k-1} \prod_{\nu=0}^{k-1} \left[\frac{1}{\varepsilon^{2\nu} z} + \sum_{n=1}^{\infty} b_n \varepsilon^{(n-1)\nu} z^n \right]. \end{aligned} \tag{2.6}$$

Since $g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n \in MS^*(\frac{k-1}{k})$, by Lemma 1 and (2.6), we can easily get the assertion of Theorem 2. □

Remark 2 From Theorem 2 and the inequality (2.2), we see that if $f \in MK^{(k)}(\beta, \gamma)$, then $f(z)$ is a meromorphic close-to-convex function. So, $MK^{(k)}(\beta, \gamma)$ is a subclass of the class MC of meromorphic close-to-convex functions.

Next, we give some coefficient inequalities for functions belonging to the class $MK^{(k)}(\beta, \gamma)$.

Theorem 3 *Let* $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$ *and* $g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n$ *be analytic in* U^*. *If for* $0 < \beta \leq 1$ *and* $0 \leq \gamma < 1$, *we have*

$$\sum_{n=1}^{\infty} n(1+\beta)|a_n| + \sum_{n=1}^{\infty} \bigl(\beta|1-2\gamma| + 1\bigr)|B_n| \leq 2\beta(1-\gamma), \tag{2.7}$$

where the coefficients B_n $(n = 1, 2, \ldots)$ *are given by* (2.9), *then* $f \in MK^{(k)}(\beta, \gamma)$.

Proof Suppose that

$$G_k(z) = z^{k-1} g_k(z). \tag{2.8}$$

By Theorem 2, we know that $G_k \in MS^*$. Hence, equality (2.6) can be written as

$$G_k(z) = z^{k-1} g_k(z) = \frac{1}{z} + \sum_{n=1}^{\infty} B_n z^n \in MS^*. \tag{2.9}$$

To prove $f \in MK^{(k)}(\beta, \gamma)$, it suffices to show that

$$\left| \frac{\frac{zf'(z)}{G_k(z)} + 1}{\frac{zf'(z)}{G_k(z)} + 2\gamma - 1} \right| < \beta,$$

where G_k is given by (2.8). From (2.7), we know that

$$\beta(2-2\gamma) - \sum_{n=1}^{\infty} n\beta|a_n| - \sum_{n=1}^{\infty} \beta|1-2\gamma||B_n| \geq \sum_{n=1}^{\infty} n|a_n| + \sum_{n=1}^{\infty} |B_n| > 0. \tag{2.10}$$

Now, by the maximum modulus principle, we deduce from (1.1), (2.9) and (2.10) that

$$\begin{aligned}\left|\frac{\frac{zf'(z)}{G_k(z)}+1}{\frac{zf'(z)}{G_k(z)}+2\gamma-1}\right| &= \left|\frac{\sum_{n=1}^{\infty} na_n z^{n+1}+\sum_{n=1}^{\infty} B_n z^{n+1}}{\sum_{n=1}^{\infty} na_n z^{n+1}-\sum_{n=1}^{\infty}(1-2\gamma)B_n z^{n+1}-(2-2\gamma)}\right| \\ &< \frac{\sum_{n=1}^{\infty} n|a_n|+\sum_{n=1}^{\infty}|B_n|}{(2-2\gamma)-\sum_{n=1}^{\infty} n|a_n|-\sum_{n=1}^{\infty}|1-2\gamma||B_n|} \leq \beta.\end{aligned}$$

This evidently completes the proof of Theorem 3. □

Theorem 4 *Let* $f(z)=\frac{1}{z}+\sum_{n=1}^{\infty} a_n z^n \in MK^{(k)}(\beta,\gamma)$. *Then*

$$1+\sum_{n=1}^{\infty}\frac{n(1-\beta e^{i\theta})}{2(1-\gamma)\beta e^{i\theta}}a_n z^{n+1}+\sum_{n=1}^{\infty}\frac{1+(1-2\gamma)\beta e^{i\theta}}{2(1-\gamma)\beta e^{i\theta}}B_n z^{n+1}\neq 0$$
$$\bigl(z\in U^*; 0<\theta<2\pi\bigr), \tag{2.11}$$

where the coefficients B_n $(n=1,2,\ldots)$ *are given by* (2.9).

Proof Suppose that $f\in MK^{(k)}(\beta,\gamma)$. Then we know that

$$-\frac{zf'(z)}{G_k(z)}\neq\frac{1+(1-2\gamma)\beta e^{i\theta}}{1-\beta e^{i\theta}}\quad\bigl(z\in U^*; 0<\theta<2\pi\bigr), \tag{2.12}$$

where G_k is given by (2.8). After a simple computation, the inequality (2.2) is equivalent to

$$zf'(z)\bigl(1-\beta e^{i\theta}\bigr)+G_k(z)\bigl(1+(1-2\gamma)\beta e^{i\theta}\bigr)\neq 0\quad\bigl(z\in U^*; 0<\theta<2\pi\bigr). \tag{2.13}$$

By substituting (1.1) and (2.9) into (2.13), we obtain the desired assertion (2.11) of Theorem 4.

Finally, we provide the following distortion theorem for the considered class of functions $MK^{(k)}(\beta,\gamma)$. □

Theorem 5 *If* $f\in MK^{(k)}(\beta,\gamma)$, *then*

$$\begin{aligned}\frac{(1-r)^2(1-(1-2\gamma)\beta r)}{r^2(1+\beta r)} &\leq \bigl|f'(z)\bigr| \\ &\leq \frac{(1+r)^2(1+(1-2\gamma)\beta r)}{r^2(1-\beta r)}\quad\bigl(|z|=r; 0<r<1\bigr).\end{aligned} \tag{2.14}$$

Proof If $f\in MK^{(k)}(\beta,\gamma)$, then there exists a function $g\in MS^*(\frac{k-1}{k})$ such that (1.3) holds true. It follows from Theorem 2 that the function G_k given by (2.8) is a meromorphic starlike function. Hence, we have (see [12])

$$\frac{(1-r)^2}{r}\leq\bigl|G_k(z)\bigr|\leq\frac{(1+r)^2}{r}\quad\bigl(|z|=r; 0<r<1\bigr). \tag{2.15}$$

Let us define $p(z)$ by

$$-\frac{zf'(z)}{G_k(z)} = p(z) \quad \big(z \in U^*\big), \tag{2.16}$$

where

$$p(z) \prec \frac{1+(1-2\gamma)\beta z}{1-\beta z}.$$

Then, by using a similar method as in [13, p.105], we have

$$\frac{1-(1-2\gamma)\beta r}{1+\beta r} \leq \big|p(z)\big| \leq \frac{1+(1-2\gamma)\beta r}{1-\beta r} \quad \big(|z| = r; 0 < r < 1\big). \tag{2.17}$$

Thus, from (2.15), (2.16) and (2.17), we readily get the inequality (2.14). The proof of Theorem 5 is thus completed. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors jointly worked on the results and they read and approved the final manuscript.

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