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Approximation properties of complex q-Balázs-Szabados operators in compact disks

Nurhayat İspir and Esma Yıldız Özkan*

*Correspondence:
esmayildiz@gazi.edu.tr
Department of Mathematics,
Faculty of Science, Gazi University,
Ankara, 06500, Turkey

Abstract

This paper deals with approximating properties and convergence results of the complex q-Balázs-Szabados operators attached to analytic functions on compact disks. The order of convergence and the Voronovskaja-type theorem with quantitative estimate of these operators and the exact degree of their approximation are given. Our study extends the approximation properties of the complex q-Balázs-Szabados operators from real intervals to compact disks in the complex plane with quantitative estimate.

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1 Introduction

In the recent years, applications of q-calculus in the area of approximation theory and number theory have been an active area of research. Details on q-calculus can be found in [1–3]. Several researchers have purposed the q-analogue of Stancu, Kantorovich and Durrmeyer type operators. Gal [4] studied some approximation properties of the complex q-Bernstein polynomials attached to analytic functions on compact disks.

Also very recently, some authors [5–7] have studied the approximation properties of some complex operators on complex disks. Balázs [8] defined the Bernstein-type rational functions and gave some convergence theorems for them. In [9], Balázs and Szabados obtained an estimate that had several advantages with respect to that given in [8]. These estimates were obtained by the usual modulus of continuity. The q-form of these operator was given by Doğru. He investigated statistical approximation properties of q-Balázs-Szabados operators [10].

The rational complex Balázs-Szabados operators were defined by Gal [4] as follows:

$$R_n(f;z) = \frac{1}{(1+a_n z)^n} \sum_{j=0}^{n} f\left(\frac{j}{b_n}\right) \binom{n}{j} (a_n z)^j,$$

where $D_R = \{z \in \mathbb{C} : |z| < R\}$ with $R > \frac{1}{2}$, $f : D_R \cup [R,\infty) \to \mathbb{C}$ is a function, $a_n = n^{\beta-1}$, $b_n = n^{\beta}$, $0 < \beta \leq \frac{2}{3}$, $n \in \mathbb{N}$, $z \in \mathbb{C}$ and $z \neq -\frac{1}{a_n}$.

He obtained the uniform convergence of $R_n(f;z)$ to $f(z)$ on compact disks and proved the upper estimate in approximation of these operators. Also, he obtained the Voronovskaja-type result and the exact degree of its approximation.

The goal of this paper is to obtain convergence results for the complex q-Balázs-Szabados operators given by

$$R_n(f;q,z)=\frac{1}{\prod_{s=0}^{n-1}(1+q^s a_n z)}\sum_{j=0}^{n} q^{j(j-1)/2} f\left(\frac{[j]_q}{b_n}\right)\begin{bmatrix} n \\ j \end{bmatrix}_q (a_n z)^j,$$

where $f: D_R\cup[R,\infty)\to\mathbb{C}$ is uniformly continuous and bounded on $[0,\infty)$, $a_n=[n]_q^{\beta-1}$, $b_n=[n]_q^{\beta}$, $q\in(0,1]$, $0<\beta\le\frac{2}{3}$, $n\in\mathbb{N}$, $z\in\mathbb{C}$ and $z\neq-\frac{1}{q^s a_n}$ for $s=0,1,2,\ldots$.

These operators are obtained simply replacing x by z in the real form of the q-Balázs-Szabados operators introduced in Doğru [10].

The complex q-Balázs-Szabados operators $R_n(f;q,z)$ are well defined, linear, and these operators are analytic for all $n\ge n_0$ and $|z|\le r<[n_0]_q^{1-\beta}$ since $|-\frac{1}{a_n}|\le|-\frac{1}{qa_n}|\le\cdots\le|-\frac{1}{q^{n-1}a_n}|$.

In this paper, we obtain the following results:

- the order of convergence for the operators $R_n(f;q,z)$,
- the Voronovskaja-type theorem with quantitative estimate,
- the exact degree of the approximation for the operators $R_n(f;q,z)$.

Throughout the paper, we denote with $\|f\|_r=\max\{|f(z)|\in\mathbb{R}:z\in\bar{D}_r\}$ the norm of f in the space of continuous functions on $\bar{D}_r$ and with $\|f\|_{B[0,\infty)}=\sup\{|f(x)|\in\mathbb{R}:x\in[0,\infty)\}$ the norm of f in the space of bounded functions on $[0,\infty)$.

Also, the many results in this study are obtained under the condition that $f: D_R\cup[R,\infty)\to\mathbb{C}$ is analytic in D_R for $r<R$, which assures the representation $f(z)=\sum_{k=0}^{\infty}c_k z^k$ for all $z\in D_R$.

2 Convergence results

The following lemmas will help in the proof of convergence results.

Lemma 1 *Let $n_0\ge 2$, $0<\beta\le\frac{2}{3}$ and $\frac{1}{2}<r<R\le\frac{[n_0]_q^{1-\beta}}{2}$. Let us define $\alpha_{k,n,q}(z)=R_n(e_k;q,z)$ for all $z\in\bar{D}_r$, where $e_k(z)=z^k$. If $f: D_R\cup[R,\infty)\to\mathbb{C}$ is uniformly continuous, bounded on $[0,\infty)$ and analytic in D_R, then we have the form*

$$R_n(f;q,z)=\sum_{k=0}^{\infty}c_k\alpha_{k,n,q}(z)$$

for all $z\in\bar{D}_r$.

Proof For any $m\in\mathbb{N}$, we define

$$f_m(z)=\sum_{k=0}^{m}c_k e_k(z)\quad\text{if } |z|\le r\quad\text{and}\quad f_m(z)=f(z)\quad\text{if } z\in(r,\infty).$$

From the hypothesis on f, it is clear that each f_m is bounded on $[0,\infty)$, that is, there exist $M(f_m)>0$ with $|f_m(z)|\le M(f_m)$, which implies that

$$\left|R_n(f_m;q,z)\right|\le\frac{1}{|\prod_{s=0}^{n-1}(1+q^s a_n z)|}\sum_{j=0}^{n} q^{j(j-1)/2} M(f_m)\begin{bmatrix} n \\ j \end{bmatrix}_q \left(a_n|z|\right)^j<\infty,$$

that is all $R_n(f_m;q,z)$ with $n\ge n_0$, $r<\frac{[n_0]_q^{1-\beta}}{2}$, $m\in\mathbb{N}$ are well defined for all $z\in\bar{D}_r$.

Defining

$$f_{m,k}(z) = c_k e_k(z) \quad \text{if } |z| \le r \quad \text{and} \quad f_{m,k}(z) = \frac{f(z)}{m+1} \quad \text{if } z \in (r,\infty),$$

it is clear that each $f_{m,k}$ is bounded on $[0,\infty)$ and that $f_m(z) = \sum_{k=0}^{m} f_{m,k}(z)$.

From the linearity of $R_n(f;q,z)$, we have

$$R_n(f_m;q,z) = \sum_{k=0}^{m} c_k \alpha_{k,n,q}(z) \quad \text{for all } |z| \le r.$$

It suffices to prove that

$$\lim_{m\to\infty} R_n(f_m;q,z) = R_n(f;q,z)$$

for any fixed $n \in \mathbb{N}$, $n \ge n_0$ and $|z| \le r$.

We have the following inequality for all $|z| \le r$:

$$\left|R_n(f_m;q,z) - R_n(f;q,z)\right| \le M_{r,n,q} \|f_m - f\|_r, \tag{1}$$

where $M_{r,n,q} = \prod_{s=0}^{n-1} \frac{(1+q^s a_n r)}{(1-q^s a_n r)}$.

Using (1), $\lim_{m\to\infty} \|f_m - f\|_r = 0$ and $\|f_m - f\|_{B[0,\infty)} \le \|f_m - f\|_r$, the proof of the lemma is finished. □

Lemma 2 *If we denote* $(\beta + z)_q^n = \prod_{s=0}^{n-1}(\beta + q^s z)$, *then the following formula holds*:

$$D_q\left[\frac{1}{(\beta+z)_q^n}\right] = -\frac{[n]_q}{(\beta+z)_q^{n+1}},$$

where β *is a fixed real number and* $z \in \mathbb{C}$.

Proof We can write $(\beta + z)_q^n$ as follows:

$$(\beta+z)_q^n = q^{n(n-1)/2}\left(z + q^{-n+1}\beta\right)_q^n. \tag{2}$$

In [3] (see p.10, Proposition 3.3), we already have the following formula:

$$D_q\left[(\beta+z)_q^n\right] = [n]_q(\beta+z)_q^{n-1}. \tag{3}$$

Using (2) and (3), we get

$$\begin{aligned} D_q\left[(\beta+z)_q^n\right] &= q^{n(n-1)/2}[n]_q\left(z + q^{-n+1}\beta\right)_q^{n-1} \\ &= [n]_q q^{n-1} q^{(n-1)(n-2)/2}\left(z + q^{-n+2}\left(q^{-1}\beta\right)\right)_q^{n-1} \\ &= [n]_q q^{n-1}\left(q^{-1}\beta + z\right)_q^{n-1} \\ &= [n]_q(\beta + qz)_q^{n-1}. \end{aligned} \tag{4}$$

From (4), we obtain the result. □

Lemma 3 *We have the following recurrence formula for the complex q-Balázs-Szabados operators* $R_n(f;q,z)$:

$$\alpha_{k+1,n,q}(z)=\frac{(1+q^n a_n z)z}{(1+a_n z)b_n}D_q\bigl[\alpha_{k,n,q}(z)\bigr]+\frac{z}{1+a_n z}\alpha_{k,n,q}(z),$$

where $\alpha_{k,n,q}(z)=R_n(e_k;q,z)$ *for all* $n\in\mathbb{N}$, $z\in\mathbb{C}$ *and* $k=0,1,2,\ldots$.

Proof Firstly, we calculate $D_q[\alpha_{k,n,q}(z)]$ as follows:

$$\begin{aligned}
&D_q\bigl[\alpha_{k,n,q}(z)\bigr]\\
&\quad=D_q\biggl[\frac{1}{\prod_{s=0}^{n-1}(1+q^s a_n z)}\biggr]\sum_{j=0}^{n}q^{j(j-1)/2}\biggl(\frac{[j]_q}{b_n}\biggr)^k\begin{bmatrix}n\\j\end{bmatrix}_q(a_n z)^j\\
&\qquad+\frac{1}{\prod_{s=0}^{n-1}(1+q^{s+1}a_n z)}\sum_{j=0}^{n}q^{j(j-1)/2}\biggl(\frac{[j]_q}{b_n}\biggr)^k\begin{bmatrix}n\\j\end{bmatrix}_q(a_n)^j D_q\bigl[z^j\bigr].
\end{aligned}\tag{5}$$

Considering Lemma 2 and using $D_q[z^j]=[j]_q z^{j-1}$ in (5), we get

$$\begin{aligned}
D_q\bigl[\alpha_{k,n,q}(z)\bigr]&=-\frac{b_n}{1+q^n a_n z}\frac{1}{\prod_{s=0}^{n-1}(1+q^s a_n z)}\alpha_{k,n,q}(z)\\
&\quad+\frac{b_n(1+a_n z)}{z(1+q^n a_n z)}\alpha_{k+1,n,q}(z).
\end{aligned}\tag{6}$$

From (6), the proof of the lemma is finished. □

Corollary 1 ([11], p.143, Corollary 1.10.4) *Let* $f(z)=\frac{p_k(z)}{\prod_{j=1}^{k}(z-a_j)}$, *where* $p_k(z)$ *is a polynomial of degree* $\le k$, *and we suppose that* $|a_j|\ge R>1$ *for all* $j=1,2,\ldots,k$. *If* $1\le r<R$, *then for all* $|z|\le r$ *we have*

$$\bigl|f'(z)\bigr|\le\frac{R+r}{R-r}\cdot\frac{k}{r}\|f\|_r.$$

Under hypothesis of the corollary above, by the mean value theorem [12] in complex analysis, we have

$$\bigl|D_q\bigl[f(z)\bigr]\bigr|\le\frac{R+r}{R-r}\cdot\frac{k}{r}\|f\|_r.\tag{7}$$

Lemma 4 *Let* $n_0\ge 2$, $0<\beta\le\frac{2}{3}$ *and* $\frac{1}{2}<r<R\le\frac{[n_0]_q^{1-\beta}}{2}$. *For all* $n\ge n_0$, $|z|\le r$ *and* $k=0,1,2,\ldots$, *we have*

$$\bigl|\alpha_{k,n,q}(z)\bigr|\le k!(20r)^k.$$

Proof Taking the absolute value of the recurrence formula in Lemma 3 and using the triangle inequality, we get

$$\bigl|\alpha_{k+1,n,q}(z)\bigr|\le\frac{(1+q^n a_n|z|)|z|}{|1-a_n|z||b_n}\bigl|D_q\bigl[\alpha_{k,n,q}(z)\bigr]\bigr|+\frac{|z|}{|1-a_n|z||}\bigl|\alpha_{k,n,q}(z)\bigr|.\tag{8}$$

In order to get an upper estimate for $|D_q[\alpha_{k,n,q}(z)]|$, by using (7), we obtain

$$\left|D_q\left[\alpha_{k,n,q}(z)\right]\right| \le \frac{[n_0]_q^{1-\beta}+r}{[n_0]_q^{1-\beta}-r}\cdot\frac{k}{r}\|\alpha_{k,n,q}\|_r.$$

Under the condition $r < \frac{[n_0]_q^{1-\beta}}{2}$, it holds $\frac{[n_0]_q^{1-\beta}+r}{[n_0]_q^{1-\beta}-r} < 3$, which implies

$$\left|D_q\left[\alpha_{k,n,q}(z)\right]\right| \le \frac{3k}{r}\|\alpha_{k,n,q}\|_r. \tag{9}$$

Applying (9) to (8) and passing to norm, we get

$$\|\alpha_{k+1,n,q}\|_r \le \frac{(1+q^n a_n r)3k}{(1-a_n r)b_n}\|\alpha_{k,n,q}\|_r + \frac{r}{1-a_n r}\|\alpha_{k,n,q}\|_r.$$

From the hypothesis of the lemma, we have $\frac{1}{1-a_n r} < 2$, $1+q^n a_n r < \frac{3}{2}$, and $\frac{1}{b_n} < 1$, which implies

$$\|\alpha_{k+1,n,q}\|_r \le 20r(k+1)\|\alpha_{k,n,q}\|_r.$$

Taking step by step $k = 0,1,2,\ldots$, we obtain

$$\|\alpha_{k+1,n,q}\|_r \le (20r)^{k+1}(k+1)!.$$

Using $|\alpha_{k+1,n,q}| \le \|\alpha_{k+1,n,q}\|_r$ and replacing $k+1$ with k, the proof of the lemma is finished. □

Let $q = \{q_n\}$ be a sequence satisfying the following conditions:

$$\lim_{n\to\infty} q_n = 1 \quad\text{and}\quad \lim_{n\to\infty} q_n^n = c \quad (0 \le c < 1). \tag{10}$$

Now we are in a position to prove the following convergence result.

Theorem 1 *Let $\{q_n\}$ be a sequence satisfying the conditions* (10) *with $q_n \in (0,1]$ for all $n \in \mathbb{N}$, and let $n_0 \ge 2$, $0 < \beta \le \frac{2}{3}$ and $\frac{1}{2} < r < R \le \frac{[n_0]_{q_n}^{1-\beta}}{2}$. If $f : D_R \cup [R,\infty) \to \mathbb{C}$ is uniformly continuous, bounded on $[0,\infty)$ and analytic in D_R and there exist $M > 0$, $0 < A < \frac{1}{20r}$ with $|c_k| \le M\frac{A^k}{k!}$ (which implies $|f(z)| \le Me^{A|z|}$ for all $z \in D_R$), then the sequence $\{R_n(f;q_n,z)\}_{n\ge n_0}$ is uniformly convergent to f in $\bar{D}_r$.*

Proof From Lemma 2 and Lemma 6, for all $n \ge n_0$ and $|z| \le r$, we have

$$\left|R_n(f;q_n,z)\right| \le \sum_{k=0}^{\infty}|c_k|\left|\alpha_{k,n,q_n}(z)\right| \le \sum_{k=0}^{\infty} M\frac{A^k}{k!}k!(20r)^k = M\sum_{k=0}^{\infty}(20Ar)^k,$$

where the series $\sum_{k=0}^{\infty}(20Ar)^k$ is convergent for $0 < A < \frac{1}{20r}$.

Since $\lim_{n\to\infty} R_n(f;q_n,x) = f(x)$ for all $x \in [0,r]$ (see [10]), by Vitali's theorem (see [13], p.112, Theorem 3.2.10), it follows that $\{R_n(f;q_n,z)\}$ uniformly converges to $f(z)$ in $\bar{D}_r$. □

We can give the following upper estimate in the approximation of $R_n(f;q_n,z)$.

Theorem 2 *Let $\{q_n\}$ be a sequence satisfying the conditions* (10) *with $q_n \in (0,1]$ for all $n \in \mathbb{N}$, and let $n_0 \geq 2$, $0 < \beta \leq \frac{2}{3}$ and $\frac{1}{2} < r < R \leq \frac{[n_0]_{q_n}^{1-\beta}}{2}$. If $f : D_R \cup [R,\infty) \to \mathbb{C}$ is uniformly continuous, bounded on $[0,\infty)$ and analytic in D_R and there exist $M > 0$, $0 < A < \frac{1}{20r}$ with $|c_k| \leq M\frac{A^k}{k!}$ (which implies $|f(z)| \leq Me^{A|z|}$ for all $z \in D_R$), then the following upper estimate holds*:

$$\left|R_n(f;q_n,z) - f(z)\right| \leq C_r^1(f)\left(a_n + \frac{1}{b_n}\right),$$

where $C_r^1(f) = \max\{9MA\sum_{k=1}^{\infty}(k-1)(20Ar)^{k-1}, 2r^2MAe^{2Ar}\}$ and $\sum_{k=1}^{\infty}(k-1)(20Ar)^{k-1} < \infty$.

Proof Using the recurrence formula in Lemma 4, we have

$$\begin{aligned}\left|\alpha_{k+1,n,q_n}(z) - z^{k+1}\right| \leq{}& \frac{(1+q_n^n a_n|z|)|z|}{|1-a_n|z||b_n}\left|D_{q_n}\left[\alpha_{k,n,q_n}(z) - z^k\right]\right| \\ &+ \frac{|z|}{|1-a_n|z||}\left|\alpha_{k,n,q_n}(z) - z^k\right| + \frac{1}{b_n}\frac{(1+q_n^n a_n|z|)}{|1-a_n|z||}[k]_{q_n}|z|^k \\ &+ \frac{a_n}{|1-a_n|z||}|z|^{k+2}.\end{aligned}$$

For $|z| \leq r$, we get

$$\begin{aligned}\left|\alpha_{k+1,n,q_n}(z) - z^{k+1}\right| \leq{}& \frac{(1+q_n^n a_n r)r}{(1-a_n r)b_n}\left|D_{q_n}\left[\alpha_{k,n,q_n}(z)\right]\right| + \frac{r}{1-a_n r}\left|\alpha_{k,n,q_n}(z) - z^k\right| \\ &+ \frac{2}{b_n}\frac{(1+q_n^n a_n r)}{(1-a_n r)}[k]_{q_n}r^k + \frac{a_n}{1-a_n r}r^{k+2}.\end{aligned}$$

Using (9), $\frac{1}{1-a_n r} < 2$, and $1 + q_n^n a_n r < \frac{3}{2}$, we obtain

$$\left|\alpha_{k+1,n,q_n}(z) - z^{k+1}\right| \leq \frac{9k\cdot k!}{b_n}(20r)^k + 2r\left|\alpha_{k,n,q_n}(z) - z^k\right| + \frac{6}{b_n}[k]_{q_n}r^k + 2a_n r^{k+2}.$$

Since $6[k]_{q_n}r^k \leq 9k\cdot k!(20r)^k$ for all $k = 0,1,2,\ldots$, we can write

$$\left|\alpha_{k+1,n,q_n}(z) - z^{k+1}\right| \leq \frac{18k\cdot k!}{b_n}(20r)^k + 2r\left|\alpha_{k,n,q_n}(z) - z^k\right| + 2a_n r^{k+2}.$$

Taking $k = 0,1,2,\ldots$ step by step, finally we arrive at

$$\left|\alpha_{k,n,q_n}(z) - z^k\right| \leq \frac{9}{b_n}(k-1)k!(20r)^{k-1} + 2a_n r^2 k(2r)^{k-1}, \tag{11}$$

which implies

$$\begin{aligned}\left|R_n(f;q_n,z) - f(z)\right| &\leq \sum_{k=1}^{\infty}|c_k|\left|\alpha_{k,n,q_n}(z) - z^k\right| \\ &\leq \sum_{k=1}^{\infty}M\frac{A^k}{k!}\left\{\frac{9}{b_n}(k-1)k!(20r)^{k-1} + 2a_n r^2 k(2r)^{k-1}\right\}\end{aligned}$$

$$= \frac{9MA}{b_n}\sum_{k=1}^{\infty}(k-1)(20Ar)^{k-1} + 2a_n r^2 MA\sum_{k=1}^{\infty}\frac{(20Ar)^{k-1}}{(k-1)!}$$
$$= \frac{9MA}{b_n}\sum_{k=1}^{\infty}(k-1)(20Ar)^{k-1} + 2a_n r^2 MAe^{2Ar}.$$

Choosing $C_r^1(f) = \max\{9MA\sum_{k=1}^{\infty}(k-1)(20Ar)^{k-1}, 2r^2MAe^{2Ar}\}$, we obtain the desired result.

Here the series $\sum_{k=0}^{\infty}(20Ar)^k$ is convergent for $0 < A < \frac{1}{20r}$ and the series is absolutely convergent in $\bar{D}_r$, it easily follows that $\sum_{k=1}^{\infty}(k-1)(20Ar)^{k-1} < \infty$. □

The following lemmas will help in the proof of the next theorem.

Lemma 5 *For all $n \in \mathbb{N}$, we have*

$$R_n(e_0;q,z) = 1, \tag{12}$$
$$R_n(e_1;q,z) = \frac{z}{1+a_n z}, \tag{13}$$
$$R_n(e_2;q,z) = \frac{(1-\frac{a_n}{b_n})qz^2}{(1+a_n z)(1+a_n qz)} + \frac{z}{b_n(1+a_n z)}, \tag{14}$$

where $e_k(z) = z^k$ for $k = 0,1,2$.

Proof (12) and (13) are obtained simply replacing x by z in Lemma 3.1 and Lemma 3.2 in [10]. Also, using $[n]_q = 1 + q[n-1]_q$ and $\frac{a_n}{b_n} = \frac{1}{[n]_q}$ and replacing x by z in Lemma 3.3 in [10], (14) is obtained. □

Lemma 6 *For all $n \in \mathbb{N}$, the following equalities for the operators $R_n(f;q,z)$ hold*:

$$\psi_{n,q}^1(z) = \frac{-a_n z^2}{1+a_n z}, \tag{15}$$
$$\begin{aligned}\psi_{n,q}^2(z) = {}& \frac{z}{b_n(1+a_n z)(1+a_n qz)} - \frac{(1-q)z^2}{(1+a_n z)(1+a_n qz)}\\ &- \frac{a_n(1-q)z^3}{(1+a_n z)(1+a_n qz)} + \frac{a_n^2 qz^4}{(1+a_n z)(1+a_n qz)},\end{aligned} \tag{16}$$

where $\psi_{n,q}^i(z) = R_n((t-e_1)^i;q,z)$ for $i = 1,2$.

Proof From Lemma 5, the proof can be easily got, so we omit it. □

Now, we present a quantitative Voronovskaja-type formula.
Let us define

$$A_{k,n,q_n}(z) = R_n(f;q_n,z) - f(z) - \psi_{n,q}^1(z)f'(z) - \frac{1}{2}\psi_{n,q}^2(z)f''(z). \tag{17}$$

Theorem 3 *Let $\{q_n\}$ be a sequence satisfying the conditions* (10) *with $q_n \in (0,1]$ for all $n \in \mathbb{N}$, $n_0 \geq 2$, $0 < \beta \leq \frac{2}{3}$ and $\frac{1}{2} < r < R \leq \frac{[n_0]_{q_n}^{1-\beta}}{2}$. If $f : D_R \cup [R,\infty) \to \mathbb{C}$ is uniformly continuous,* *bounded on* $[0,\infty)$ *and analytic in* D_R *and there exist* $M>0$, $0<A<\frac{1}{20r}$ *with* $|c_k|\le M\frac{A^k}{k!}$ (*which implies* $|f(z)|\le Me^{A|z|}$ *for all* $z\in D_R$), *then for all* $n\ge n_0$ *and* $|z|\le r$, *we have*

$$\left|A_{k,n,q_n}(z)\right|\le C_r^2(f)\left(a_n+\frac{1}{b_n}\right)^2,$$

where $C_r^2(f)=C_*Mr^3\sum_{k=3}^{\infty}(k-2)(k-1)k(k+1)(20rA)^{k-3}<\infty$ *and* C_* *is a fixed real number.*

Proof From Lemma 1 and the analyticity of f, we can write

$$\left|A_{k,n,q_n}(z)\right|\le\sum_{k=2}^{\infty}|c_k|\left|E_{k,n,q_n}(z)\right|, \tag{18}$$

where

$$\begin{aligned}E_{k,n,q_n}(z)={}&\alpha_{k,n,q_n}(z)-z^k+\frac{a_nkz^{k+1}}{1+a_nz}-\frac{(k-1)kz^{k-1}}{2b_n(1+a_nz)(1+a_nq_nz)}\\&+\frac{(1-q_n)(k-1)kz^k}{2(1+a_nz)(1+a_nq_nz)}+\frac{a_n(1-q_n)(k-1)kz^{k+1}}{2(1+a_nz)(1+a_nq_nz)}\\&-\frac{a_n^2q_n(k-1)kz^{k+2}}{2(1+a_nz)(1+a_nq_nz)}.\end{aligned} \tag{19}$$

Using Lemma 5, we easily obtain that $E_{0,n,q}(z)=E_{1,n,q}(z)=E_{2,n,q}(z)=0$.

Combining (19) with the recurrence formula in Lemma 3, a simple calculation leads us to the following recurrence formula:

$$E_{k+1,n,q_n}(z)=\frac{(1+q_n^na_nz)z}{b_n(1+a_nz)}D_{q_n}\left[E_{k,n,q_n}(z)\right]+\frac{z}{1+a_nz}E_{k,n,q_n}(z)+F_{k,n,q_n}(z), \tag{20}$$

where

$$\begin{aligned}F_{k,n,q_n}(z)={}&-\frac{(k-[k]_{q_n})z^k}{b_n(1+a_nz)^2(1+a_nq_nz)}+\frac{a_n^2kz^{k+3}}{(1+a_nz)^2}-\frac{(1-q_n)kz^{k+1}}{(1+a_nz)^2(1+a_nq_nz)}\\&+\frac{a_n(1-q_n)kz^{k+2}}{(1+a_nz)^2(1+a_nq_nz)}-\frac{a_n^2q_nkz^{k+3}}{(1+a_nz)^2(1+a_nq_nz)}\\&-\frac{a_nk(k+1)z^{k+1}}{2b_n(1+a_nz)^2(1+a_nq_nz)}+\frac{a_n(1-q_n)k(k+1)z^{k+2}}{2(1+a_nz)^2(1+a_nq_nz)}\\&+\frac{a_n^2(1-q_n)k(k+1)z^{k+3}}{2(1+a_nz)^2(1+a_nq_nz)}-\frac{a_n^3q_nk(k+1)z^{k+4}}{2(1+a_nz)^2(1+a_nq_nz)}\\&-\frac{a_n(1+q_n^na_nz)((k-1)[k+1]_{q_n}-q_n[k-1]_{q_n})z^{k+1}}{b_n(1+a_nz)^2(1+a_nq_nz)}\\&-\frac{a_n^2(1+q_n^na_nz)(k-1)q_n[k]_{q_n}z^{k+2}}{b_n(1+a_nz)^2(1+a_nq_nz)}+\frac{a_nq_n^n[k]_{q_n}z^{k+1}}{b_n(1+a_nz)^2(1+a_nq_nz)}\\&-\frac{(1+q_n^na_nz)[k-1]_{q_n}(k-1)kz^{k-1}}{2b_n^2(1+a_nz)(1+a_nq_nz)(1+a_nq_n^2z)}\\&+\frac{(1-q_n)(1+q_n^na_nz)[k]_{q_n}(k-1)kz^k}{2b_n(1+a_nz)(1+a_nq_nz)(1+a_nq_n^2z)}\end{aligned}$$

$$
\begin{aligned}
&+\frac{a_n(1-q_n)(1+q_n^na_nz)[k+1]_{q_n}(k-1)kz^{k+1}}{2b_n(1+a_nz)(1+a_nq_nz)(1+a_nq_n^2z)}\\
&-\frac{a_n^2q_n(1+q_n^na_nz)[k+2]_{q_n}(k-1)kz^{k+2}}{2b_n(1+a_nz)(1+a_nq_nz)(1+a_nq_n^2z)}\\
&+\frac{a_n(1+q_n^na_nz)(1+q_n)(k-1)kz^{k}}{2b_n^2(1+a_nz)^2(1+a_nq_nz)(1+a_nq_n^2z)}\\
&-\frac{a_n(1-q_n)(1+q_n)(1+q_n^na_nz)(k-1)kz^{k+1}}{2b_n(1+a_nz)^2(1+a_nq_nz)(1+a_nq_n^2z)}\\
&-\frac{a_n^2(1-q_n)(1+q_n)(1+q_n^na_nz)(k-1)kz^{k+2}}{2b_n(1+a_nz)^2(1+a_nq_nz)(1+a_nq_n^2z)}\\
&-\frac{a_n^3q_n(1+q_n)(1+q_n^na_nz)(k-1)kz^{k+3}}{2b_n(1+a_nz)^2(1+a_nq_nz)(1+a_nq_n^2z)}.
\end{aligned}
$$

In the following results, C_i will denote fixed real numbers for $i=1,2,3$.

Under the hypothesis of Theorem 3, we have

$$\left|\frac{1}{1+q_n^sa_nz}\right|\le\frac{1}{1-q_n^sa_nr}<2\quad\text{for } s=0,1,2, \tag{21}$$

$$a_nr<\frac{1}{2}\quad\text{and}\quad 1+q_n^na_nr<\frac{3}{2}, \tag{22}$$

$$1-q_n\le\frac{a_n}{b_n}\quad\text{and}\quad k-[k]_{q_n}\le\frac{a_n}{b_n}\frac{(k-1)k}{2}, \tag{23}$$

$$[k]_{q_n}\le k,\qquad a_n<1\quad\text{and}\quad\frac{1}{b_n}<1. \tag{24}$$

Using (21)-(24), for $|z|\le r$, we get

$$
\begin{aligned}
\left|F_{k,n,q_n}(z)\right| &\le C_1\left(a_n^2+\frac{a_n}{b_n}+\frac{1}{b_n^2}\right)k(k+1)(k+2)\\
&\quad\times\max\left\{r^{k-1},r^{k},r^{k+1},r^{k+2},r^{k+3},r^{k+4}\right\}\\
&\le C_1\left(a_n+\frac{1}{b_n}\right)^2k(k+1)(k+2)(2r)^{k+4}.
\end{aligned}
\tag{25}
$$

On the other hand, for $|z|\le r$, we have

$$
\begin{aligned}
\left|\frac{(1+q_n^na_nz)z}{b_n(1+a_nz)}D_{q_n}\left[E_{k,n,q_n}(z)\right]\right| &\le \frac{(1+q_n^na_nr)r}{b_n(1-a_nr)}\frac{3k}{r}\|E_{k,n,q_n}\|_r\\
&\le\frac{3k(1+q_n^na_nr)}{b_n(1-a_nr)}\Biggl\{\|\alpha_{k,n,q_n}-e_k\|_r\\
&\quad+\frac{a_nkr^{k+1}}{1-a_nr}+\frac{(k-1)kr^{k-1}}{2b_n(1-a_nr)(1-a_nq_nr)}\\
&\quad+\frac{(1-q_n)(k-1)kr^{k}}{2(1-a_nr)(1-a_nq_nr)}\\
&\quad+\frac{a_n(1-q_n)(k-1)kr^{k+1}}{2(1-a_nr)(1-a_nq_nr)}+\frac{a_n^2q_n(k-1)kr^{k+2}}{2(1-a_nr)(1-a_nq_nr)}\Biggr\}.
\end{aligned}
$$

Taking into account (11) in the proof of Theorem 2, we obtain

$$\begin{aligned}\left|\frac{(1+q_n^n a_n z)z}{b_n(1+a_n z)}D_{q_n}\left[E_{k,n,q_n}(z)\right]\right| &\leq C_2\frac{1}{b_n}\left(a_n+\frac{1}{b_n}\right)(k-1)k(k+1)\\ &\quad\times(k!)(20r)^{k+2}\\ &\leq C_2\left(a_n+\frac{1}{b_n}\right)^2(k-1)k(k+1)(k!)(20r)^{k+2}.\end{aligned} \tag{26}$$

Considering (25) and (26) in (20), we get

$$\left|E_{k+1,n,q_n}(z)\right| \leq 2r\left|E_{k,n,q_n}(z)\right| + C_3\left(a_n+\frac{1}{b_n}\right)^2 k(k+1)(k+2)(k+1)!(20r)^{k+4}.$$

Since $E_{0,n,q}(z) = E_{1,n,q}(z) = E_{2,n,q}(z) = 0$, taking $k = 2,3,4,\ldots$ in the last inequality step by step, finally we arrive at

$$\left|E_{k,n,q_n}(z)\right| \leq C_3\left(a_n+\frac{1}{b_n}\right)^2(k-2)(k-1)k(k+1)(k!)(20r)^{k+3}. \tag{27}$$

Finally, considering (27) in (18) and using $20rA < 1$, the proof of the theorem is complete. □

Remark 1 For $0 < q \leq 1$, since $\frac{1}{[n]_q} \to 1-q$ as $n \to \infty$, therefore $a_n = (\frac{1}{[n]_q})^{1-\beta} \to (1-q)^{1-\beta}$ and $\frac{1}{b_n} = (\frac{1}{[n]_q})^{\beta} \to (1-q)^{\beta}$ as $n \to \infty$. If a sequence $\{q_n\}$ satisfies the conditions (10), then $\frac{1}{[n]_q} \to 0$ as $n \to \infty$; therefore $a_n = (\frac{1}{[n]_q})^{1-\beta} \to 0$ and $\frac{1}{b_n} = (\frac{1}{[n]_q})^{\beta} \to 0$ as $n \to \infty$.

Under the conditions (10), Theorem 2 and Theorem 3 show that $\{R_n(f;q_n,z)\}_{n\geq n_0}$ uniformly converges to $f(z)$ in $\bar{D}_r$.

From Theorem 2 and Theorem 3, we get the following consequence.

Theorem 4 *Let $\{q_n\}$ be a sequence satisfying the conditions* (10) *with $q_n \in (0,1]$ for all $n \in \mathbb{N}$, $n_0 \geq 2$, $0 < \beta \leq \frac{2}{3}$, $\beta \neq \frac{1}{2}$ and $\frac{1}{2} < r < R \leq \frac{[n_0]_{q_n}^{1-\beta}}{2}$. Suppose that $f : D_R \cup [R,\infty) \to \mathbb{C}$ is uniformly continuous, bounded on $[0,\infty)$ and analytic in D_R and there exist $M > 0$, $0 < A < \frac{1}{20r}$ with $|c_k| \leq M\frac{A^k}{k!}$ (which implies $|f(z)| \leq Me^{A|z|}$ for all $z \in D_R$). If f is not a polynomial of degree ≤ 1, then for all $n \geq n_0$ we have*

$$\left\|R_n(f;q_n,\cdot)-f\right\|_r \sim \left(a_n+\frac{1}{b_n}\right).$$

Proof We can write

$$R_n(f;q_n,z)-f(z) = \left(a_n+\frac{1}{b_n}\right)\left\{G(z)+H_n(z)\right\}, \tag{28}$$

where

$$\begin{aligned}G(z) = &-\frac{a_n}{a_n+1/b_n}\frac{z^2f'(z)}{1+a_nz}\\ &+\frac{1}{a_nb_n+1}\frac{zf''(z)}{2(1+a_nz)(1+a_nq_nz)}\end{aligned}$$

$$\begin{aligned}&-\frac{1-q_n}{a_n+1/b_n}\frac{z^2f''(z)}{2(1+a_nz)(1+a_nq_nz)}\\&-\frac{a_n(1-q_n)}{a_n+1/b_n}\frac{z^3f''(z)}{2(1+a_nz)(1+a_nq_nz)}\\&+\frac{a_n^2}{a_n+1/b_n}\frac{q_nz^4f''(z)}{2(1+a_nz)(1+a_nq_nz)}\end{aligned}\tag{29}$$

and

$$H_n(z)=\left(a_n+\frac{1}{b_n}\right)\left[\frac{1}{(a_n+\frac{1}{b_n})^2}A_{k,n,q_n}(z)\right],\tag{30}$$

and also $(H_n(z))_{n\in\mathbb{N}}$ is a sequence of analytic functions uniformly convergent to zero for all $|z|\le r$.

Since $a_n+\frac{1}{b_n}\to 0$ as $n\to\infty$, and taking into account Theorem 3, it remains only to show that for sufficiently large n and for all $|z|\le r$, we have $|G(z)|>\rho>0$, where ρ is independent of n.

If $2\beta-1<0$, then the term $\frac{1}{a_nb_n+1}\to 1$ as $n\to\infty$, while the other terms converge to zero, so there exists a natural number $n_1\in\mathbb{N}$ with $n_1\ge n_0$ so that for all $n\ge n_1$ and $|z|\le r$, we have

$$\left|G(z)\right|\ge\frac{1}{2}\left|\frac{zf''(z)}{2(1+a_nz)(1+a_nq_nz)}\right|\ge\frac{1}{4}\frac{|zf''(z)|}{(1+r)^2}.\tag{31}$$

If $2\beta-1>0$, then the term $\frac{a_n}{a_n+1/b_n}\to 1$ as $n\to\infty$, while the other terms converge to zero. So, there exists a natural number $n_2\in\mathbb{N}$ with $n_2\ge n_0$ so that for all $n\ge n_2$ and $|z|\le r$, we have

$$\left|G(z)\right|\ge\frac{1}{2}\left|\frac{z^2f'(z)}{1+a_nz}\right|\ge\frac{1}{2}\frac{|z^2f'(z)|}{1+r}.\tag{32}$$

In the case of $2\beta-1=0$, that is, $\beta=\frac{1}{2}$, we obtain $\frac{a_n^2}{a_n+1/b_n}=[n]_{q_n}^{1/2}\to\infty$, as $n\to\infty$, so that the case $\beta=\frac{1}{2}$ remains unsettled.

Choosing $n_3=\max\{n_1,n_2\}$, considering (31) and (32), for all $n\ge n_3$, we get

$$\left\|R_n(f;q_n,\cdot)-f\right\|_r\ge\left(a_n+\frac{1}{b_n}\right)\left|\|G\|_r-\|H_n\|_r\right|\ge\left(a_n+\frac{1}{b_n}\right)\frac{1}{2}\|G\|_r.$$

For all $n\in\{n_0,\dots,n_3-1\}$, we get

$$\left\|R_n(f;q_n,\cdot)-f\right\|_r\ge\left(a_n+\frac{1}{b_n}\right)M_{r,n,q_n}(z)$$

with $M_{r,n,q_n}(z)=\frac{1}{a_n+1/b_n}\|R_n(f;q_n,\cdot)-f\|_r>0$, which finally implies

$$\left\|R_n(f;q_n,\cdot)-f\right\|_r\ge\left(a_n+\frac{1}{b_n}\right)C_r(f)\tag{33}$$

for all $n\ge n_0$, with $C_r(f)=\min\{M_{r,n_0,q_n}(z),\dots,M_{r,n_3-1,q_n}(z),\frac{1}{2}\|G\|_r\}$.

From (33) and Theorem 3, the proof is complete. □

Remark 2 Recently, it is much more interesting to study these operators in the case $q > 1$. Authors continue to study that case.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The main idea of this paper is proposed by NI. All authors contributed equally in writing this article and read and approved the final manuscript.

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