



Well-posedness for systems of generalized mixed quasivariational inclusion problems and optimization problems with constraints

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Communicated by G. Marino

Abstract

In this paper, several metric characterizations of well-posedness for systems of generalized mixed quasivariational inclusion problems and for optimization problems with systems of generalized mixed quasivariational inclusion problems as constraints are given. The equivalence between the well-posedness of systems of generalized mixed quasivariational inclusion problems and the existence of solutions of systems of generalized mixed quasivariational inclusion problems is given under suitable conditions. ©2017 All rights reserved.

Keywords: Well-posedness, metric characterization, system of generalized mixed quasivariational inclusion problems, optimization problem with constraint.

2010 MSC: 49J40, 49J27, 90C31.

1. Introduction

It is well-known that the concept of well-posedness has played an important role in the study of optimization theory. The classical concept of well-posedness for a minimization problem is due to Tykhonov [25], which is known as the Tykhonov well-posedness. A minimization problem is called Tykhonov well-posed if there hold two aspects: the existence and uniqueness of minimizers and the convergence of every minimizing sequence toward the unique minimizer. It is clear that the concept of Tykhonov well-posedness is inspired by the numerical methods producing optimizing sequences for optimization

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doi:[10.22436/jnsa.010.10.23](https://doi.org/10.22436/jnsa.010.10.23)

Received 2017-08-07

problems. The concept of generalized Tykhonov well-posedness is also introduced for a minimization problem having more than one solution, which means the existence of minimizers and the convergence of some subsequence of every minimizing sequence toward a minimizer. Another important concept of well-posedness for a minimization problem is well-posedness by perturbations or extended well-posedness due to Zolezzi [32, 33]. The concept of well-posedness by perturbations establishes a form of continuous dependence of the solutions upon a parameter. There are many other concepts of well-posedness in optimization problems. For more details, we refer the reader to [6, 19, 23] and the references therein.

The concept of well-posedness has been generalized to some other problems: variational inequality problems [5, 6, 9, 10, 13, 16, 19, 29], saddle point problems [4], Nash equilibrium problems [20, 21], inclusion problems [9, 15], and fixed point problems [9, 15, 22].

Let P be a nonempty closed subset of a normed space, C be a nonempty closed convex subset of a real Banach space, $g : P \times C \times C \rightarrow \mathbf{R}$ and $h : P \times C \rightarrow \mathbf{R}$ be functions. Recently, Fang et al. [9] considered the following problems: for $p \in P$,

$$(EP)_p: \quad \text{Find } x \in C \text{ such that } g(p, x, u) \geq 0, \text{ for all } u \in C,$$

and

$$(OPEP): \quad \min h(p, x) \text{ subject to } x \text{ is a solution of } (EP)_p.$$

They established some metric characterizations of well-posedness for equilibrium problems and for optimization problems with equilibrium constraints. Furthermore, they proved that under suitable conditions, the well-posedness is equivalent to the existence and uniqueness of solutions, and that the well-posedness in the generalized sense is equivalent to the existence of solutions.

In recent years, the study of well-posedness for equilibrium problems has received so much attention and many results of well-posedness have been obtained for variational equilibrium problems (see e.g., [2, 8, 18] and the references therein). In 2009, Lin and Chuang [17] studied the well-posedness in the generalized sense for variational inclusion problems and variational disclusion problems, the well-posedness for optimization problems with variational inclusion problems, variational disclusion problems and scalar equilibrium problems as constraints.

The quasivariational inclusion problem is an important generalization of the variational inclusion problem, which contains a lot of important problems as special cases and has many applications, like variational disclusion problems, minimax inequalities, equilibrium problems, saddle point problems, optimization problems, bilevel problems, mathematical program with equilibrium constraint, variational inequalities, fixed point problems, coincidence point problems, Ekeland's variational principle, etc. For more details, we refer the reader to [4, 11, 12, 24, 27, 28, 30, 31] and the references therein. In 2009, Long and Huang [18] generalized the concept of α -well-posedness to symmetric quasiequilibrium problems in Banach spaces, which include equilibrium problems, Nash equilibrium problems, quasivariational inequalities, variational inequalities, and fixed-point problems as special cases. Under some suitable conditions, they established some metric characterizations of α -well-posedness for symmetric quasiequilibrium problems. In 2012, Ceng and Lin [3] generalized the concept of α -well-posedness to systems of mixed quasivariational-like inequalities in Banach spaces, which include symmetric quasiequilibrium problems as special cases. Moreover, they derived some metric characterizations of α -well-posedness for the system of mixed quasivariational-like inequalities. Very recently, Wang et al. [26] introduced and studied well-posedness of generalized quasivariational inclusion problems and of optimization problems with generalized quasivariational inclusion problems as constraints.

Motivated and inspired by the works mentioned above, in this paper, we shall investigate well-posedness for systems of generalized mixed quasivariational inclusion problems and for optimization problems with systems of generalized mixed quasivariational inclusion problems as constraints. We establish some metric characterizations of well-posedness for systems of generalized mixed quasivariational inclusion problems and for optimization problems with systems of generalized mixed quasivariational inclusion problems as constraints. We also prove that under mild conditions, the well-posedness of systems

of generalized mixed quasivariational inclusion problems is equivalent to the existence and uniqueness of solutions, and that the well-posedness in the generalized sense of systems of generalized mixed quasivariational inclusion problems is equivalent to the existence of solutions.

2. Preliminaries

In this section, we recall some definitions and lemmas used in the sequel.

Definition 2.1 ([1]). Let X and Y be two topological spaces. A multivalued mapping $T : X \rightarrow 2^Y$ is said to be

- (i) upper semicontinuous (for short, u.s.c.) at $x \in X$, if for each open set V in Y with $T(x) \subseteq V$, there exists an open neighborhood $U(x)$ of x such that $T(u) \subseteq V$ for all $u \in U(x)$;
- (ii) lower semicontinuous (for short, l.s.c.) at $x \in X$, if for each open set V in Y with $T(x) \cap V \neq \emptyset$, there exists an open neighborhood $U(x)$ of x such that $T(u) \cap V \neq \emptyset$ for all $u \in U(x)$;
- (iii) u.s.c. (resp., l.s.c.) on X if it is u.s.c. (resp., l.s.c.) at every point $x \in X$;
- (iv) continuous on X if it is both u.s.c. and l.s.c. on X ;
- (v) closed if the graph of T is closed, i.e., the set $\text{Gr}(T) = \{(x, y) \in X \times Y : y \in T(x)\}$ is closed in $X \times Y$.

Lemma 2.2 ([1]). Let X and Y be two topological spaces and $F : X \rightarrow 2^Y$ be a multivalued mapping.

- (i) If F is u.s.c. and closed-valued, then F is closed.
- (ii) If F is compact-valued, then F is u.s.c. at $x \in X$ if and only if for any net $\{x_\alpha\} \subseteq X$ with $x_\alpha \rightarrow x$ and for any net $\{y_\alpha\} \subseteq Y$ with $y_\alpha \in F(x_\alpha)$ for all α , there exist $y \in F(x)$ and a subnet $\{y_\beta\}$ of $\{y_\alpha\}$ such that $y_\beta \rightarrow y$.
- (iii) F is l.s.c. at $x \in X$ if and only if for any $y \in F(x)$ and for any net $\{x_\alpha\}$ with $x_\alpha \rightarrow x$, there exists a net $\{y_\alpha\}$ with $y_\alpha \in F(x_\alpha)$ for all α such that $y_\alpha \rightarrow y$.

Definition 2.3 ([14]). Let S be a nonempty subset of a complete metric space (X, d) . The Kuratowski measure of noncompactness of S is defined by

$$\mu(S) = \inf\{\varepsilon > 0 : S \subseteq \bigcup_{i=1}^n S_i, \text{diam} S_i < \varepsilon, i = 1, 2, \dots, n\},$$

where $\text{diam} S_i$ denotes the diameter of set S_i and is defined by $\text{diam} S_i = \sup\{d(x_1, x_2) : x_1, x_2 \in S_i\}$.

Definition 2.4. Let (X, d) be a metric space and let U, V be nonempty subsets of X . The Hausdorff metric $\mathcal{H}(\cdot, \cdot)$ between U and V is defined by

$$\mathcal{H}(U, V) := \max\{e(U, V), e(V, U)\},$$

where $e(U, V) := \sup_{u \in U} d(u, V)$ with $d(u, V) = \inf_{v \in V} d(u, v)$. Let $\{U_n\}$ be a sequence of nonempty subsets of X . One says that U_n converges to U in the sense of Hausdorff metric if $\mathcal{H}(U_n, U) \rightarrow 0$. It is easy to see that $e(U_n, U) \rightarrow 0$ if and only if $d(u_n, U) \rightarrow 0$ for all sections $u_n \in U_n$. For more details on this topic, we refer the readers to [14].

Definition 2.5. Let (E, ρ) be a metric space which is also a vector space. (E, ρ) is said to be a linear metric space if E is a vector topological space in the topology induced by ρ .

3. Well-posedness for systems of generalized mixed quasivariational inclusion problems

Let (\mathcal{P}, d_0) be a metric space and P be a nonempty closed subset of \mathcal{P} . Let (X, \hat{d}) and (Y, \bar{d}) be two linear metric spaces, Z be a Hausdorff topological vector space, $C \subseteq X$ and $D \subseteq Y$ be two nonempty closed subsets. Let $f, g : P \times C \times D \rightarrow Z$ be two single-valued mappings. Let $S : C \times D \rightarrow 2^C$, $T : C \times D \rightarrow 2^D$, $G : P \times X \times Y \times X \rightarrow 2^Z$ and $F : P \times Y \times X \times Y \rightarrow 2^Z$ be multivalued mappings. Let $\hat{e} : C \rightarrow Z$ and $\bar{e} : D \rightarrow Z$ be two continuous mappings. Throughout this paper, unless otherwise specified, we use these notations and assumptions.

We consider the following parametric system of generalized mixed quasivariational inclusion problems. For $p \in P$, $(\text{SGMQVIP})_p$: find a point $(x, y) \in S(x, y) \times T(x, y)$ such that

$$\begin{cases} 0 \in G(p, x, y, u) + f(p, x, y) - f(p, u, y), & \forall u \in S(x, y), \\ 0 \in F(p, y, x, v) + g(p, x, y) - g(p, x, v), & \forall v \in T(x, y). \end{cases}$$

Denote by (SGMQVIP) the family $\{(\text{SGMQVIP})_p : p \in P\}$. For each $p \in P$, let S_p be the solution set of $(\text{SGMQVIP})_p$.

In particular, if we set $f = 0$ and $g = 0$, the parametric system of generalized mixed quasivariational inclusion problems reduces to the following parametric system of generalized quasivariational inclusion problems. For $p \in P$, $(\text{SGQVIP})_p$: find a point $(x, y) \in S(x, y) \times T(x, y)$ such that

$$\begin{cases} 0 \in G(p, x, y, u), & \forall u \in S(x, y), \\ 0 \in F(p, y, x, v), & \forall v \in T(x, y). \end{cases}$$

Denote by (SGQVIP) the family $\{(\text{SGQVIP})_p : p \in P\}$. For each $p \in P$, let S_p be the solution set of $(\text{SGQVIP})_p$.

Let \mathcal{D} be a metric space. For each $a \in \mathcal{D}$ and each $r > 0$, we denote by $B(a, r)$ the closed ball centered at a with radius r . When $\mathcal{D} = \mathbf{R}$, we denote by $B^+(0, r)$ the closed interval $[0, r]$.

Definition 3.1. Let $p \in P$ and let $\{p_n\} \subseteq P$ be any sequence such that $p_n \rightarrow p$. A sequence $\{(x_n^*, y_n^*)\} \subset C \times D$ is called an approximating solution sequence for $(\text{SGMQVIP})_p$ corresponding to $\{p_n\}$ if for each $n \in \mathbf{N}$, $(x_n^*, y_n^*) \in S(x_n^*, y_n^*) \times T(x_n^*, y_n^*)$, and there exist $(x_n, y_n) \in S(x_n^*, y_n^*) \times T(x_n^*, y_n^*)$ and a sequence $\varepsilon_n > 0$ with $\varepsilon_n \rightarrow 0$ such that

$$\begin{cases} \hat{d}(x_n, x_n^*) \leq \hat{d}(x_n, S(x_n, y_n)), & \bar{d}(y_n, y_n^*) \leq \bar{d}(y_n, T(x_n, y_n)), \\ 0 \in G(p_n, x_n^*, y_n, u) + f(p_n, x_n^*, y_n) - f(p_n, u, y_n) + B^+(0, \varepsilon_n)\hat{e}(x_n^*), & \forall u \in S(x_n^*, y_n^*), \\ 0 \in F(p_n, y_n^*, x_n, v) + g(p_n, x_n, y_n^*) - g(p_n, x_n, v) + B^+(0, \varepsilon_n)\bar{e}(y_n^*), & \forall v \in T(x_n^*, y_n^*). \end{cases}$$

Definition 3.2. (SGMQVIP) is said to be well-posed if for every $p \in P$, $(\text{SGMQVIP})_p$ has a unique solution (x_p, y_p) , and for every sequence $\{p_n\} \subseteq P$ with $p_n \rightarrow p$, every approximating solution sequence for $(\text{SGMQVIP})_p$ corresponding to $\{p_n\}$ converges to (x_p, y_p) , and (SGMQVIP) is said to be well-posed in the generalized sense if for every $p \in P$, $(\text{SGMQVIP})_p$ has a nonempty solution set S_p , and for every sequence $\{p_n\} \subseteq P$ with $p_n \rightarrow p$, every approximating solution sequence for $(\text{SGMQVIP})_p$ corresponding to $\{p_n\}$ has a subsequence which converges to some point of S_p .

Remark 3.3.

- (a) When $(\text{SGMQVIP})_p$ reduces to $(\text{SGQVIP})_p$, we have a definition of approximating solution sequence for $(\text{SGQVIP})_p$ and definitions of well-posedness and well-posedness in the generalized sense for (SGQVIP) .
- (b) The definition of approximating solution sequence for $(\text{SGQVIP})_p$ generalizes that of Wang et al. [26].
- (c) The definition of approximating solution sequence for $(\text{GQVIP})_p$ in [26] generalizes that of Lin and Chuang [17].

For each $p \in P$, the approximating solution set for $(\text{SGMQVIP})_p$ is defined by

$$\Omega_p(\delta, \varepsilon) = \bigcup_{p' \in B(p, \delta)} \left\{ (x^*, y^*) \in C \times D : (x^*, y^*) \in S(x^*, y^*) \times T(x^*, y^*), \text{ and } \begin{aligned} & \exists (x, y) \in S(x^*, y^*) \times T(x^*, y^*) \text{ s.t. } \hat{d}(x, x^*) \leq \hat{d}(x, S(x, y)), \bar{d}(y, y^*) \leq \bar{d}(y, T(x, y)), \\ & 0 \in G(p', x^*, y, u) + f(p', x^*, y) - f(p', u, y) + B^+(0, \varepsilon)\hat{e}(x^*), \quad \forall u \in S(x^*, y^*), \\ & 0 \in F(p', y^*, x, v) + g(p', x, y^*) - g(p', x, v) + B^+(0, \varepsilon)\bar{e}(y^*), \quad \forall v \in T(x^*, y^*) \end{aligned} \right\},$$

for all $\delta, \varepsilon > 0$. Clearly, we have, for every $p \in P$,

- (i) $S_p \subseteq \Omega_p(\delta, \varepsilon)$ for all $\delta, \varepsilon > 0$;
- (ii) if $0 < \delta_1 \leq \delta_2$ and $0 < \varepsilon_1 \leq \varepsilon_2$, then $\Omega_p(\delta_1, \varepsilon_1) \subseteq \Omega_p(\delta_2, \varepsilon_2)$.

Next, we further establish some properties of $\Omega_p(\delta, \varepsilon)$.

Proposition 3.4. Assume that

- (i) S and T are compact-valued;
- (ii) for each $(x, y) \in C \times D$, single-valued mappings $(p, v) \mapsto f(p, x, v)$ and $(p, u) \mapsto g(p, u, y)$ are continuous;
- (iii) for each $(x, u, y, v) \in C \times C \times D \times D$, multivalued mappings $(p, t) \mapsto G(p, x, t, u)$ and $(p, s) \mapsto F(p, y, s, v)$ are closed;
- (iv) for each $(x^*, y^*) \in \bigcap_{\delta > 0, \varepsilon > 0} \Omega_p(\delta, \varepsilon)$, the convergence of $(x_n, y_n) \in S(x^*, y^*) \times T(x^*, y^*)$, $n \in \mathbf{N}$ implies $\hat{d}(x_n, S(x_n, y_n)) \rightarrow 0$ and $\bar{d}(y_n, T(x_n, y_n)) \rightarrow 0$.

Then $S_p = \bigcap_{\delta > 0, \varepsilon > 0} \Omega_p(\delta, \varepsilon)$ for each $p \in P$.

Proof. Take a fixed $p \in P$ arbitrarily. Clearly, $S_p \subseteq \bigcap_{\delta > 0, \varepsilon > 0} \Omega_p(\delta, \varepsilon)$. Hence, we only need to show that $\bigcap_{\delta > 0, \varepsilon > 0} \Omega_p(\delta, \varepsilon) \subseteq S_p$. If not, then there exists $(x^*, y^*) \in \bigcap_{\delta > 0, \varepsilon > 0} \Omega_p(\delta, \varepsilon)$ such that $(x^*, y^*) \notin S_p$. Then, for each $\delta > 0$ and each $\varepsilon > 0$, $(x^*, y^*) \in \Omega_p(\delta, \varepsilon) \setminus S_p$. Thus, for each $n \in \mathbf{N}$, $(x^*, y^*) \in \Omega_p(\frac{1}{n}, \frac{1}{n}) \setminus S_p$, and so there exist $p_n \in B(p, \frac{1}{n})$ and $(x_n, y_n) \in S(x^*, y^*) \times T(x^*, y^*)$ such that

$$\hat{d}(x_n, x^*) \leq \hat{d}(x_n, S(x_n, y_n)), \quad \bar{d}(y_n, y^*) \leq \bar{d}(y_n, T(x_n, y_n)),$$

$$\begin{cases} 0 \in G(p_n, x^*, y_n, u) + f(p_n, x^*, y_n) - f(p_n, u, y_n) + B^+(0, \frac{1}{n})\hat{e}(x^*), & \forall u \in S(x^*, y^*), \\ 0 \in F(p_n, y^*, x_n, v) + g(p_n, x_n, y^*) - g(p_n, x_n, v) + B^+(0, \frac{1}{n})\bar{e}(y^*), & \forall v \in T(x^*, y^*). \end{cases}$$

Clearly, $p_n \rightarrow p$. Since $\{y_n\} \subseteq T(x^*, y^*)$ and $T(x^*, y^*)$ is a compact set, there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k} \rightarrow \bar{y}$ as $k \rightarrow \infty$, and so, for each $k \in \mathbf{N}$,

$$0 \in G(p_{n_k}, x^*, y_{n_k}, u) + f(p_{n_k}, x^*, y_{n_k}) - f(p_{n_k}, u, y_{n_k}) + B^+(0, \frac{1}{n_k})\hat{e}(x^*), \quad \forall u \in S(x^*, y^*). \quad (3.1)$$

For each $u \in S(x^*, y^*)$, by (3.1), for every $k \in \mathbf{N}$, there exists $\hat{\gamma}_k \in B^+(0, \frac{1}{n_k})$ such that

$$0 \in G(p_{n_k}, x^*, y_{n_k}, u) + f(p_{n_k}, x^*, y_{n_k}) - f(p_{n_k}, u, y_{n_k}) + \hat{\gamma}_k \hat{e}(x^*). \quad (3.2)$$

Meantime, for each $v \in T(x^*, y^*)$ and each $k \in \mathbf{N}$, we know that there exists $\bar{\gamma}_k \in B^+(0, \frac{1}{n_k})$ such that

$$0 \in F(p_{n_k}, y^*, x_{n_k}, v) + g(p_{n_k}, x_{n_k}, y^*) - g(p_{n_k}, x_{n_k}, v) + \bar{\gamma}_k \bar{e}(y^*). \quad (3.3)$$

Clearly, $p_{n_k} \rightarrow p$ as $k \rightarrow \infty$. Since $\{x_{n_k}\} \subseteq T(x^*, y^*)$ and $T(x^*, y^*)$ is a compact set, there exists a subsequence of $\{x_{n_k}\}$, denoted still by $\{x_{n_k}\}$, such that $x_{n_k} \rightarrow \hat{x}$ as $k \rightarrow \infty$.

Clearly, $\hat{\gamma}_k \rightarrow 0$ and $\bar{\gamma}_k \rightarrow 0$ as $k \rightarrow \infty$. Since $(p, y) \mapsto G(p, x^*, y, u)$ is closed, $(p, y) \mapsto f(p, x^*, y)$ and $(p, y) \mapsto f(p, u, y)$ are continuous and (3.2) holds, we get

$$0 \in G(p, x^*, \bar{y}, u) + f(p, x^*, \bar{y}) - f(p, u, \bar{y}). \quad (3.4)$$

Meantime, since $(p, x) \mapsto G(p, y^*, x, v)$ is closed, $(p, x) \mapsto g(p, x, y^*)$ and $(p, x) \mapsto g(p, x, v)$ are continuous and (3.3) holds, we get

$$0 \in F(p, y^*, \hat{x}, v) + g(p, \hat{x}, y^*) - g(p, \hat{x}, v). \quad (3.5)$$

Furthermore, by the definition of metric $d(\cdot, \cdot)$ in the product space $(X \times Y, d)$, we deduce that

$$d((x_{n_k}, y_{n_k}), (\hat{x}, \bar{y})) = \sqrt{(\hat{d}(x_{n_k}, \hat{x}))^2 + (\bar{d}(y_{n_k}, \bar{y}))^2} \leq \hat{d}(x_{n_k}, \hat{x}) + \bar{d}(y_{n_k}, \bar{y}) \rightarrow 0.$$

From condition (iv) and the convergence of $\{(x_{n_k}, y_{n_k})\} \subseteq S(x^*, y^*) \times T(x^*, y^*)$ to (\hat{x}, \bar{y}) , it follows that $\hat{d}(x_{n_k}, S(x_{n_k}, y_{n_k})) \rightarrow 0$ and $\bar{d}(y_{n_k}, T(x_{n_k}, y_{n_k})) \rightarrow 0$ as $k \rightarrow \infty$. Thus,

$$\hat{d}(\hat{x}, x^*) = \lim_{k \rightarrow \infty} \hat{d}(x_{n_k}, x^*) \leq \lim_{k \rightarrow \infty} \hat{d}(x_{n_k}, S(x_{n_k}, y_{n_k})) = 0,$$

and

$$\bar{d}(\bar{y}, y^*) = \lim_{k \rightarrow \infty} \bar{d}(y_{n_k}, y^*) \leq \lim_{k \rightarrow \infty} d(y_{n_k}, T(x_{n_k}, y_{n_k})) = 0.$$

So, we get $\hat{x} = x^*$ and $\bar{y} = y^*$. Consequently, from (3.4) and (3.5) we have

$$\begin{cases} 0 \in G(p, x^*, y^*, u) + f(p, x^*, y^*) - f(p, u, y^*), & \forall u \in S(x^*, y^*), \\ 0 \in F(p, y^*, x^*, v) + g(p, x^*, y^*) - g(p, x^*, v), & \forall v \in T(x^*, y^*). \end{cases}$$

That is, $(x^*, y^*) \in S_p$, which is a contradiction. Hence, $\bigcap_{\delta>0, \varepsilon>0} \Omega_p(\delta, \varepsilon) \subseteq S_p$ and so

$$S_p = \bigcap_{\delta>0, \varepsilon>0} \Omega_p(\delta, \varepsilon).$$

This completes the proof. □

Proposition 3.5. Assume that

- (i) S and T are continuous and compact-valued;
- (ii) for each $p \in P$, single-valued mappings $(x, y) \mapsto f(p, x, y)$ and $(x, y) \mapsto g(p, x, y)$ are continuous;
- (iii) for each $p \in P$, multivalued mappings $(x, y, u) \mapsto G(p, x, y, u)$ and $(y, x, v) \mapsto F(p, y, x, v)$ are closed;
- (iv) for each convergent sequence $\{(x_n^*, y_n^*)\} \subseteq S_p$, the convergence of $(x_n, y_n) \in S(x_n^*, y_n^*) \times T(x_n^*, y_n^*)$, $n \in \mathbf{N}$ implies $\hat{d}(x_n, S(x_n, y_n)) \rightarrow 0$ and $\bar{d}(y_n, T(x_n, y_n)) \rightarrow 0$.

Then S_p and

$$\left\{ (x^*, y^*) \in C \times D : (x^*, y^*) \in S(x^*, y^*) \times T(x^*, y^*), \text{ and } \exists (x, y) \in S(x^*, y^*) \times T(x^*, y^*) \text{ s.t.} \right. \\ \hat{d}(x, x^*) \leq \hat{d}(x, S(x, y)), \bar{d}(y, y^*) \leq \bar{d}(y, T(x, y)), \\ 0 \in G(p, x^*, y, u) + f(p, x^*, y) - f(p, u, y) + B^+(0, \varepsilon)\hat{e}(x^*), \forall u \in S(x^*, y^*), \\ \left. 0 \in F(p, y^*, x, v) + g(p, x, y^*) - g(p, x, v) + B^+(0, \varepsilon)\bar{e}(y^*), \forall v \in T(x^*, y^*) \right\},$$

are closed subsets of $C \times D$.

Proof. (i) Take a fixed $p \in P$ arbitrarily. If $(x^*, y^*) \in \text{cl}S_p$, then there exists a sequence $\{(x_n^*, y_n^*)\}$ in S_p such that $(x_n^*, y_n^*) \rightarrow (x^*, y^*)$ (i.e., $x_n^* \rightarrow x^*$ and $y_n^* \rightarrow y^*$) as $n \rightarrow \infty$. It follows that, for each $n \in \mathbf{N}$, $(x_n^*, y_n^*) \in S(x_n^*, y_n^*) \times T(x_n^*, y_n^*)$ and there exists some $(x_n, y_n) \in S(x_n^*, y_n^*) \times T(x_n^*, y_n^*)$ such that $\hat{d}(x_n, x_n^*) \leq \hat{d}(x_n, S(x_n, y_n))$, $\bar{d}(y_n, y_n^*) \leq \bar{d}(y_n, T(x_n, y_n))$,

$$\begin{cases} 0 \in G(p, x_n^*, y_n, u) + f(p, x_n^*, y_n) - f(p, u, y_n), & \forall u \in S(x_n^*, y_n^*), \\ 0 \in F(p, y_n^*, x_n, v) + g(p, x_n, y_n^*) - g(p, x_n, v), & \forall v \in T(x_n^*, y_n^*). \end{cases}$$

Since S is u.s.c. and closed-valued, S is closed and so $x^* \in S(x^*, y^*)$. Since T is u.s.c. and compact-valued, there exist a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ and $\bar{y} \in T(x^*, y^*)$ such that $y_{n_k} \rightarrow \bar{y}$ as $k \rightarrow \infty$. It follows

that, for each $k \in \mathbf{N}$,

$$0 \in G(p, x_{n_k}^*, y_{n_k}, u) + f(p, x_{n_k}^*, y_{n_k}) - f(p, u, y_{n_k}), \quad \forall u \in S(x_{n_k}^*, y_{n_k}^*). \quad (3.6)$$

For each $u \in S(x^*, y^*)$, since S is l.s.c., there exists a sequence $\{u_{n_k}\}$ with $u_{n_k} \in S(x_{n_k}^*, y_{n_k}^*)$ such that $u_{n_k} \rightarrow u$ as $k \rightarrow \infty$, and so

$$0 \in G(p, x_{n_k}^*, y_{n_k}, u_k) + f(p, x_{n_k}^*, y_{n_k}) - f(p, u_k, y_{n_k}). \quad (3.7)$$

Next, for each $k \in \mathbf{N}$, we get the following relation associated with (3.6)

$$0 \in F(p, y_{n_k}^*, x_{n_k}, v) + g(p, x_{n_k}, y_{n_k}^*) - g(p, x_{n_k}, v), \quad \forall v \in T(x_{n_k}^*, y_{n_k}^*).$$

Since T is u.s.c. and closed-valued, T is closed and so $y^* \in T(x^*, y^*)$. Since S is u.s.c. and compact-valued, there exist a subsequence of $\{x_{n_k}\}$, denoted still by $\{x_{n_k}\}$, and $\hat{x} \in S(x^*, y^*)$ such that $x_{n_k} \rightarrow \hat{x}$ as $k \rightarrow \infty$. Moreover, for each $v \in T(x^*, y^*)$, since T is l.s.c., there exists a sequence $\{v_k\}$ with $v_k \in T(x_{n_k}^*, y_{n_k}^*)$ such that $v_k \rightarrow v$ as $k \rightarrow \infty$, and so

$$0 \in F(p, y_{n_k}^*, x_{n_k}, v_k) + g(p, x_{n_k}, y_{n_k}^*) - g(p, x_{n_k}, v_k). \quad (3.8)$$

Since multivalued mappings $(x, y, u) \mapsto G(p, x, y, u)$ and $(y, x, v) \mapsto F(p, y, x, v)$ are closed and single-valued mappings $(u, y) \mapsto f(p, u, y)$ and $(x, v) \mapsto g(p, x, v)$ are continuous, we obtain from (3.7) and (3.8) that

$$\begin{cases} 0 \in G(p, x^*, \bar{y}, u) + f(p, x^*, \bar{y}) - f(p, u, \bar{y}), & \forall u \in S(x^*, y^*), \\ 0 \in F(p, y^*, \hat{x}, v) + g(p, \hat{x}, y^*) - g(p, \hat{x}, v), & \forall v \in T(x^*, y^*). \end{cases} \quad (3.9)$$

From condition (iv) and the convergence of $(x_{n_k}, y_{n_k}) \in S(x_{n_k}^*, y_{n_k}^*) \times T(x_{n_k}^*, y_{n_k}^*)$, $k \in \mathbf{N}$ to (\hat{x}, \bar{y}) , it follows that $\hat{d}(x_{n_k}, S(x_{n_k}, y_{n_k})) \rightarrow 0$ and $\bar{d}(y_{n_k}, T(x_{n_k}, y_{n_k})) \rightarrow 0$ as $k \rightarrow \infty$. Thus,

$$\hat{d}(\hat{x}, x^*) = \lim_{k \rightarrow \infty} \hat{d}(x_{n_k}, x_{n_k}^*) \leq \lim_{k \rightarrow \infty} \hat{d}(x_{n_k}, S(x_{n_k}, y_{n_k})) = 0,$$

and

$$\bar{d}(\bar{y}, y^*) = \lim_{k \rightarrow \infty} \bar{d}(y_{n_k}, y_{n_k}^*) \leq \lim_{k \rightarrow \infty} \bar{d}(y_{n_k}, T(x_{n_k}, y_{n_k})) = 0.$$

So, we get $\hat{x} = x^*$ and $\bar{y} = y^*$. Consequently, from (3.9) we have

$$\begin{cases} 0 \in G(p, x^*, y^*, u) + f(p, x^*, y^*) - f(p, u, y^*), & \forall u \in S(x^*, y^*), \\ 0 \in F(p, y^*, x^*, v) + g(p, x^*, y^*) - g(p, x^*, v), & \forall v \in T(x^*, y^*). \end{cases}$$

That is, $(x^*, y^*) \in S_p$, which hence implies $\text{cl}S_p \subseteq S_p$. Therefore, $S_p = \text{cl}S_p$.

(ii) Take any $p \in P$ and any $\varepsilon > 0$ and let (p, ε) be fixed. Let

$$A = \left\{ (x^*, y^*) \in C \times D : (x^*, y^*) \in S(x^*, y^*) \times T(x^*, y^*), \text{ and } \exists (x, y) \in S(x^*, y^*) \times T(x^*, y^*) \text{ s.t.} \right. \\ \left. \begin{aligned} &\hat{d}(x, x^*) \leq \hat{d}(x, S(x, y)), \quad \bar{d}(y, y^*) \leq \bar{d}(y, T(x, y)), \\ &0 \in G(p, x^*, y, u) + f(p, x^*, y) - f(p, u, y) + B^+(0, \varepsilon)\hat{e}(x^*), \quad \forall u \in S(x^*, y^*), \\ &0 \in F(p, y^*, x, v) + g(p, x, y^*) - g(p, x, v) + B^+(0, \varepsilon)\bar{e}(y^*), \quad \forall v \in T(x^*, y^*) \end{aligned} \right\}.$$

If $(x^*, y^*) \in \text{cl}A$, then there exists a sequence $\{(x_n^*, y_n^*)\}$ in A such that $(x_n^*, y_n^*) \rightarrow (x^*, y^*)$ as $n \rightarrow \infty$. It follows that, for each $n \in \mathbf{N}$, $(x_n^*, y_n^*) \in S(x_n^*, y_n^*) \times T(x_n^*, y_n^*)$ and there exists some $(x_n, y_n) \in S(x_n^*, y_n^*) \times T(x_n^*, y_n^*)$ such that $\hat{d}(x_n, x_n^*) \leq \hat{d}(x_n, S(x_n, y_n))$, $\bar{d}(y_n, y_n^*) \leq \bar{d}(y_n, T(x_n, y_n))$,

$$\begin{cases} 0 \in G(p, x_n^*, y_n, u) + f(p, x_n^*, y_n) - f(p, u, y_n) + B^+(0, \varepsilon)\hat{e}(x_n^*), & \forall u \in S(x_n^*, y_n^*), \\ 0 \in F(p, y_n^*, x_n, v) + g(p, x_n, y_n^*) - g(p, x_n, v) + B^+(0, \varepsilon)\bar{e}(y_n^*), & \forall v \in T(x_n^*, y_n^*). \end{cases}$$

Since S is u.s.c. and closed-valued, S is closed and so $x^* \in S(x^*, y^*)$. Since T is u.s.c. and compact-valued, there exist a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ and $\bar{y} \in T(x^*, y^*)$ such that $y_{n_k} \rightarrow \bar{y}$ as $k \rightarrow \infty$. It follows that, for each $k \in \mathbf{N}$,

$$0 \in G(p, x_{n_k}^*, y_{n_k}, u) + f(p, x_{n_k}^*, y_{n_k}) - f(p, u, y_{n_k}) + B^+(0, \varepsilon) \hat{e}(x_{n_k}^*), \quad \forall u \in S(x_{n_k}^*, y_{n_k}^*). \quad (3.10)$$

For each $u \in S(x^*, y^*)$, since S is l.s.c., there exists a sequence $\{u_{n_k}\}$ with $u_{n_k} \in S(x_{n_k}^*, y_{n_k}^*)$ such that $u_{n_k} \rightarrow u$ as $k \rightarrow \infty$, and so

$$0 \in G(p, x_{n_k}^*, y_{n_k}, u_k) + f(p, x_{n_k}^*, y_{n_k}) - f(p, u_k, y_{n_k}) + B^+(0, \varepsilon) \hat{e}(x_{n_k}^*). \quad (3.11)$$

Next, for each $k \in \mathbf{N}$, we get the following relation associated with (3.10)

$$0 \in F(p, y_{n_k}^*, x_{n_k}, v) + g(p, x_{n_k}, y_{n_k}^*) - g(p, x_{n_k}, v) + B^+(0, \varepsilon) \bar{e}(y_{n_k}^*), \quad \forall v \in T(x_{n_k}^*, y_{n_k}^*). \quad (3.12)$$

Since T is u.s.c. and closed-valued, T is closed and so $y^* \in T(x^*, y^*)$. Since S is u.s.c. and compact-valued, there exist a subsequence of $\{x_{n_k}\}$, denoted still by $\{x_{n_k}\}$, and $\hat{x} \in S(x^*, y^*)$ such that $x_{n_k} \rightarrow \hat{x}$ as $k \rightarrow \infty$. Moreover, for each $v \in T(x^*, y^*)$, since T is l.s.c., there exists a sequence $\{v_k\}$ with $v_k \in T(x_{n_k}^*, y_{n_k}^*)$ such that $v_k \rightarrow v$ as $k \rightarrow \infty$, and so

$$0 \in F(p, y_{n_k}^*, x_{n_k}, v_k) + g(p, x_{n_k}, y_{n_k}^*) - g(p, x_{n_k}, v_k) + B^+(0, \varepsilon) \bar{e}(y_{n_k}^*). \quad (3.12)$$

So, it follows from (3.11) and (3.12) that there exist two sequences $\{\hat{\gamma}_k\}$ and $\{\bar{\gamma}_k\}$ in $B^+(0, \varepsilon)$ such that for all $k \in \mathbf{N}$

$$\begin{cases} 0 \in G(p, x_{n_k}^*, y_{n_k}, u_k) + f(p, x_{n_k}^*, y_{n_k}) - f(p, u_k, y_{n_k}) + \hat{\gamma}_k \hat{e}(x_{n_k}^*), \\ 0 \in F(p, y_{n_k}^*, x_{n_k}, v_k) + g(p, x_{n_k}, y_{n_k}^*) - g(p, x_{n_k}, v_k) + \bar{\gamma}_k \bar{e}(y_{n_k}^*). \end{cases} \quad (3.13)$$

Observe that $B^+(0, \varepsilon) = [0, \varepsilon] \subseteq \mathbf{R}$ is compact. We may assume that $\hat{\gamma}_k \rightarrow \hat{\gamma} \in B^+(0, \varepsilon)$ and $\bar{\gamma}_k \rightarrow \bar{\gamma} \in B^+(0, \varepsilon)$ as $k \rightarrow \infty$. Since multivalued mappings $(x, y, u) \mapsto G(p, x, y, u)$ and $(y, x, v) \mapsto F(p, y, x, v)$ are closed and single-valued mappings $(u, y) \mapsto f(p, u, y)$ and $(x, v) \mapsto g(p, x, v)$ are continuous, we obtain from (3.13) that

$$\begin{cases} 0 \in G(p, x^*, \bar{y}, u) + f(p, x^*, \bar{y}) - f(p, u, \bar{y}) + \hat{\gamma} \hat{e}(x^*), & \forall u \in S(x^*, y^*), \\ 0 \in F(p, y^*, \hat{x}, v) + g(p, \hat{x}, y^*) - g(p, \hat{x}, v) + \bar{\gamma} \bar{e}(y^*), & \forall v \in T(x^*, y^*), \end{cases}$$

which immediately yields

$$\begin{cases} 0 \in G(p, x^*, \bar{y}, u) + f(p, x^*, \bar{y}) - f(p, u, \bar{y}) + B^+(0, \varepsilon) \hat{e}(x^*), & \forall u \in S(x^*, y^*), \\ 0 \in F(p, y^*, \hat{x}, v) + g(p, \hat{x}, y^*) - g(p, \hat{x}, v) + B^+(0, \varepsilon) \bar{e}(y^*), & \forall v \in T(x^*, y^*). \end{cases} \quad (3.14)$$

From condition (iv) and the convergence of $(x_{n_k}, y_{n_k}) (\in S(x_{n_k}^*, y_{n_k}^*) \times T(x_{n_k}^*, y_{n_k}^*), k \in \mathbf{N})$ to (\hat{x}, \bar{y}) , it follows that $\hat{d}(x_{n_k}, S(x_{n_k}, y_{n_k})) \rightarrow 0$ and $\bar{d}(y_{n_k}, T(x_{n_k}, y_{n_k})) \rightarrow 0$ as $k \rightarrow \infty$. Thus,

$$\hat{d}(\hat{x}, x^*) = \lim_{k \rightarrow \infty} \hat{d}(x_{n_k}, x_{n_k}^*) \leq \lim_{k \rightarrow \infty} \hat{d}(x_{n_k}, S(x_{n_k}, y_{n_k})) = 0,$$

and

$$\bar{d}(\bar{y}, y^*) = \lim_{k \rightarrow \infty} \bar{d}(y_{n_k}, y_{n_k}^*) \leq \lim_{k \rightarrow \infty} \bar{d}(y_{n_k}, T(x_{n_k}, y_{n_k})) = 0.$$

So, we get $\hat{x} = x^*$ and $\bar{y} = y^*$. Consequently, from (3.14) we have

$$\begin{cases} 0 \in G(p, x^*, y^*, u) + f(p, x^*, y^*) - f(p, u, y^*) + B^+(0, \varepsilon) \hat{e}(x^*), & \forall u \in S(x^*, y^*), \\ 0 \in F(p, y^*, x^*, v) + g(p, x^*, y^*) - g(p, x^*, v) + B^+(0, \varepsilon) \bar{e}(y^*), & \forall v \in T(x^*, y^*). \end{cases}$$

That is, $(x^*, y^*) \in A$, which hence implies $\text{cl}A \subseteq A$. Therefore, $A = \text{cl}A$. □

Proposition 3.6. Assume that

- (i) P is finite-dimensional;
- (ii) S and T are continuous and compact-valued;
- (iii) single-valued mappings f and g are continuous;
- (iv) multivalued mappings G and F are closed;
- (v) for every $p \in P$, there exists $\Omega_p(\delta, \varepsilon)$ such that for each convergent sequence $\{(x_n^*, y_n^*)\} \subseteq \Omega_p(\delta, \varepsilon)$, the convergence of $(x_n, y_n) \in S(x_n^*, y_n^*) \times T(x_n^*, y_n^*)$, $n \in \mathbf{N}$ implies $\hat{d}(x_n, S(x_n, y_n)) \rightarrow 0$ and

$$\bar{d}(y_n, T(x_n, y_n)) \rightarrow 0.$$

Then, for each $p \in P$, $\Omega_p(\delta, \varepsilon)$ is a closed subset of $C \times D$.

Proof. Take any $p \in P$ and any $\delta, \varepsilon > 0$ and let (p, δ, ε) be fixed. If $(x^*, y^*) \in \text{cl}(\Omega_p(\delta, \varepsilon))$, then there exists a sequence $\{(x_n^*, y_n^*)\}$ in $\Omega_p(\delta, \varepsilon)$ such that $(x_n^*, y_n^*) \rightarrow (x^*, y^*)$ (i.e., $x_n^* \rightarrow x^*$ and $y_n^* \rightarrow y^*$) as $n \rightarrow \infty$. It follows that, for each $n \in \mathbf{N}$, $(x_n^*, y_n^*) \in S(x_n^*, y_n^*) \times T(x_n^*, y_n^*)$ and there exist $p_n \in B(p, \delta)$ and $(x_n, y_n) \in S(x_n^*, y_n^*) \times T(x_n^*, y_n^*)$ such that $\hat{d}(x_n, x_n^*) \leq \hat{d}(x_n, S(x_n, y_n))$, $\bar{d}(y_n, y_n^*) \leq \bar{d}(y_n, T(x_n, y_n))$,

$$\begin{cases} 0 \in G(p_n, x_n^*, y_n, u) + f(p_n, x_n^*, y_n) - f(p_n, u, y_n) + B^+(0, \varepsilon)\hat{e}(x_n^*), & \forall u \in S(x_n^*, y_n^*), \\ 0 \in F(p_n, y_n^*, x_n, v) + g(p_n, x_n, y_n^*) - g(p_n, x_n, v) + B^+(0, \varepsilon)\bar{e}(y_n^*), & \forall v \in T(x_n^*, y_n^*). \end{cases}$$

Note that P is finite-dimensional. We may assume that $p_n \rightarrow \tilde{p} \in B(p, \delta)$. Then, by the similar arguments to those in the second part of the proof of Proposition 3.5, we can show that $(x^*, y^*) \in S(x^*, y^*) \times T(x^*, y^*)$ and

$$\begin{cases} 0 \in G(\tilde{p}, x^*, y^*, u) + f(\tilde{p}, x^*, y^*) - f(\tilde{p}, u, y^*) + B^+(0, \varepsilon)\hat{e}(x^*), & \forall u \in S(x^*, y^*), \\ 0 \in F(\tilde{p}, y^*, x^*, v) + g(\tilde{p}, x^*, y^*) - g(\tilde{p}, x^*, v) + B^+(0, \varepsilon)\bar{e}(y^*), & \forall v \in T(x^*, y^*). \end{cases}$$

That is, $(x^*, y^*) \in \Omega_p(\delta, \varepsilon)$, which hence implies $\text{cl}(\Omega_p(\delta, \varepsilon)) \subseteq \Omega_p(\delta, \varepsilon)$. Therefore, $\Omega_p(\delta, \varepsilon) = \text{cl}(\Omega_p(\delta, \varepsilon))$. This completes the proof. \square

Theorem 3.7. Let X and Y be complete. Assume that

- (i) S and T are continuous and compact-valued;
- (ii) single-valued mappings f and g are continuous;
- (iii) multivalued mappings G and F are closed;
- (iv) for every $p \in P$, there exists $\Omega_p(\delta, \varepsilon)$ such that for each convergent sequence $\{(x_n^*, y_n^*)\} \subseteq \Omega_p(\delta, \varepsilon)$, the convergence of $(x_n, y_n) \in S(x_n^*, y_n^*) \times T(x_n^*, y_n^*)$, $n \in \mathbf{N}$ implies $\hat{d}(x_n, S(x_n, y_n)) \rightarrow 0$ and

$$\bar{d}(y_n, T(x_n, y_n)) \rightarrow 0.$$

Then (SGMQVIP) is well-posed if and only if for every $p \in P$,

$$\Omega_p(\delta, \varepsilon) \neq \emptyset, \quad \forall \delta, \varepsilon > 0, \quad \text{and} \quad \text{diam}(\Omega_p(\delta, \varepsilon)) \rightarrow 0 \quad \text{as} \quad (\delta, \varepsilon) \rightarrow (0, 0). \quad (3.15)$$

Proof. Suppose that (SGMQVIP) is well-posed. Then, for every $p \in P$, (SGMQVIP) $_p$ has a unique solution (x_p, y_p) , and so $\Omega_p(\delta, \varepsilon) \neq \emptyset$ since $(x_p, y_p) \in S_p \subseteq \Omega_p(\delta, \varepsilon)$ for all $\delta, \varepsilon > 0$.

Now we shall show that

$$\text{diam}(\Omega_p(\delta, \varepsilon)) \rightarrow 0 \quad \text{as} \quad (\delta, \varepsilon) \rightarrow (0, 0). \quad (3.16)$$

If not, then there exist $r > 0$, sequences $\{\delta_n\}$ and $\{\varepsilon_n\}$ of positive real numbers with $(\delta_n, \varepsilon_n) \rightarrow (0, 0)$ as $n \rightarrow \infty$ and sequences $\{(x_n^{1*}, y_n^{1*})\}$ and $\{(x_n^{2*}, y_n^{2*})\}$ with $(x_n^{1*}, y_n^{1*}), (x_n^{2*}, y_n^{2*}) \in \Omega_p(\delta_n, \varepsilon_n)$ for each $n \in \mathbf{N}$ such that

$$r < d((x_n^{1*}, y_n^{1*}), (x_n^{2*}, y_n^{2*})) = \sqrt{(\hat{d}(x_n^{1*}, x_n^{2*}))^2 + (\bar{d}(y_n^{1*}, y_n^{2*}))^2}, \quad n \in \mathbf{N}. \quad (3.17)$$

For each $n \in \mathbf{N}$ and $i = 1, 2$, since $(x_n^{i*}, y_n^{i*}) \in \Omega_p(\delta_n, \varepsilon_n)$, we know that

$$(x_n^{i*}, y_n^{i*}) \in S(x_n^{i*}, y_n^{i*}) \times T(x_n^{i*}, y_n^{i*}),$$

and there exist $p_n^i \in B(p, \delta_n)$ and $(x_n^i, y_n^i) \in S(x_n^{i*}, y_n^{i*}) \times T(x_n^{i*}, y_n^{i*})$ such that

$$\begin{aligned} \hat{d}(x_n^i, x_n^{i*}) &\leq \hat{d}(x_n^i, S(x_n^i, y_n^i)), \quad \bar{d}(y_n^i, y_n^{i*}) \leq \bar{d}(y_n^i, T(x_n^i, y_n^i)), \\ \begin{cases} 0 \in G(p_n^i, x_n^{i*}, y_n^i, u) + f(p_n^i, x_n^{i*}, y_n^i) - f(p_n^i, u, y_n^i) + B^+(0, \varepsilon_n)\hat{e}(x_n^{i*}), & \forall u \in S(x_n^{i*}, y_n^{i*}), \\ 0 \in F(p_n^i, y_n^{i*}, x_n^i, v) + g(p_n^i, x_n^i, y_n^{i*}) - g(p_n^i, x_n^i, v) + B^+(0, \varepsilon_n)\bar{e}(y_n^{i*}), & \forall v \in T(x_n^{i*}, y_n^{i*}). \end{cases} \end{aligned}$$

Clearly $p_n^1 \rightarrow p$ and $p_n^2 \rightarrow p$ as $n \rightarrow \infty$. Hence $\{(x_n^{1*}, y_n^{1*})\}$ and $\{(x_n^{2*}, y_n^{2*})\}$ are approximating solution sequences for $(\text{SGMQVIP})_p$ corresponding to $\{p_n^1\}$ and $\{p_n^2\}$, respectively. Then, by the well-posedness of $(\text{SGMQVIP})_p$, they have to converge to the unique solution (x_p, y_p) of $(\text{SGMQVIP})_p$, a contradiction to (3.17). Thus (3.16) holds.

Conversely, suppose that condition (3.15) holds. Take any $p \in P$ and let p be fixed. Let $\{p_n\}$ be any sequence in P with $p_n \rightarrow p$ as $n \rightarrow \infty$. If $\{(x_n^*, y_n^*)\}$ is an approximating solution sequence for $(\text{SGMQVIP})_p$ corresponding to $\{p_n\}$, then for each $n \in \mathbf{N}$, $(x_n^*, y_n^*) \in S(x_n^*, y_n^*) \times T(x_n^*, y_n^*)$, and there exist $(x_n, y_n) \in S(x_n^*, y_n^*) \times T(x_n^*, y_n^*)$ and a sequence $\varepsilon_n > 0$ with $\varepsilon_n \rightarrow 0$ such that

$$\begin{aligned} \hat{d}(x_n, x_n^*) &\leq \hat{d}(x_n, S(x_n, y_n)), \quad \bar{d}(y_n, y_n^*) \leq \bar{d}(y_n, T(x_n, y_n)), \\ \begin{cases} 0 \in G(p_n, x_n^*, y_n, u) + f(p_n, x_n^*, y_n) - f(p_n, u, y_n) + B^+(0, \varepsilon_n)\hat{e}(x_n^*), & \forall u \in S(x_n^*, y_n^*), \\ 0 \in F(p_n, y_n^*, x_n, v) + g(p_n, x_n, y_n^*) - g(p_n, x_n, v) + B^+(0, \varepsilon_n)\bar{e}(y_n^*), & \forall v \in T(x_n^*, y_n^*). \end{cases} \end{aligned}$$

For each $n \in \mathbf{N}$, let $\delta_n = d_0(p_n, p)$. Then, $p_n \in B(p, \delta_n)$ and $(x_n^*, y_n^*) \in \Omega_p(\delta_n, \varepsilon_n)$ for each $n \in \mathbf{N}$, and $\delta_n \rightarrow 0$ as $n \rightarrow \infty$. It follows from (3.15) that $\{(x_n^*, y_n^*)\}$ is a Cauchy sequence and so it converges to a point $(x^*, y^*) \in C \times D$. Then, by the similar arguments to those in the second part of the proof of Proposition 3.5, we can show that $(x^*, y^*) \in S(x^*, y^*) \times T(x^*, y^*)$ and

$$\begin{cases} 0 \in G(p, x^*, y^*, u) + f(p, x^*, y^*) - f(p, u, y^*), & \forall u \in S(x^*, y^*), \\ 0 \in F(p, y^*, x^*, v) + g(p, x^*, y^*) - g(p, x^*, v), & \forall v \in T(x^*, y^*). \end{cases}$$

Thus (x^*, y^*) is a solution of $(\text{SGMQVIP})_p$.

To complete the proof, it is sufficient to prove that $(\text{SGMQVIP})_p$ has a unique solution. If $(\text{SGMQVIP})_p$ has two distinct solutions (x^1, y^1) and (x^2, y^2) , it is easy to see that $(x^1, y^1), (x^2, y^2) \in \Omega_p(\delta, \varepsilon)$ for all $\delta, \varepsilon > 0$. It follows that

$$0 < d((x^1, y^1), (x^2, y^2)) \leq \text{diam}(\Omega_p(\delta, \varepsilon)), \quad \forall \delta, \varepsilon > 0,$$

a contradiction to (3.15). Therefore, $(\text{SGMQVIP})_p$ has a unique solution. This completes the proof. \square

Theorem 3.8. *Let X and Y be complete. Assume that*

- (i) P is finite-dimensional;
- (ii) S and T are continuous and compact-valued;
- (iii) single-valued mappings f and g are continuous;
- (iv) multivalued mappings G and F are closed;
- (v) for every $p \in P$, there exists $\Omega_p(\delta, \varepsilon)$ such that for each convergent sequence $\{(x_n^*, y_n^*)\} \subseteq \Omega_p(\delta, \varepsilon)$, the convergence of $(x_n, y_n) \in S(x_n^*, y_n^*) \times T(x_n^*, y_n^*)$, $n \in \mathbf{N}$ implies

$$\begin{cases} \hat{d}(x_n, S(x_n, y_n)) \rightarrow 0, \\ \bar{d}(y_n, T(x_n, y_n)) \rightarrow 0. \end{cases}$$

Then (SGMQVIP) is well-posed in the generalized sense if and only if for every $p \in P$,

$$\Omega_p(\delta, \varepsilon) \neq \emptyset, \quad \forall \delta, \varepsilon > 0, \quad \text{and} \quad \mu(\Omega_p(\delta, \varepsilon)) \rightarrow 0 \quad \text{as} \quad (\delta, \varepsilon) \rightarrow (0, 0). \quad (3.18)$$

Proof. Suppose that (SGMQVIP) is well-posed in the generalized sense. Take any $p \in P$ and let p be fixed. Then S_p is nonempty. Now we show that S_p is compact. Indeed, let $\{(x_n^*, y_n^*)\}$ be any sequence in S_p . Then $\{(x_n^*, y_n^*)\}$ is an approximating solution sequence for (SGMQVIP) $_p$. By the well-posedness in the generalized sense of (SGMQVIP), $\{(x_n^*, y_n^*)\}$ has a subsequence which converges to some point of S_p . Thus S_p is compact. Clearly, for each $\delta, \varepsilon > 0$, $S_p \subseteq \Omega_p(\delta, \varepsilon)$, and so $\Omega_p(\delta, \varepsilon) \neq \emptyset$.

Now we shall show that

$$\mu(\Omega_p(\delta, \varepsilon)) \rightarrow 0 \quad \text{as} \quad (\delta, \varepsilon) \rightarrow (0, 0). \quad (3.19)$$

Observe that for every $\delta, \varepsilon > 0$,

$$\mathcal{H}(\Omega_p(\delta, \varepsilon), S_p) = \max\{e(\Omega_p(\delta, \varepsilon), S_p), e(S_p, \Omega_p(\delta, \varepsilon))\} = e(\Omega_p(\delta, \varepsilon), S_p).$$

Taking into account the compactness of S_p , we get

$$\mu(\Omega_p(\delta, \varepsilon)) \leq 2\mathcal{H}(\Omega_p(\delta, \varepsilon), S_p) + \mu(S_p) = 2e(\Omega_p(\delta, \varepsilon), S_p).$$

To prove (3.19), it is sufficient to show that

$$e(\Omega_p(\delta, \varepsilon), S_p) \rightarrow 0 \quad \text{as} \quad (\delta, \varepsilon) \rightarrow (0, 0). \quad (3.20)$$

As a matter of fact, if (3.20) does not hold, then there exist $r > 0$, sequences $\{\delta_n\}$ and $\{\varepsilon_n\}$ of positive real numbers with $(\delta_n, \varepsilon_n) \rightarrow (0, 0)$ as $n \rightarrow \infty$ and sequence $\{(x_n^*, y_n^*)\}$ with $(x_n^*, y_n^*) \in \Omega_p(\delta_n, \varepsilon_n)$ for every $n \in \mathbf{N}$ such that

$$(x_n^*, y_n^*) \notin S_p + B(0, r), \quad \forall n \in \mathbf{N}. \quad (3.21)$$

For each $n \in \mathbf{N}$, since $(x_n^*, y_n^*) \in \Omega_p(\delta_n, \varepsilon_n)$, we know that $(x_n^*, y_n^*) \in S(x_n^*, y_n^*) \times T(x_n^*, y_n^*)$, and there exist $p_n \in B(p, \delta_n)$ and $(x_n, y_n) \in S(x_n^*, y_n^*) \times T(x_n^*, y_n^*)$ such that

$$\hat{d}(x_n, x_n^*) \leq \hat{d}(x_n, S(x_n, y_n)), \quad \bar{d}(y_n, y_n^*) \leq \bar{d}(y_n, T(x_n, y_n)),$$

$$\begin{cases} 0 \in G(p_n, x_n^*, y_n, u) + f(p_n, x_n^*, y_n) - f(p_n, u, y_n) + B^+(0, \varepsilon_n)\hat{e}(x_n^*), & \forall u \in S(x_n^*, y_n^*), \\ 0 \in F(p_n, y_n^*, x_n, v) + g(p_n, x_n, y_n^*) - g(p_n, x_n, v) + B^+(0, \varepsilon_n)\bar{e}(y_n^*), & \forall v \in T(x_n^*, y_n^*). \end{cases}$$

Clearly $p_n \rightarrow p$ as $n \rightarrow \infty$. Hence $\{(x_n^*, y_n^*)\}$ is an approximating solution sequence for (SGMQVIP) $_p$ corresponding to $\{p_n\}$. Then, by the well-posedness in the generalized sense of (SGMQVIP), $\{(x_n^*, y_n^*)\}$ has a subsequence $\{(x_{n_k}^*, y_{n_k}^*)\}$ which converges to some point of S_p . This contradicts (3.21), and so (3.20) holds.

Conversely, suppose that condition (3.18) holds. Take a fixed $p \in P$ arbitrarily. Then, by Propositions 3.4 and 3.6, $\Omega_p(\delta, \varepsilon)$ is closed for $\delta, \varepsilon > 0$ small enough and $S_p = \bigcap_{\delta > 0, \varepsilon > 0} \Omega_p(\delta, \varepsilon)$. Since $\mu(\Omega_p(\delta, \varepsilon)) \rightarrow 0$ as $(\delta, \varepsilon) \rightarrow (0, 0)$, by the Kuratowski theorem [14], S_p is nonempty and compact and

$$e(\Omega_p(\delta, \varepsilon), S_p) \rightarrow 0 \quad \text{as} \quad (\delta, \varepsilon) \rightarrow (0, 0). \quad (3.22)$$

Let $\{p_n\}$ be any sequence in P with $p_n \rightarrow p$ as $n \rightarrow \infty$. If $\{(x_n^*, y_n^*)\}$ is an approximating solution sequence for (SGMQVIP) $_p$ corresponding to $\{p_n\}$, then for each $n \in \mathbf{N}$, $(x_n^*, y_n^*) \in S(x_n^*, y_n^*) \times T(x_n^*, y_n^*)$, and there exist $(x_n, y_n) \in S(x_n^*, y_n^*) \times T(x_n^*, y_n^*)$ and a sequence $\varepsilon_n > 0$ with $\varepsilon_n \rightarrow 0$ such that

$$\hat{d}(x_n, x_n^*) \leq \hat{d}(x_n, S(x_n, y_n)), \quad \bar{d}(y_n, y_n^*) \leq \bar{d}(y_n, T(x_n, y_n)),$$

$$\begin{cases} 0 \in G(p_n, x_n^*, y_n, u) + f(p_n, x_n^*, y_n) - f(p_n, u, y_n) + B^+(0, \varepsilon_n)\hat{e}(x_n^*), & \forall u \in S(x_n^*, y_n^*), \\ 0 \in F(p_n, y_n^*, x_n, v) + g(p_n, x_n, y_n^*) - g(p_n, x_n, v) + B^+(0, \varepsilon_n)\bar{e}(y_n^*), & \forall v \in T(x_n^*, y_n^*). \end{cases}$$

For each $n \in \mathbf{N}$, let $\delta_n = d_0(p_n, p)$. Then, $p_n \in B(p, \delta_n)$ and $(x_n^*, y_n^*) \in \Omega_p(\delta_n, \varepsilon_n)$ for each $n \in \mathbf{N}$, and $\delta_n \rightarrow 0$ as $n \rightarrow \infty$. It follows from (3.22) that $d((x_n^*, y_n^*), S_p) \leq e(\Omega_p(\delta_n, \varepsilon_n), S_p) \rightarrow 0$ as $n \rightarrow \infty$. Since S_p is compact, for each $n \in \mathbf{N}$, there exists $(x'_n, y'_n) \in S_p$ such that $d((x_n^*, y_n^*), (x'_n, y'_n)) = d((x_n^*, y_n^*), S_p) \rightarrow 0$ as $n \rightarrow \infty$. Again from the compactness of S_p , $\{(x'_n, y'_n)\}$ has a subsequence $\{(x'_{n_k}, y'_{n_k})\}$ which converges to a point $(x^*, y^*) \in S_p$. Hence, the corresponding subsequence $\{(x_{n_k}^*, y_{n_k}^*)\}$ of $\{(x_n^*, y_n^*)\}$ converges to (x^*, y^*) . Therefore, (SGMQVIP) is well-posed in the generalized sense. This completes the proof. \square

Utilizing the similar reasoning to that in the proof of Theorem 3.8, we have the following characterization by considering the Hausdorff distance between the solution set and the approximating solution set.

Theorem 3.9. Assume that all the conditions of Theorem 3.8 are satisfied. Then (SGMQVIP) is well-posed in the generalized sense if and only if for every $p \in P$, S_p is nonempty and compact and $e(\Omega_p(\delta, \varepsilon), S_p) \rightarrow 0$ as $(\delta, \varepsilon) \rightarrow (0, 0)$.

Proof. By Propositions 3.4 and 3.6, $\Omega_p(\delta, \varepsilon)$ is closed for every $\delta, \varepsilon > 0$ small enough, and

$$S_p = \bigcap_{\delta > 0, \varepsilon > 0} \Omega_p(\delta, \varepsilon).$$

For the remainder of the proof, using the similar reasoning to that in the proof of Theorem 3.8, we obtain the desired result. \square

Theorem 3.10. Let X and Y be finite-dimensional. Assume that

- (i) S and T are continuous and compact-valued;
- (ii) single-valued mappings f and g are continuous;
- (iii) multivalued mappings G and F are closed;
- (iv) for every $p \in P$, there exists a nonempty and bounded set $\Omega_p(\varepsilon, \varepsilon)$ such that for each convergent sequence $\{(x_n^*, y_n^*)\} \subseteq \Omega_p(\varepsilon, \varepsilon)$, the convergence of $(x_n, y_n) \in S(x_n^*, y_n^*) \times T(x_n^*, y_n^*)$, $n \in \mathbf{N}$ implies $\hat{d}(x_n, S(x_n, y_n)) \rightarrow 0$ and $\bar{d}(y_n, T(x_n, y_n)) \rightarrow 0$.

Then $(\text{SGMQVIP})_p$ is well-posed if and only if for every $p \in P$, $(\text{SGMQVIP})_p$ has a unique solution.

Proof. The necessity is obvious. For the sufficiency, suppose that $(\text{SGMQVIP})_p$ has a unique solution (x_p, y_p) for every $p \in P$. Take a fixed $p \in P$ arbitrarily. Let $\{p_n\}$ be any sequence in P with $p_n \rightarrow p$ as $n \rightarrow \infty$. If $\{(x_n^*, y_n^*)\}$ is an approximating solution sequence for $(\text{SGMQVIP})_p$ corresponding to $\{p_n\}$, then for each $n \in \mathbf{N}$, $(x_n^*, y_n^*) \in S(x_n^*, y_n^*) \times T(x_n^*, y_n^*)$, and there exist $(x_n, y_n) \in S(x_n^*, y_n^*) \times T(x_n^*, y_n^*)$ and a sequence $\varepsilon_n > 0$ with $\varepsilon_n \rightarrow 0$ such that

$$\hat{d}(x_n, x_n^*) \leq \hat{d}(x_n, S(x_n, y_n)), \quad \bar{d}(y_n, y_n^*) \leq \bar{d}(y_n, T(x_n, y_n)),$$

$$\begin{cases} 0 \in G(p_n, x_n^*, y_n, u) + f(p_n, x_n^*, y_n) - f(p_n, u, y_n) + B^+(0, \varepsilon_n)\hat{e}(x_n^*), & \forall u \in S(x_n^*, y_n^*), \\ 0 \in F(p_n, y_n^*, x_n, v) + g(p_n, x_n, y_n^*) - g(p_n, x_n, v) + B^+(0, \varepsilon_n)\bar{e}(y_n^*), & \forall v \in T(x_n^*, y_n^*). \end{cases}$$

For each $n \in \mathbf{N}$, let $\delta_n = d_0(p_n, p)$. Then, $p_n \in B(p, \delta_n)$ and $(x_n^*, y_n^*) \in \Omega_p(\delta_n, \varepsilon_n)$ for each $n \in \mathbf{N}$, and $\delta_n \rightarrow 0$ as $n \rightarrow \infty$. Let $\varepsilon > 0$ be such that $\Omega_p(\varepsilon, \varepsilon)$ is a nonempty and bounded set satisfying condition (iv). Then there exists some $n_0 \in \mathbf{N}$ such that $(x_n^*, y_n^*) \in \Omega_p(\delta_n, \varepsilon_n) \subseteq \Omega_p(\varepsilon, \varepsilon)$ for all $n \geq n_0$. Thus $\{(x_n^*, y_n^*)\}$ is bounded. Let $\{(x_{n_k}^*, y_{n_k}^*)\}$ be any subsequence of $\{(x_n^*, y_n^*)\}$ such that $(x_{n_k}^*, y_{n_k}^*) \rightarrow (x^*, y^*)$ as $k \rightarrow \infty$. Then, by the similar arguments to those in the second part of the proof of Proposition 3.5, we can show that $(x^*, y^*) \in S(x^*, y^*) \times T(x^*, y^*)$ and

$$\begin{cases} 0 \in G(p, x^*, y^*, u) + f(p, x^*, y^*) - f(p, u, y^*), & \forall u \in S(x^*, y^*), \\ 0 \in F(p, y^*, x^*, v) + g(p, x^*, y^*) - g(p, x^*, v), & \forall v \in T(x^*, y^*). \end{cases}$$

Thus (x^*, y^*) is a solution of $(\text{SGMQVIP})_p$. By the solution uniqueness to $(\text{SGMQVIP})_p$, we have $(x^*, y^*) = (x_p, y_p)$. Thus, the whole sequence $\{(x_n^*, y_n^*)\}$ converges to (x_p, y_p) and so (SGMQVIP) is well-posed. \square

Theorem 3.11. *Let X and Y be finite-dimensional. Assume that*

- (i) S and T are continuous and compact-valued;
- (ii) single-valued mappings f and g are continuous;
- (iii) multivalued mappings G and F are closed;
- (iv) for every $p \in P$, there exists a nonempty and bounded set $\Omega_p(\varepsilon, \varepsilon)$ such that for each convergent sequence $\{(x_n^*, y_n^*)\} \subseteq \Omega_p(\varepsilon, \varepsilon)$, the convergence of $(x_n, y_n) \in S(x_n^*, y_n^*) \times T(x_n^*, y_n^*)$, $n \in \mathbf{N}$ implies $\hat{d}(x_n, S(x_n, y_n)) \rightarrow 0$ and $\bar{d}(y_n, T(x_n, y_n)) \rightarrow 0$.

Then $(\text{SGMQVIP})_p$ is well-posed in the generalized sense if and only if for every $p \in P$, $(\text{SGMQVIP})_p$ has a nonempty solution set S_p .

Proof. The necessity is obvious. For the sufficiency, suppose that $(\text{SGMQVIP})_p$ has a nonempty solution set S_p for every $p \in P$. Take a fixed $p \in P$ arbitrarily. Let $\{p_n\}$ be any sequence in P with $p_n \rightarrow p$ as $n \rightarrow \infty$. If $\{(x_n^*, y_n^*)\}$ is an approximating solution sequence for $(\text{SGMQVIP})_p$ corresponding to $\{p_n\}$, then for each $n \in \mathbf{N}$, $(x_n^*, y_n^*) \in S(x_n^*, y_n^*) \times T(x_n^*, y_n^*)$, and there exist $(x_n, y_n) \in S(x_n^*, y_n^*) \times T(x_n^*, y_n^*)$ and a sequence $\varepsilon_n > 0$ with $\varepsilon_n \rightarrow 0$ such that

$$\begin{aligned} \hat{d}(x_n, x_n^*) &\leq \hat{d}(x_n, S(x_n, y_n)), \quad \bar{d}(y_n, y_n^*) \leq \bar{d}(y_n, T(x_n, y_n)), \\ \begin{cases} 0 \in G(p_n, x_n^*, y_n, u) + f(p_n, x_n^*, y_n) - f(p_n, u, y_n) + B^+(0, \varepsilon_n)\hat{e}(x_n^*), & \forall u \in S(x_n^*, y_n^*), \\ 0 \in F(p_n, y_n^*, x_n, v) + g(p_n, x_n, y_n^*) - g(p_n, x_n, v) + B^+(0, \varepsilon_n)\bar{e}(y_n^*), & \forall v \in T(x_n^*, y_n^*). \end{cases} \end{aligned}$$

For each $n \in \mathbf{N}$, let $\delta_n = d_0(p_n, p)$. Then, $p_n \in B(p, \delta_n)$ and $(x_n^*, y_n^*) \in \Omega_p(\delta_n, \varepsilon_n)$ for each $n \in \mathbf{N}$, and $\delta_n \rightarrow 0$ as $n \rightarrow \infty$. Let $\varepsilon > 0$ be such that $\Omega_p(\varepsilon, \varepsilon)$ is a nonempty and bounded set satisfying condition (iv). Then there exists some $n_0 \in \mathbf{N}$ such that $(x_n^*, y_n^*) \in \Omega_p(\delta_n, \varepsilon_n) \subseteq \Omega_p(\varepsilon, \varepsilon)$, for all $n \geq n_0$. Thus $\{(x_n^*, y_n^*)\}$ is bounded, and so there exists a subsequence $\{(x_{n_k}^*, y_{n_k}^*)\}$ of $\{(x_n^*, y_n^*)\}$ such that $(x_{n_k}^*, y_{n_k}^*) \rightarrow (x^*, y^*)$ as $k \rightarrow \infty$. Then, by the similar arguments to those in the second part of the proof of Proposition 3.5, we can show that $(x^*, y^*) \in S(x^*, y^*) \times T(x^*, y^*)$ and

$$\begin{cases} 0 \in G(p, x^*, y^*, u) + f(p, x^*, y^*) - f(p, u, y^*), & \forall u \in S(x^*, y^*), \\ 0 \in F(p, y^*, x^*, v) + g(p, x^*, y^*) - g(p, x^*, v), & \forall v \in T(x^*, y^*). \end{cases}$$

Therefore (x^*, y^*) is a solution of $(\text{SGMQVIP})_p$ and so $(\text{SGMQVIP})_p$ is well-posed in the generalized sense. This completes the proof. \square

If we set $f = 0$ and $g = 0$, then (SGMQVIP) reduces to (SGQVIP) . For each $p \in P$, the approximating solution set for $(\text{SGQVIP})_p$ is defined by

$$\begin{aligned} \Omega_p(\delta, \varepsilon) = \bigcup_{p' \in B(p, \delta)} \bigg\{ (x^*, y^*) \in C \times D : (x^*, y^*) \in S(x^*, y^*) \times T(x^*, y^*), \text{ and} \\ \exists (x, y) \in S(x^*, y^*) \times T(x^*, y^*) \text{ s.t. } \hat{d}(x, x^*) \leq \hat{d}(x, S(x, y)), \quad \bar{d}(y, y^*) \leq \bar{d}(y, T(x, y)), \\ 0 \in G(p', x^*, y, u) + B^+(0, \varepsilon)\hat{e}(x^*), \quad \forall u \in S(x^*, y^*), \\ 0 \in F(p', y^*, x, v) + B^+(0, \varepsilon)\bar{e}(y^*), \quad \forall v \in T(x^*, y^*) \bigg\}, \end{aligned}$$

for all $\delta, \varepsilon > 0$. Thus, by Theorems 3.7–3.11, we can obtain the following results of well-posedness for (SGQVIP) .

Theorem 3.12. Let X and Y be complete. Assume that

- (i) S and T are continuous and compact-valued;
- (ii) multivalued mappings G and F are closed;
- (iii) for every $p \in P$, there exists $\Omega_p(\delta, \varepsilon)$ such that for each convergent sequence $\{(x_n^*, y_n^*)\} \subseteq \Omega_p(\delta, \varepsilon)$,

$$\text{the convergence of } (x_n, y_n) \in S(x_n^*, y_n^*) \times T(x_n^*, y_n^*), n \in \mathbf{N} \Rightarrow \begin{cases} \hat{d}(x_n, S(x_n, y_n)) \rightarrow 0, \\ \bar{d}(y_n, T(x_n, y_n)) \rightarrow 0. \end{cases}$$

Then (SGQVIP) is well-posed if and only if for every $p \in P$, $\Omega_p(\delta, \varepsilon) \neq \emptyset$ for all $\delta, \varepsilon > 0$, and $\text{diam}(\Omega_p(\delta, \varepsilon)) \rightarrow 0$ as $(\delta, \varepsilon) \rightarrow (0, 0)$.

Theorem 3.13. Let X and Y be complete. Assume that

- (i) P is finite-dimensional;
- (ii) S and T are continuous and compact-valued;
- (iii) multivalued mappings G and F are closed;
- (iv) for every $p \in P$, there exists $\Omega_p(\delta, \varepsilon)$ such that for each convergent sequence $\{(x_n^*, y_n^*)\} \subseteq \Omega_p(\delta, \varepsilon)$,

$$\text{the convergence of } (x_n, y_n) \in S(x_n^*, y_n^*) \times T(x_n^*, y_n^*), n \in \mathbf{N} \Rightarrow \begin{cases} \hat{d}(x_n, S(x_n, y_n)) \rightarrow 0, \\ \bar{d}(y_n, T(x_n, y_n)) \rightarrow 0. \end{cases}$$

Then (SGQVIP) is well-posed in the generalized sense if and only if for every $p \in P$, $\Omega_p(\delta, \varepsilon) \neq \emptyset$ for all $\delta, \varepsilon > 0$, and $\mu(\Omega_p(\delta, \varepsilon)) \rightarrow 0$ as $(\delta, \varepsilon) \rightarrow (0, 0)$.

Theorem 3.14. Assume that all the conditions of Theorem 3.13 are satisfied. Then (SGQVIP) is well-posed in the generalized sense if and only if for every $p \in P$, S_p is nonempty and compact and $e(\Omega_p(\delta, \varepsilon), S_p) \rightarrow 0$ as $(\delta, \varepsilon) \rightarrow (0, 0)$.

Theorem 3.15. Let X and Y be finite-dimensional. Assume that

- (i) S and T are continuous and compact-valued;
- (ii) multivalued mappings G and F are closed;
- (iii) for every $p \in P$, there exists a nonempty and bounded set $\Omega_p(\varepsilon, \varepsilon)$ such that for each convergent sequence $\{(x_n^*, y_n^*)\} \subseteq \Omega_p(\varepsilon, \varepsilon)$, the convergence of $(x_n, y_n) \in S(x_n^*, y_n^*) \times T(x_n^*, y_n^*), n \in \mathbf{N}$ implies $\hat{d}(x_n, S(x_n, y_n)) \rightarrow 0$ and $\bar{d}(y_n, T(x_n, y_n)) \rightarrow 0$.

Then $(\text{SGQVIP})_p$ is well-posed if and only if for every $p \in P$, $(\text{SGQVIP})_p$ has a unique solution.

Theorem 3.16. Let X and Y be finite-dimensional. Assume that

- (i) S and T are continuous and compact-valued;
- (ii) multivalued mappings G and F are closed;
- (iii) for every $p \in P$, there exists a nonempty and bounded set $\Omega_p(\varepsilon, \varepsilon)$ such that for each convergent sequence $\{(x_n^*, y_n^*)\} \subseteq \Omega_p(\varepsilon, \varepsilon)$, the convergence of $(x_n, y_n) \in S(x_n^*, y_n^*) \times T(x_n^*, y_n^*), n \in \mathbf{N}$ implies $\hat{d}(x_n, S(x_n, y_n)) \rightarrow 0$ and $\bar{d}(y_n, T(x_n, y_n)) \rightarrow 0$.

Then $(\text{SGQVIP})_p$ is well-posed in the generalized sense if and only if for every $p \in P$, $(\text{SGQVIP})_p$ has a nonempty solution set S_p .

Remark 3.17.

- (a) Theorems 3.11–3.16 generalize and extend [26, Theorems 3.1–3.5] from (GQVIP) to (SGQVIP), respectively.
- (b) Compared with [26, Theorem 3.3], our Theorem 3.14 is its corresponding one, which shows that if (SGQVIP) is well-posed in the generalized sense, then S_p is compact for every $p \in P$, but such a corresponding conclusion cannot be concluded in [17].

4. Well-posedness for optimization problems with constraints

In this section, we study the well-posedness of optimization problems with systems of generalized mixed quasivariational inclusion problems as constraints.

Let \mathcal{P}, X, Y and W be normed spaces, and $K \subseteq W$ be a pointed, closed and convex cone with $\text{int}K \neq \emptyset$. Let $H : P \times C \times D \rightarrow 2^W$ be a multivalued mapping with nonempty values. We consider the following optimization problem with the system of generalized mixed quasivariational inclusion problems as constraint:

$$(\text{OPSGMQVIP}) : \min H(p, x, y) \text{ subject to } (x, y) \in S_p.$$

Let $M = \{(p, x, y) \in P \times C \times D : (x, y) \in S_p\}$. Suppose that M is nonempty closed subset of $P \times C \times D$. A point $w \in H(M)$ is called a minimal point of $H(M)$ if $H(M) \cap (w - K \setminus \{0\}) = \emptyset$. A point $(p, x, y) \in M$ is called an efficient solution of (OPSGMQVIP) if there exists $w \in H(p, x, y)$ such that w is a minimal point of $H(M)$.

For each $w, q \in W$ and each $\delta, \varepsilon > 0$, let

$$Q(w) := \{(p, x, y) \in P \times C \times D : (x, y) \in S_p, w \in H(p, x, y) \text{ and } H(M) \cap (w - K \setminus \{0\}) = \emptyset\},$$

and $L(w, q, \delta, \varepsilon) := A(w, q, \delta) \cap D(\varepsilon)$, where

$$A(w, q, \delta) := \{(p, x, y) \in P \times C \times D : H(p, x, y) \cap (w + \delta q - K) \neq \emptyset\},$$

and

$$D(\varepsilon) := \left\{ (p, x^*, y^*) \in P \times C \times D : (x^*, y^*) \in S(x^*, y^*) \times T(x^*, y^*), \text{ and } \exists (x, y) \in S(x^*, y^*) \times T(x^*, y^*) \text{ s.t.} \right. \\ \hat{d}(x, x^*) \leq \hat{d}(x, S(x, y)), \bar{d}(y, y^*) \leq \bar{d}(y, T(x, y)), \\ 0 \in G(p, x^*, y, u) + f(p, x^*, y) - f(p, u, y) + B^+(0, \varepsilon)\hat{e}(x^*), \forall u \in S(x^*, y^*), \\ \left. 0 \in F(p, y^*, x, v) + g(p, x, y^*) - g(p, x, v) + B^+(0, \varepsilon)\bar{e}(y^*), \forall v \in T(x^*, y^*) \right\}.$$

Clearly, for any $(w, q) \in W \times \text{int}K$, $\delta_1, \delta_2 > 0$ with $\delta_1 \leq \delta_2$ and $\varepsilon_1, \varepsilon_2 > 0$ with $\varepsilon_1 \leq \varepsilon_2$, we have

- (i) $A(w, q, \delta_1) \subseteq A(w, q, \delta_2)$;
- (ii) $M \subseteq D(\varepsilon_1) \subseteq D(\varepsilon_2)$;
- (iii) $L(w, q, \delta_1, \varepsilon_1) \subseteq L(w, q, \delta_2, \varepsilon_2)$.

Definition 4.1. Let $\{k_n\} \subseteq W$ be a sequence with $k_n \rightarrow 0$ and (p, x, y) be an efficient solution of (OPSGMQVIP). A sequence $\{(p_n, x_n, y_n)\} \subseteq P \times C \times D$ is said to be an approximating solution sequence for (OPSGMQVIP) corresponding to $\{k_n\}$ at (p, x, y) if

- (i) there exists $w \in H(p, x, y)$ with w being a minimal point of $H(M)$ such that

$$H(p_n, x_n, y_n) \cap (w + k_n - K) \neq \emptyset, \quad \forall n \in \mathbf{N};$$

- (ii) there exists a sequence $\{\varepsilon_n\}$ of positive real numbers with $\varepsilon_n \rightarrow 0$ such that $(p_n, x_n, y_n) \in D(\varepsilon_n)$ for all $n \in \mathbf{N}$.

Definition 4.2. Let $q \in \text{int}K$, $\{k_n\} \subseteq W$ be any sequence with $k_n \rightarrow 0$, and (p, x, y) be an efficient solution of (OPSGMQVIP). Then (OPSGMQVIP) is said to be well-posed at (p, x, y) if every approximating solution sequence for (OPSGMQVIP) corresponding to $\{k_n\}$ at (p, x, y) converges to (p, x, y) .

Remark 4.3.

- (a) When $(\text{SGMQVIP})_p$ reduces to $(\text{SGQVIP})_p$, (OPSGMQVIP) reduces to (OPSGQVIP) and therefore, by Definitions 4.1 and 4.2, we have definitions of approximating solution sequence and well-posedness for (OPSGQVIP).

- (b) The definition of approximating solution sequence for (OPSGQVIP) generalizes and extends that of Wang et al. [26].
- (c) The definition of approximating solution sequence for (OPGQVIP) (including (OPVIP) as a special case) generalizes and extends that of Lin and Chuang [17].

Lemma 4.4 ([7]). *Let $\{k_n\} \subseteq W$ be any sequence with $k_n \rightarrow 0$. Then, for every $q \in \text{int}K$, there exists a sequence $\{\delta_n\}$ of positive real numbers with $\delta_n \rightarrow 0$ such that $\delta_n q - k_n \in \text{int}K$ for all $n \in \mathbf{N}$.*

Proposition 4.5. *Let $q \in \text{int}K$. If $(p, x^*, y^*) \in M$ is an efficient solution of (OPSGMQVIP) and $w \in H(p, x^*, y^*)$ is a minimal point of $H(M)$, then $(p, x^*, y^*) \in L(w, q, \delta, \varepsilon)$ for all $\delta, \varepsilon > 0$.*

Proof. Let $(p, x^*, y^*) \in M$ be an efficient solution of (OPSGMQVIP) and $w \in H(p, x^*, y^*)$ be a minimal point of $H(M)$. Then $(x^*, y^*) \in S_p$ and $H(M) \cap (w - K \setminus \{0\}) = \emptyset$. It follows that

$$(x^*, y^*) \in S(x^*, y^*) \times T(x^*, y^*),$$

and there exists $(x, y) = (x^*, y^*) \in S(x^*, y^*) \times T(x^*, y^*)$ such that

$$\hat{d}(x, x^*) \leq \hat{d}(x, S(x, y)), \quad \bar{d}(y, y^*) \leq \bar{d}(y, T(x, y)),$$

$$\begin{cases} 0 \in G(p, x^*, y, u) + f(p, x^*, y) - f(p, u, y) + B^+(0, \varepsilon)\hat{e}(x^*), & \forall u \in S(x^*, y^*), \\ 0 \in F(p, y^*, x, v) + g(p, x, y^*) - g(p, x, v) + B^+(0, \varepsilon)\bar{e}(y^*), & \forall v \in T(x^*, y^*), \end{cases}$$

for any $\varepsilon > 0$. Moreover, it is also clear that

$$H(p, x^*, y^*) \cap (w - K) = \{w\}. \quad (4.1)$$

Hence $(p, x^*, y^*) \in D(\varepsilon)$. Taking into account $q \in \text{int}K$, we have $-K \subseteq \delta q - K$ for any $\delta > 0$. It follows from (4.1) that $w \in H(p, x^*, y^*) \cap (w - K) \subseteq H(p, x^*, y^*) \cap (w + \delta q - K)$. This implies that

$$H(p, x^*, y^*) \cap (w + \delta q - K) \neq \emptyset.$$

Thus, $(p, x^*, y^*) \in A(w, q, \delta)$ and so $(p, x^*, y^*) \in L(w, q, \delta, \varepsilon)$. This completes the proof. \square

Theorem 4.6. *Let $q \in \text{int}K$ and $(p, x^*, y^*) \in M$ be an efficient solution of (OPSGMQVIP). Then (OPSGMQVIP) is well-posed at (p, x^*, y^*) if and only if for every $w \in H(p, x^*, y^*)$ with w being a minimal point of $H(M)$,*

$$\text{diam}(L(w, q, \delta, \varepsilon)) \rightarrow 0 \text{ as } (\delta, \varepsilon) \rightarrow (0, 0). \quad (4.2)$$

Proof. Suppose that (OPSGMQVIP) is well-posed at (p, x^*, y^*) . Take any $w \in H(p, x^*, y^*)$ with w being a minimal point of $H(M)$ and let w be fixed. If $\text{diam}(L(w, q, \delta, \varepsilon)) \not\rightarrow 0$ as $(\delta, \varepsilon) \rightarrow (0, 0)$, then there exist a positive number r and two sequences $\{\delta_n\}$ and $\{\varepsilon_n\}$ of positive real numbers and two sequences $\{(p_n^1, x_n^{1*}, y_n^{1*})\}$ and $\{(p_n^2, x_n^{2*}, y_n^{2*})\}$ in $P \times C \times D$ such that

- (a) $(\delta_n, \varepsilon_n) \rightarrow (0, 0)$ as $n \rightarrow \infty$;
 (b) for each $n \in \mathbf{N}$, $(p_n^1, x_n^{1*}, y_n^{1*}), (p_n^2, x_n^{2*}, y_n^{2*}) \in L(w, q, \delta_n, \varepsilon_n)$ and $\|(p_n^1, x_n^{1*}, y_n^{1*}) - (p_n^2, x_n^{2*}, y_n^{2*})\| \geq r$;
 (c) $H(p_n^1, x_n^{1*}, y_n^{1*}) \cap (w + \delta_n q - K) \neq \emptyset$ and $H(p_n^2, x_n^{2*}, y_n^{2*}) \cap (w + \delta_n q - K) \neq \emptyset$;
 (d) $(p_n^1, x_n^{1*}, y_n^{1*}), (p_n^2, x_n^{2*}, y_n^{2*}) \in D(\varepsilon_n)$.

By (a), $\delta_n q \rightarrow 0$ as $n \rightarrow \infty$. Then, by (c) and (d), $\{(p_n^1, x_n^{1*}, y_n^{1*})\}$ and $\{(p_n^2, x_n^{2*}, y_n^{2*})\}$ are both approximating solution sequences for (OPSGMQVIP) corresponding to $\{\delta_n q\}$ at (p, x^*, y^*) . Since (OPSGMQVIP) is well-posed at (p, x^*, y^*) , $(p_n^1, x_n^{1*}, y_n^{1*}) \rightarrow (p, x^*, y^*)$ and $(p_n^2, x_n^{2*}, y_n^{2*}) \rightarrow (p, x^*, y^*)$ as $n \rightarrow \infty$. This leads to a contradiction. Thus $\text{diam}(L(w, q, \delta, \varepsilon)) \rightarrow 0$ as $(\delta, \varepsilon) \rightarrow (0, 0)$.

Conversely, suppose that condition (4.2) holds. Let $\{k_n\} \subseteq W$ be any sequence with $k_n \rightarrow 0$. If $\{(p_n, x_n^*, y_n^*)\}$ is an approximating solution sequence for (OPSGMQVIP) corresponding to $\{k_n\}$ at (p, x^*, y^*) , then

(a) there exists $w \in H(p, x^*, y^*)$ with w being a minimal point of $H(M)$ such that

$$H(p_n, x_n^*, y_n^*) \cap (w + k_n - K) \neq \emptyset, \quad \forall n \in \mathbf{N};$$

(b) there exists a sequence $\{\varepsilon_n\}$ of positive real numbers with $\varepsilon_n \rightarrow 0$ such that $(p_n, x_n^*, y_n^*) \in D(\varepsilon_n)$ for all $n \in \mathbf{N}$.

By (a), for each $n \in \mathbf{N}$, there exists $w_n \in H(p_n, x_n^*, y_n^*)$ such that $w_n \in w + k_n - K$. Note that $\{k_n\} \subseteq W$ and $k_n \rightarrow 0$. By Lemma 4.4, there exists a sequence $\{\delta_n\}$ of positive real numbers with $\delta_n \rightarrow 0$ such that $k_n \in \delta_n q - \text{int}K$. It follows that

$$w_n \in w + k_n - K \subseteq w + \delta_n q - \text{int}K - K \subseteq w + \delta_n q - \text{int}K.$$

Hence $w_n \in H(p_n, x_n^*, y_n^*) \cap (w + \delta_n q - \text{int}K)$. This together with (b) implies that

$$(p_n, x_n^*, y_n^*) \in L(w, q, \delta_n, \varepsilon_n).$$

Moreover, for each $n \in \mathbf{N}$, $(p, x^*, y^*) \in M \subseteq D(\varepsilon_n)$ and $w \in H(p, x^*, y^*) \cap (w + \delta_n q - K)$. Hence, $(p, x^*, y^*) \in L(w, q, \delta_n, \varepsilon_n)$ for every $n \in \mathbf{N}$. Then, we have

$$\|(p_n, x_n^*, y_n^*) - (p, x^*, y^*)\| \leq \text{diam}(L(w, q, \delta_n, \varepsilon_n)), \quad \forall n \in \mathbf{N}.$$

By the assumption, we get $(p_n, x_n^*, y_n^*) \rightarrow (p, x^*, y^*)$ and so (OPSGMQVIP) is well-posed at (p, x^*, y^*) . This completes the proof. \square

Theorem 4.7. Let P, C and D be finite-dimensional. Let $q \in \text{int}K$ and let $(p, x^*, y^*) \in M$ be an efficient solution of (OPSGMQVIP). Assume that

- (i) H is u.s.c. and compact-valued and S, T are continuous and compact-valued;
- (ii) single-valued mappings f and g are continuous;
- (iii) multivalued mappings G and F are closed;
- (iv) for every $w \in H(p, x^*, y^*)$ with w being a minimal point of $H(M)$, there exists a nonempty and bounded set $L(w, q, \varepsilon, \varepsilon)$ such that for each convergent sequence $\{(p_n, x_n^*, y_n^*)\} \subseteq L(w, q, \varepsilon, \varepsilon)$,

$$\text{the convergence of } (x_n, y_n) \in S(x_n^*, y_n^*) \times T(x_n^*, y_n^*), n \in \mathbf{N} \Rightarrow \begin{cases} \hat{d}(x_n, S(x_n, y_n)) \rightarrow 0, \\ \bar{d}(y_n, T(x_n, y_n)) \rightarrow 0. \end{cases}$$

Then (OPSGMQVIP) is well-posed at (p, x^*, y^*) if and only if for every $w \in H(p, x^*, y^*)$ with w being a minimal point of $H(M)$, $Q(w) = \{(p, x^*, y^*)\}$.

Proof. Suppose that (OPSGMQVIP) is well-posed at (p, x^*, y^*) . Take any $w \in H(p, x^*, y^*)$ with w being a minimal point of $H(M)$ and let w be fixed. Clearly, $(p, x^*, y^*) \in Q(w)$. Moreover, if $(p', x', y') \in Q(w)$, then $(x', y') \in S_{p'}$, $w \in H(p', x', y')$ and $H(M) \cap (w - K \setminus \{0\}) = \emptyset$. It follows that

$$(p', x', y') \in M \quad \text{and} \quad H(p', x', y') \cap (w - K) = \{w\}.$$

For each $n \in \mathbf{N}$, let $k_n = 0$, $p_n = p'$, $x_n^* = x'$ and $y_n^* = y'$. Then $k_n \rightarrow 0$ and $\{(p_n, x_n^*, y_n^*)\}$ is an approximating solution sequence for (OPSGMQVIP) corresponding to $\{k_n\}$ at (p, x^*, y^*) . Since (OPSGMQVIP) is well-posed at (p, x^*, y^*) , $(p_n, x_n^*, y_n^*) \rightarrow (p, x^*, y^*)$ as $n \rightarrow \infty$. This implies that $(p', x', y') = (p, x^*, y^*)$ and so $Q(w) = \{(p, x^*, y^*)\}$.

Conversely, suppose that, for every $w \in H(p, x^*, y^*)$ with w being a minimal point of $H(M)$, $Q(w) = \{(p, x^*, y^*)\}$. Let $\{k_n\} \subseteq W$ be any sequence with $k_n \rightarrow 0$. If $\{(p_n, x_n^*, y_n^*)\} \subseteq P \times C \times D$ is an approximating solution sequence for (OPSGMQVIP) corresponding to $\{k_n\}$ at (p, x^*, y^*) , then

(a) there exists $w \in H(p, x^*, y^*)$ with w being a minimal point of $H(M)$ such that

$$H(p_n, x_n^*, y_n^*) \cap (w + k_n - K) \neq \emptyset, \quad \forall n \in \mathbf{N};$$

(b) there exists a sequence $\{\varepsilon_n\}$ of positive real numbers with $\varepsilon_n \rightarrow 0$ such that $(p_n, x_n^*, y_n^*) \in D(\varepsilon_n)$ for all $n \in \mathbf{N}$.

By (a), for each $n \in \mathbf{N}$, there exists $z_n \in H(p_n, x_n^*, y_n^*)$ such that $z_n \in w + k_n - K$. Note that $\{k_n\} \subseteq W$ and $k_n \rightarrow 0$. By Lemma 4.4, there exists a sequence $\{\delta_n\}$ of positive real numbers with $\delta_n \rightarrow 0$ such that $k_n \in \delta_n q - \text{int}K$. Hence

$$z_n \in w + k_n - K \subseteq w + \delta_n q - \text{int}K - K \subseteq w + \delta_n q - \text{int}K.$$

It follows that

$$z_n \in H(p_n, x_n^*, y_n^*) \cap (w + \delta_n q - \text{int}K). \quad (4.3)$$

This together with (b) implies that $(p_n, x_n^*, y_n^*) \in L(w, q, \delta_n, \varepsilon_n)$. Let $L(w, q, \varepsilon, \varepsilon)$ be a nonempty and bounded set satisfying condition (iv). Then there exists some $n_0 \in \mathbf{N}$ such that

$$(p_n, x_n^*, y_n^*) \in L(w, q, \delta_n, \varepsilon_n) \subseteq L(w, q, \varepsilon, \varepsilon), \quad \forall n \geq n_0,$$

and so $\{(p_n, x_n^*, y_n^*)\}$ is bounded. Also, since for each $n \in \mathbf{N}$, $(p_n, x_n^*, y_n^*) \in L(w, q, \delta_n, \varepsilon_n) \subseteq D(\varepsilon_n)$, it follows that $(x_n, y_n) \in S(x_n^*, y_n^*) \times T(x_n^*, y_n^*)$, and there exists $(x_n, y_n) \in S(x_n^*, y_n^*) \times T(x_n^*, y_n^*)$ such that

$$\hat{d}(x_n, x_n^*) \leq \hat{d}(x_n, S(x_n, y_n)), \quad \bar{d}(y_n, y_n^*) \leq \bar{d}(y_n, T(x_n, y_n)),$$

$$\begin{cases} 0 \in G(p_n, x_n^*, y_n, u) + f(p_n, x_n^*, y_n) - f(p_n, u, y_n) + B^+(0, \varepsilon_n)\hat{e}(x_n^*), & \forall u \in S(x_n^*, y_n^*), \\ 0 \in F(p_n, y_n^*, x_n, v) + g(p_n, x_n, y_n^*) - g(p_n, x_n, v) + B^+(0, \varepsilon_n)\bar{e}(y_n^*), & \forall v \in T(x_n^*, y_n^*). \end{cases}$$

Let $\{(p_{n_k}, x_{n_k}^*, y_{n_k}^*)\}$ be any subsequence of $\{(p_n, x_n^*, y_n^*)\}$ such that $(p_{n_k}, x_{n_k}^*, y_{n_k}^*) \rightarrow (\tilde{p}, \tilde{x}, \tilde{y})$ as $k \rightarrow \infty$ for some $(\tilde{p}, \tilde{x}, \tilde{y}) \in P \times C \times D$. Then, by the similar arguments to those in the second part of the proof of Proposition 3.5, we can show that $(\tilde{x}, \tilde{y}) \in S(\tilde{x}, \tilde{y}) \times T(\tilde{x}, \tilde{y})$ such that

$$\begin{cases} 0 \in G(\tilde{p}, \tilde{x}, \tilde{y}, u) + f(\tilde{p}, \tilde{x}, \tilde{y}) - f(\tilde{p}, u, \tilde{y}), & \forall u \in S(\tilde{x}, \tilde{y}), \\ 0 \in F(\tilde{p}, \tilde{y}, \tilde{x}, v) + g(\tilde{p}, \tilde{x}, \tilde{y}) - g(\tilde{p}, \tilde{x}, v), & \forall v \in T(\tilde{x}, \tilde{y}). \end{cases}$$

It follows that $(\tilde{x}, \tilde{y}) \in S_{\tilde{p}}$, i.e., $(\tilde{p}, \tilde{x}, \tilde{y}) \in M$. Moreover, by (4.3), we have $z_{n_k} \in H(p_{n_k}, x_{n_k}^*, y_{n_k}^*)$ and $z_{n_k} - w - \delta_{n_k} q \in -K$. Since H is u.s.c. and compact-valued, we may assume that $z_{n_k} \rightarrow \tilde{z} \in H(\tilde{p}, \tilde{x}, \tilde{y})$. Clearly $\delta_{n_k} q \rightarrow 0$ as $k \rightarrow \infty$. Then, we have $\tilde{z} - w \in -K$, i.e., $\tilde{z} \in w - K$. This implies that $\tilde{z} \in H(\tilde{p}, \tilde{x}, \tilde{y}) \cap (w - K)$. Noting that $w \in H(p, x^*, y^*)$ is a minimal point of $H(M)$, we get

$$\tilde{z} \in H(\tilde{p}, \tilde{x}, \tilde{y}) \cap (w - K) \subseteq H(M) \cap (w - K) = \{w\}.$$

It follows that $\tilde{z} = w$, and so $H(\tilde{p}, \tilde{x}, \tilde{y}) \cap (w - K) = H(M) \cap (w - K) = \{w\}$. Hence,

$$w \in H(\tilde{p}, \tilde{x}, \tilde{y}) \quad \text{and} \quad H(M) \cap (w - K \setminus \{0\}) = \emptyset.$$

This implies that $(\tilde{p}, \tilde{x}, \tilde{y}) \in Q(w) = \{(p, x^*, y^*)\}$. Thus, $(\tilde{p}, \tilde{x}, \tilde{y}) = (p, x^*, y^*)$, and so the whole sequence $\{(p_n, x_n^*, y_n^*)\}$ converges to (p, x^*, y^*) . Therefore (OPSGMQVIP) is well-posed at (p, x^*, y^*) . This completes the proof. \square

If we set $f = 0$ and $g = 0$, then (SGMQVIP) reduces to (SGQVIP). For each $p \in P$, by S_p we denote the solution set of (SGQVIP). In this case, for all $\varepsilon > 0$ we have

$$D(\varepsilon) := \left\{ (p, x^*, y^*) \in P \times C \times D : (x^*, y^*) \in S(x^*, y^*) \times T(x^*, y^*), \text{ and } \exists (x, y) \in S(x^*, y^*) \times T(x^*, y^*) \text{ s.t.} \right. \\ \hat{d}(x, x^*) \leq \hat{d}(x, S(x, y)), \bar{d}(y, y^*) \leq \bar{d}(y, T(x, y)), \\ 0 \in G(p, x^*, y, u) + B^+(0, \varepsilon)\hat{e}(x^*), \forall u \in S(x^*, y^*), \\ \left. 0 \in F(p, y^*, x, v) + B^+(0, \varepsilon)\bar{e}(y^*), \forall v \in T(x^*, y^*) \right\}.$$

Thus, by Theorems 4.6 and 4.7, we can obtain the following results of well-posedness for (OPSGQVIP).

Theorem 4.8. *Let $q \in \text{int}K$ and let $(p, x^*, y^*) \in M$ be an efficient solution of (OPSGQVIP). Then (OPSGQVIP) is well-posed at (p, x^*, y^*) if and only if for every $w \in H(p, x^*, y^*)$ with w being a minimal point of $H(M)$,*

$$\text{diam}(L(w, q, \delta, \varepsilon)) \rightarrow 0 \text{ as } (\delta, \varepsilon) \rightarrow (0, 0).$$

Theorem 4.9. *Let $q \in \text{int}K$ and let $(p, x^*, y^*) \in M$ be an efficient solution of (OPSGQVIP). Assume that*

- (i) P, C and D are finite-dimensional;
- (ii) H is u.s.c. and compact-valued and S, T are continuous and compact-valued;
- (iii) multivalued mappings G and F are closed;
- (iv) for every $w \in H(p, x^*, y^*)$ with w being a minimal point of $H(M)$, there exists a nonempty and bounded set $L(w, q, \varepsilon, \varepsilon)$ such that for each convergent sequence $\{(p_n, x_n^*, y_n^*)\} \subseteq L(w, q, \varepsilon, \varepsilon)$,

$$\text{the convergence of } (x_n, y_n) \in S(x_n^*, y_n^*) \times T(x_n^*, y_n^*), n \in \mathbf{N} \Rightarrow \begin{cases} \hat{d}(x_n, S(x_n, y_n)) \rightarrow 0, \\ \bar{d}(y_n, T(x_n, y_n)) \rightarrow 0. \end{cases}$$

Then (OPSGQVIP) is well-posed at (p, x^*, y^*) if and only if for every $w \in H(p, x^*, y^*)$ with w being a minimal point of $H(M)$, $Q(w) = \{(p, x^*, y^*)\}$.

Remark 4.10.

- (a) Theorems 4.8–4.9 generalize and extend [26, Theorems 4.1–4.2] from (OPGQVIP) to (OPSGQVIP), respectively.
- (b) Compared with [26, Theorems 4.1–4.2], our Theorems 4.6–4.7 are their corresponding ones, which show that for the well-posedness of (OPSGQVIP), the necessary condition coincides with the sufficient one. But, for the well-posedness of (OPVIP), Lin and Chuang [17] gave necessary condition and sufficient condition, respectively. It is worth mentioning that their necessary condition is different from the sufficient one.

Acknowledgment

This research was partially supported by the Innovation Program of Shanghai Municipal Education Commission (15ZZ068), Ph.D. Program Foundation of Ministry of Education of China (20123127110002) and Program for Outstanding Academic Leaders in Shanghai City (15XD1503100). Yeong-Cheng Liou was supported in part by the grand form Kaohsiung Medical University Research Foundation (KMU-Q106005), Taiwan-Thailand joint grant MOST 106-2218-E-040-001-MY2 and Taiwan-Russian joint grant MOST 106-2923-E-039-001-MY3. J. C. Yao was partially supported by the Grant MOST 106-2923-E-039-001-MY3.

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