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Operator iteration on the Young inequality

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Abstract

In this paper, we employ iteration on operator version of the famous Young inequality and obtain more arithmetic-geometric mean inequalities and the reverse versions for positive operators. Concretely, we obtain refined Young inequalities with the Kantorovich constant, the reverse ratio type and difference type inequalities for arithmetic-geometric operator mean under different conditions.

MSC: 47A30; 47A63

Keywords: operator iteration; Young inequality; arithmetic-geometric means; Kantorovich constant

1 Introduction

Throughout this paper, A, B are both positive operators on a Hilbert space H, and $\mathcal{B}_h(H)$ is the semi-space of all bounded linear self-adjoint operators on H. In addition, notation $\mathcal{B}^+(H)$ is written as the set of all positive operators in $\mathcal{B}_h(H)$. Besides, we may assume that A and B are invertible without loss of generality,

$$A\nabla_\mu B = (1-\mu)A + \mu B \quad \text{and} \quad A\sharp_\mu B = A^{1/2}\big(A^{-1/2}BA^{-1/2}\big)^\mu A^{1/2}, \quad \text{where } 0 \le \mu \le 1.$$

When $\mu = 1/2$ we write $A\nabla B$ and $A\sharp B$ for brevity, respectively; see Kubo and Ando [1]. The Kantorovich constant is defined by $K(t,2) = \frac{(t+1)^2}{4t}$ for $t > 0$, while the Specht ratio [2] is denoted by

$$S(t) = \frac{t^{\frac{1}{t-1}}}{e \log t^{\frac{1}{t-1}}} \quad \text{for } t > 0, t \ne 1, \quad \text{and} \quad S(1) = \lim_{t \to 1} S(t) = 1,$$

and has the following properties:

(i) $S(t) = S(\frac{1}{t}) \ge 1$ for $t > 0$.
(ii) $S(t)$ is a monotone increasing function on $(1, +\infty)$.
(iii) $S(t)$ is a monotone decreasing function on $(0, 1)$.

We start from an improvement of the famous Young inequality as follows.

Theorem ZS ([3]) *For $a, b > 0$, we have*

$$a\nabla_\mu b \ge K(h,2)^r a^{1-\mu} b^\mu$$

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for all $\mu \in [0,1]$, where $r = \min\{\mu, 1-\mu\}$ and $h = \frac{b}{a}$. It admits an operator extension

$$A\nabla_\mu B \geq K(h,2)^r A\sharp_\mu B$$

for positive operators A and B on a Hilbert space.

Next, we show the reverse arithmetic-geometric mean inequality with the Specht ratio for two positive operators.

Theorem T ([4]) *For invertible operators A and B with $0 < aI_H \leq A, B \leq bI_H$, we have*

   (i) $A\nabla_\mu B \leq S(h)A\sharp_\mu B$,

   (ii) $A\nabla_\mu B - A\sharp_\mu B \leq L(1,h)\log S(h)A$,

where $L(1,h)$ is defined by $L(a,b) = \frac{a-b}{\log a - \log b}(a \neq b); L(a,a) = a, h = \frac{b}{a}$.

These inequalities have recently been improved by Furuichi [5] as follows.

Theorem F ([5]) *If $0 < aI_H \leq A, B \leq bI_H$, then*

   (i) $A\nabla_\mu B - 2r(A\nabla B - A\sharp B) \leq S(\sqrt{h})A\sharp_\mu B$,

   (ii) $A\nabla_\mu B - A\sharp_\mu B - 2r(A\nabla B - A\sharp B) \leq L(\sqrt{h},1)\log S(\sqrt{h})bI_H$,

where $r = \min\{\mu, 1-\mu\}$, $L(a,b) = \frac{a-b}{\log a - \log b}$, $h = \frac{b}{a}$.

Afterwards, Krnić *et al.* [6] introduced Jensen's operator and established some bounds for the spectra of Jensen's operator. The obtained results were then applied to operator means. In such a way, they get refinements and converses of numerous mean inequalities for Hilbert space operators. See [1, 2, 4, 6–10] for more related developments.

See also [11] for another improvement of the reverse weighted arithmetic-geometric operator mean inequalities. Their proof is independent of [5].

Theorem ZF ([11]) *If $0 < aA \leq B \leq bA$ with $0 < a < 1 < b$, then*

   (i) $A\nabla_\mu B - 2r(A\nabla B - A\sharp B) \leq \max\{S(\sqrt{a}), S(\sqrt{b})\}A\sharp_\mu B$,

   (ii) $A\nabla_\mu B - A\sharp_\mu B - 2r(A\nabla B - A\sharp B)$

$$\leq \max\left\{L\left(\frac{1}{\sqrt{a}}, 1\right)\log S(\sqrt{a}), L\left(\frac{1}{\sqrt{b}}, 1\right)\log S(\sqrt{b})\right\}bA,$$

where $r = \min\{\mu, 1-\mu\}$, $L(a,b) = \frac{a-b}{\log a - \log b}$.

This paper aims to provide a method to obtain more arithmetic-geometric mean inequalities and the reverse version for positive operators. In Section 2, we introduce the main lemmas. In Section 3, utilizing the refined Young inequality and iteration method, we establish some weighted arithmetic-geometric mean inequality for two positive operators. We also obtain reverse ratio type and difference type inequalities for positive operators by means of iteration under different conditions in Section 4 and Section 5, respectively.

2 Main lemmas

First of all, the main lemmas including operator iteration can be stated as follows.

Lemma 2.1 ([6]) *If $A, B \in \mathcal{B}^+(H)$, $0 \le \mu, \nu \le 1$, then*

$$2\min\{\mu, 1-\mu\}(A\nabla B - A\sharp B) \le A\nabla_\mu B - A\sharp_\mu B \le 2\max\{\mu, 1-\mu\}(A\nabla B - A\sharp B). \quad (2.1)$$

Lemma 2.2 *If A and B are positive operators on a Hilbert space, $0 \le \mu, \nu \le 1$, then*

$$A\nabla_\mu(A\sharp_\nu B) = A\nabla_{\mu\nu}B - \mu(A\nabla_\nu B - A\sharp_\nu B).$$

Proof

$$
\begin{aligned}
A\nabla_\mu(A\sharp_\nu B) &= (1-\mu)A + \mu A\sharp_\nu B \\
&= A - \mu A + \mu\nu A - \mu\nu A + \mu\nu B - \mu\nu B + \mu A\sharp_\nu B \\
&= \mu\nu B + (1-\mu\nu)A - \mu\big[(1-\nu)A + \nu B - A\sharp_\nu B\big] \\
&= A\nabla_{\mu\nu}B - \mu(A\nabla_\nu B - A\sharp_\nu B). \qquad \square
\end{aligned}
$$

3 Further refinement of the Young inequalities

In this section, we use the scalar ratio type arithmetic-geometric mean inequality to get a series of operator versions.

Lemma 3.1 ([3]) *If $a, b > 0$ and $\mu \in [0,1]$, then*

$$a\nabla_\mu b \ge K(h,2)^r a^{1-\mu}b^\mu,$$

where $r = \min\{\mu, 1-\mu\}$ and $h = \frac{b}{a}$.

Using the method of the proof of Theorem 7 in [3] we can obtain the following theorem.

Theorem 3.2 *If $A, B \ge 0$, $1 < h \le A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \le h'$ or $0 < h' \le A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \le h < 1$, then*

$$A\nabla_\mu B \ge K(h,2)^r A\sharp_\mu B \qquad (3.1)$$

for all $\mu \in [0,1]$, where $r = \min\{\mu, 1-\mu\}$.

Proof For the case of $1 < h \le A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \le h'$, by Lemma 3.1 we have

$$(1-\mu) + \mu x \ge K(x,2)^r x^\mu$$

for any $x > 0$. And hence

$$(1-\mu)I + \mu X \ge \min_{1 < h \le x \le h'} K(x,2)^r X^\mu$$

for a positive operator X such that $1 < hI \le X \le h'I$.

Substituting $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ for X in the above inequality we have

$$(1-\mu)I + \mu A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \geq \min_{1<h\leq x\leq h'} K(x,2)^r \big(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\big)^\mu.$$

It is easy to check that the function $K(x,2)$ is increasing for $x > 1$, then

$$(1-\mu)I + \mu A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \geq K(h,2)^r \big(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\big)^\mu. \tag{3.2}$$

Multiplying both sides by $A^{\frac{1}{2}}$ to inequality (3.2), we obtain the required inequality.

For the case of $0 < h' \leq A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \leq h < 1$, since the function $K(x,2)$ is decreasing for $0 < x < 1$ and $K(\frac{1}{h},2) = K(h,2)$, similarly we obtain inequality (3.1). □

Theorem 3.3 *For two operators A, $B \geq 0$ and $1 < h \leq A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \leq h'$ or $0 < h' \leq A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \leq h < 1$, we have*

$$A\nabla_\mu B - 2r(A\nabla B - A\sharp B) \geq K(\sqrt{h},2)^R A\sharp_\mu B \tag{3.3}$$

for all $\mu \in [0,1]$, where $r = \min\{\mu, 1-\mu\}$ and $R = \min\{2r, 1-2r\}$.

Proof If $0 \leq \mu \leq \frac{1}{2}$, then $0 \leq 2\mu \leq 1$.

Here $1 < h \leq A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \leq h'$ ensures that $1 < \sqrt{h} \leq A^{-\frac{1}{2}}(A\sharp B)A^{-\frac{1}{2}} \leq \sqrt{h'}$, and $0 < h' \leq A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \leq h < 1$ ensures that $0 < \sqrt{h'} \leq A^{-\frac{1}{2}}(A\sharp B)A^{-\frac{1}{2}} \leq \sqrt{h} < 1$, respectively. Substituting B by $A\sharp B$ and μ by 2μ in (3.1), it follows that

$$A\nabla_{2\mu}(A\sharp B) \geq K(\sqrt{h},2)^{\min\{2\mu,1-2\mu\}} A\sharp_{2\mu}(A\sharp B).$$

By Lemma 2.1 and $A\sharp_{2\mu}(A\sharp B) = A\sharp_\mu B$, we have

$$A\nabla_{2\mu}(A\sharp B) = A\nabla_\mu B - 2\mu(A\nabla B - A\sharp B),$$

and then

$$A\nabla_\mu B - 2\mu(A\nabla B - A\sharp B) \geq K(\sqrt{h},2)^{\min\{2\mu,1-2\mu\}} A\sharp_\mu B.$$

If $\frac{1}{2} \leq \mu \leq 1$, then $0 \leq 1-\mu \leq \frac{1}{2}$. By the above inequality we have

$$B\nabla_{1-\mu}A - 2(1-\mu)(B\nabla A - B\sharp A) \geq K(\sqrt{h},2)^{\min\{2(1-\mu),1-2(1-\mu)\}} B\sharp_{1-\mu}A.$$

Therefore, for $0 \leq \mu \leq 1$, we have

$$A\nabla_\mu B - 2\min\{\mu,1-\mu\}(A\nabla B - A\sharp B) \geq K(\sqrt{h},2)^{\min\{2r,1-2r\}} A\sharp_\mu B.$$

This completes the proof. □

Now, applying the same iteration method as in Theorem 3.2 to inequality (3.3), we obtain the following.

Corollary 3.4 *Assume the conditions as in Theorem 3.3. Then*

$$A\nabla_\mu B - (2r-R)(A\nabla B - A\sharp B) - 2R(A\nabla_{\frac{2[2\mu]+1}{4}}B - A\sharp_{\frac{2[2\mu]+1}{4}}B) \geq K(\sqrt{h},2)^{R'}A\sharp_\mu B$$

for all $\mu \in [0,1]$, where $r = \min\{\mu, 1-\mu\}$, $R = \min\{2r, 1-2r\}$, $R' = \min\{2R, 1-2R\}$, $h = \frac{M}{m}$, and $[x]$ is the greatest integer less than or equal to x.

Corollary 3.5 ([12]) *Suppose that two operators A, B and positive real numbers m, m', M, M' satisfy either $0 < m'I \leq A \leq mI < MI \leq B \leq M'I$ or $0 < m'I \leq B \leq mI < MI \leq A \leq M'I$.*

(I) *If $0 \leq \mu \leq \frac{1}{2}$, then*

$$A\nabla_\mu B - (2r-R)(A\nabla B - A\sharp B) - 2R(A\nabla_{\frac{1}{4}}B - A\sharp_{\frac{1}{4}}B) \geq K(\sqrt{h},2)^{R'}A\sharp_\mu B.$$

(II) *If $\frac{1}{2} \leq \mu \leq 1$, then*

$$A\nabla_\mu B - (2r-R)(A\nabla B - A\sharp B) - 2R(A\nabla_{\frac{3}{4}}B - A\sharp_{\frac{3}{4}}B) \geq K(\sqrt{h},2)^{R'}A\sharp_\mu B,$$

where $r = \min\{\mu, 1-\mu\}$, $R = \min\{2r, 1-2r\}$, $R' = \min\{2R, 1-2R\}$, $h = \frac{M}{m}$, and $h' = \frac{M'}{m'}$.

Proof In the case of (i), $1 < h = \frac{M}{m} \leq A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \leq \frac{M'}{m'} = h'$; In the case of (ii), $0 < \frac{1}{h'} = \frac{m'}{M'} \leq A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \leq \frac{m}{M} = \frac{1}{h} < 1$. Then Corollary 3.4 leads to the required inequality. This completes the proof. □

4 Reverse ratio type arithmetic-geometric mean inequalities

In the following, we show a refinement of reverse arithmetic-geometric mean inequality by applying the main lemmas in Section 2.

Theorem 4.1 *If $0 < aI \leq A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \leq bI$ with $a < 1 < b$, and $0 \leq \mu \leq 1$, then*

$$A\nabla_\mu B - (2r-R)(A\nabla B - A\sharp B) - 2R(A\nabla_{\frac{2[2\mu]+1}{4}}B - A\sharp_{\frac{2[2\mu]+1}{4}}B)$$

$$\leq \max\{S(\sqrt[4]{a}), S(\sqrt[4]{b})\}A\sharp_\mu B,$$

where $r = \min\{\mu, 1-\mu\}$, $R = \min\{2r, 1-2r\}$.

Proof If $0 \leq \mu \leq \frac{1}{2}$, then $0 \leq 2\mu \leq 1$. Since $0 < aI \leq A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \leq bI$ ensures that $\sqrt{a}A \leq A\sharp B \leq \sqrt{b}A$, by substituting B by $A\sharp B$ and μ by 2μ in (i) of Theorem ZF, it follows that

$$A\nabla_{2\mu}(A\sharp B) - 2\min\{2\mu, 1-2\mu\}[A\nabla(A\sharp B) - A\sharp(A\sharp B)]$$

$$\leq \max\{S(\sqrt[4]{a}), S(\sqrt[4]{b})\}A\sharp_{2\mu}(A\sharp B).$$

Lemma 2.2 leads to the following equalities:

$$A\nabla_{2\mu}(A\sharp B) = A\nabla_\mu B - 2\mu(A\nabla B - A\sharp B),$$

$$A\nabla(A\sharp B) - A\sharp(A\sharp B) = A\nabla_{\frac{1}{4}}B - A\sharp_{\frac{1}{4}}B - \frac{1}{2}(A\nabla B - A\sharp B).$$

Then it follows by Lemma 2.1 that

$$A\nabla_\mu B - 2\mu(A\nabla B - A\sharp B) - 2\min\{2\mu, 1 - 2\mu\}\left[A\nabla_{\frac{1}{4}}B - A\sharp_{\frac{1}{4}}B - \frac{1}{2}(A\nabla B - A\sharp B)\right]$$

$$\leq \max\left\{S(\sqrt[4]{a}), S(\sqrt[4]{b})\right\}A\sharp_\mu B. \tag{4.1}$$

If $\frac{1}{2} \leq \mu \leq 1$, then $0 \leq 1 - \mu \leq \frac{1}{2}$. The hypothesis $0 < aI \leq A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \leq bI$ admits $\frac{1}{\sqrt{b}}B \leq B\sharp A \leq \frac{1}{\sqrt{a}}B$. Then by the inequality (4.1) we have

$$B\nabla_{1-\mu}A - 2(1 - \mu)(B\nabla A - B\sharp A) - 2R\left[B\nabla_{\frac{1}{4}}A - B\sharp_{\frac{1}{4}}A - \frac{1}{2}(B\nabla A - B\sharp A)\right]$$

$$\leq \max\left\{S\left(\frac{1}{\sqrt[4]{b}}\right), S\left(\frac{1}{\sqrt[4]{a}}\right)\right\}B\sharp_{1-\mu}A.$$

Notice that $S(\frac{1}{\sqrt[4]{a}}) = S(\sqrt[4]{a})$ and $S(\frac{1}{\sqrt[4]{b}}) = S(\sqrt[4]{b})$, so it follows that

$$A\nabla_\mu B - 2(1 - \mu)(A\nabla B - A\sharp B) - 2R\left[A\nabla_{\frac{3}{4}}B - A\sharp_{\frac{3}{4}}B - \frac{1}{2}(A\nabla B - A\sharp B)\right]$$

$$\leq \max\left\{S(\sqrt[4]{a}), S(\sqrt[4]{b})\right\}A\sharp_\mu B.$$

This completes the proof. □

Since $0 < aI_H \leq A, B \leq bI_H$ with $a < b$ admits that $\frac{1}{\sqrt{h}} \leq A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \leq \sqrt{h}$ where $\frac{1}{\sqrt{h}} < 1 < \sqrt{h}$, we obtain the counterpart of Theorem 4.1.

Corollary 4.2 *If* $0 < aI \leq A, B \leq bI$ *with* $a < b$ *and* $0 \leq \mu \leq 1$, *then*

$$A\nabla_\mu B - (2r - R)(A\nabla B - A\sharp B) - 2R(A\nabla_{\frac{2[2\mu]+1}{4}}B - A\sharp_{\frac{2[2\mu]+1}{4}}B) \leq S(\sqrt[4]{h})A\sharp_\mu B,$$

where $r = \min\{\mu, 1 - \mu\}$, $R = \min\{2r, 1 - 2r\}$, *and* $h = \frac{b}{a}$.

5 Reverse difference type arithmetic-geometric mean inequalities

In the following theorem we show the corresponding difference type analogs of Theorem 4.1.

Theorem 5.1 *If* $0 \leq \mu \leq 1$ *and* $0 < aI \leq A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \leq bI$ *with* $a < 1 < b$, *then*

$$A\nabla_\mu B - A\sharp_\mu B - (2r - R)(A\nabla B - A\sharp B) - 2R(A\nabla_{\frac{2[2\mu]+1}{4}}B - A\sharp_{\frac{2[2\mu]+1}{4}}B)$$

$$\leq \max\left\{L(\sqrt[4]{a}, 1)\log S(\sqrt[4]{a}), L(\sqrt[4]{b}, 1)\log S(\sqrt[4]{b})\right\}\frac{b}{\sqrt{a}}A,$$

where $r = \min\{\mu, 1 - \mu\}$, $R = \min\{2r, 1 - 2r\}$.

Proof (I) If $0 \leq \mu \leq \frac{1}{2}$, then $0 \leq 2\mu \leq 1$. Since $0 < aI \leq A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \leq bI$ ensures that $\sqrt{a}A \leq A\sharp B \leq \sqrt{b}A$, substitute B by $A\sharp B$ and μ by 2μ in (ii) of Theorem ZF, so we obtain

$$A\nabla_{2\mu}(A\sharp B) - A\sharp_{2\mu}(A\sharp B) - 2\min\{2\mu, 1 - 2\mu\}\big[A\nabla(A\sharp B) - A\sharp(A\sharp B)\big]$$

$$\leq \max\Big\{L\big(\sqrt[4]{1/a}, 1\big)\log S\big(\sqrt[4]{a}\big), L\big(\sqrt[4]{1/b}, 1\big)\log S\big(\sqrt[4]{b}\big)\Big\}\sqrt{b}A.$$

As showed in the proof of Theorem 4.1, the following inequality holds:

$$A\nabla_{\mu}B - A\sharp_{\mu}B - (2r - R)(A\nabla B - A\sharp B) - 2R(A\nabla_{\frac{1}{4}}B - A\sharp_{\frac{1}{4}}B)$$

$$\leq \max\Big\{L\big(\sqrt[4]{1/a}, 1\big)\log S\big(\sqrt[4]{a}\big), L\big(\sqrt[4]{1/b}, 1\big)\log S\big(\sqrt[4]{b}\big)\Big\}\sqrt{b}A$$

$$\leq \max\Big\{L\big(\sqrt[4]{a}, 1\big)\log S\big(\sqrt[4]{a}\big), L\big(\sqrt[4]{b}, 1\big)\log S\big(\sqrt[4]{b}\big)\Big\}b/\sqrt{a}A, \tag{5.1}$$

since $a < 1 < b$ and $L(\frac{1}{\sqrt[4]{a}}, 1) = \frac{1}{\sqrt[4]{a}}L(\sqrt[4]{a}, 1)$.

(II) If $\frac{1}{2} \leq \mu \leq 1$, then $0 \leq 1 - \mu \leq \frac{1}{2}$. The hypothesis $0 < aI \leq A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \leq bI$ ensures $\frac{1}{\sqrt{b}}B \leq B\sharp A \leq \frac{1}{\sqrt{a}}B$. Then by the inequality (5.1) and $S(\frac{1}{\sqrt[4]{a}}) = S(\sqrt[4]{a})$, we have

$$A\nabla_{\mu}B - A\sharp_{\mu}B - (2r - R)(A\nabla B - A\sharp B) - 2R(A\nabla_{\frac{3}{4}}B - A\sharp_{\frac{3}{4}}B)$$

$$= B\nabla_{1-\mu}A - B\sharp_{1-\mu}A - (2r - R)(B\nabla A - B\sharp A) - 2R(B\nabla_{\frac{1}{4}}A - B\sharp_{\frac{1}{4}}A)$$

$$\leq \max\Big\{L\big(\sqrt[4]{b}, 1\big)\log S\big(\sqrt[4]{1/b}\big), L\big(\sqrt[4]{a}, 1\big)\log S\big(\sqrt[4]{1/a}\big)\Big\}\sqrt{1/a}B$$

$$\leq \max\Big\{L\big(\sqrt[4]{b}, 1\big)\log S\big(\sqrt[4]{b}\big), L\big(\sqrt[4]{a}, 1\big)\log S\big(\sqrt[4]{a}\big)\Big\}b/\sqrt{a}A.$$

The proof is done. □

Similarly, we replace the hypothesis $0 < aI \leq A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \leq bI$, where $a < 1 < b$ by $0 < aI \leq A, B \leq bI$ with $a < b$, and we obtain the following.

Corollary 5.2 *If* $0 < aI \leq A, B \leq bI$ *with* $a < b$, *and* $0 \leq \mu \leq 1$, *then*

$$A\nabla_{\mu}B - A\sharp_{\mu}B - (2r - R)(A\nabla B - A\sharp B) - 2R(A\nabla_{\frac{2[2\mu]+1}{4}}B - A\sharp_{\frac{2[2\mu]+1}{4}}B)$$

$$\leq b\sqrt{h}L\big(\sqrt[4]{h}, 1\big)\log S\big(\sqrt[4]{h}\big)I,$$

where $r = \min\{\mu, 1 - \mu\}$, $R = \min\{2r, 1 - 2r\}$, *and* $h = \frac{b}{a}$.

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