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Power series inequalities via Young's inequality with applications

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Abstract

In this paper, we establish some inequalities for power series with real coefficients by utilizing Young's inequality for sequences of complex numbers. Some applications for special functions such as polylogarithm, hypergeometric and Bessel functions are also presented.

MSC: 26D15

Keywords: Young's inequality; Hölder's inequality; power series

1 Introduction

Let $a_k, b_k \in \mathbb{C}$, $k \in \{1, 2, \dots, n\}$, $p > 1$ and let q satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Then the classical Hölder's inequality [1, pp.19-21] states that

$$\left|\sum_{k=1}^{n} a_k b_k\right| \le \left(\sum_{k=1}^{n} |a_k|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} |b_k|^q\right)^{\frac{1}{q}} \tag{1.1}$$

with equality holds if and only if the sequences $\{|a_k|^p\}$ and $\{|b_k|^q\}$ for $k \in \{1, 2, \dots, n\}$ are proportional and the $\arg(a_k b_k)$ is independent of k. The inequality (1.1) is reversed if $p < 1$.

The weighted version of Hölder's inequality also holds, namely

$$\left|\sum_{k=1}^{n} p_k a_k b_k\right| \le \left(\sum_{k=1}^{n} p_k |a_k|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} p_k |b_k|^q\right)^{\frac{1}{q}}, \tag{1.2}$$

where $p_k \ge 0$, $a_k, b_k \in \mathbb{C}$, $k \in \{1, 2, \dots, n\}$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

Tolsted in [2] (see also [3, p.457], [4, pp.63-64]) showed that Hölder's inequality (also known in the literature as the Rogers inequality) can be easily proved by using Young's inequality [5], namely

$$\frac{1}{q} x^q + \frac{1}{p} y^p \ge xy \tag{1.3}$$

for any positive numbers x, y and $p > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Equality holds in (1.3) if and only if $x^q = y^p$. For other applications and extensions of Young's inequality, see [6, 7], [8, pp.379-389] and references therein.

It is well known that Hölder's inequality is one of the most important inequalities in real and complex analysis. For example, the celebrated Cauchy-Bunyakovsky-Schwarz (CBS) inequality (see [9, p.16], [8, p.83]) is a special case of Hölder's inequality (1.1) for $p = q = 2$. Some other inequalities such as Minkowski's inequality can be proved by using Hölder's inequality.

Various extensions, generalizations, refinements, *etc.* of Hölder's inequality have been obtained by several authors (see [10–20] and references therein). For instance, it comes to our attention that an interesting generalizations of Hölder's inequality (1.1) by utilizing Young's inequality (1.3), which was established by Dragomir and Sándor in [21] (see also [22, pp.10-16]), is as follows:

$$\sum_{k=1}^{n} p_k|x_k y_k| \sum_{k=1}^{n} q_k|x_k y_k| \leq \frac{1}{p}\sum_{k=1}^{n} p_k|x_k|^p \sum_{k=1}^{n} q_k|y_k|^p + \frac{1}{q}\sum_{k=1}^{n} q_k|x_k|^q \sum_{k=1}^{n} p_k|y_k|^q \tag{1.4}$$

for $x_k, y_k \in \mathbb{C}$, $p_k, q_k \geq 0$, $k \in \{1, 2, \dots, n\}$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

If now we consider an analytic function defined by the power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{1.5}$$

with real coefficients and convergent on the disk $D(0, R)$, $R > 0$ and apply the weighted version of Hölder's inequality (1.2), then we can state that

$$\begin{aligned} \left|f(xy)\right| &= \left|\sum_{n=0}^{\infty} a_n x^n y^n\right| \\ &\leq \left(\sum_{n=0}^{\infty} |a_n||x|^{pn}\right)^{\frac{1}{p}} \left(\sum_{n=0}^{\infty} |a_n||y|^{qn}\right)^{\frac{1}{q}} = f_A^{\frac{1}{p}}\left(|x|^p\right) f_A^{\frac{1}{q}}\left(|y|^q\right) \end{aligned} \tag{1.6}$$

for any $x, y \in \mathbb{C}$ with $xy, |x|^p, |y|^q \in D(0, R)$ and $f_A(z)$ is a new power series defined by $\sum_{n=0}^{\infty} |a_n|z^n$, where $a_n = |a_n| \operatorname{sgn}(a_n)$ with $\operatorname{sgn}(x)$ is the real signum function defined to be 1 if $x > 0$, -1 if $x < 0$ and 0 if $x = 0$. The power series $f_A(z)$ have the same radius of convergence as the original power series $f(z)$.

Motivated by the above results (1.6), (1.4) and the results from [21], and utilizing Young's inequality, we established in this paper some inequalities for functions defined by power series with real coefficients. Particular examples that are related to some fundamental complex functions such as the exponential, logarithm, trigonometric and hyperbolic functions are presented. Applications for some special functions such as polylogarithm, hypergeometric and Bessel functions for the first kind are presented as well.

2 Some inequalities via Young's inequality

On utilizing Young's inequality (1.3) for power series with real coefficients, we establish the following result.

Theorem 1 *Let* $f(z) = \sum_{n=0}^{\infty} p_n z^n$ *and* $g(z) = \sum_{n=0}^{\infty} q_n z^n$ *be two power series with real coefficients and convergent on the open disk* $D(0, R)$, $R > 0$. *If* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ *and* $x, y \in \mathbb{C}$, $x, y \neq 0$

so that $xy, |x|^p, |x|^q, |y|^p, |y|^q \in D(0,R)$, *then*

$$\frac{1}{p} g_A\bigl(|x|^p\bigr) f_A\bigl(|y|^p\bigr) + \frac{1}{q} f_A\bigl(|x|^q\bigr) g_A\bigl(|y|^q\bigr) \geq \bigl|f(xy)g(xy)\bigr| \tag{2.1}$$

and

$$\frac{1}{p} g_A\bigl(|x|^p\bigr) f_A\bigl(|y|^q\bigr) + \frac{1}{q} f_A\bigl(|x|^q\bigr) g_A\bigl(|y|^p\bigr) \geq \bigl|f\bigl(x|y|^{q-1}\bigr) g\bigl(x|y|^{p-1}\bigr)\bigr|. \tag{2.2}$$

Proof If we choose $x = |x|^j |y|^k$, $y = |x|^k |y|^j$, $j,k \in \{0,1,2,\ldots,n\}$ in (1.3), then we have

$$p|x|^{qj}|y|^{qk} + q|x|^{pk}|y|^{pj} \geq pq|xy|^j|xy|^k \tag{2.3}$$

for any $j,k \in \{0,1,2,\ldots,n\}$.

Now, if we multiply this inequality (2.3) with positive quantities $|p_j||q_k|$ and summing over j and k from 0 to n, then we derive

$$\begin{aligned} &p\sum_{j=0}^{n} |p_j||x|^{qj} \sum_{k=0}^{n} |q_k||y|^{qk} + q\sum_{k=0}^{n} |q_k||x|^{pk} \sum_{j=0}^{n} |p_j||y|^{pj} \\ &\quad \geq pq\left|\sum_{j=0}^{n} p_j (xy)^j \sum_{k=0}^{n} q_k (xy)^k\right|. \end{aligned} \tag{2.4}$$

Since all the series whose partial sums are involved in inequality (2.4) are convergent on the disk $D(0,R)$, taking the limit as $n \to \infty$ in (2.4), we deduce the desired result (2.1).

Further, if we choose in (1.3), $x = \frac{|x|^j}{|y|^j}$, $y = \frac{|x|^k}{|y|^k}$, then we get

$$p\left(\frac{|x|^j}{|y|^j}\right)^q + q\left(\frac{|x|^k}{|y|^k}\right)^p \geq pq\frac{|x|^j}{|y|^j}\frac{|x|^k}{|y|^k} \tag{2.5}$$

for any $|y|^j, |y|^k \neq 0$, $j,k \in \{0,1,2,\ldots,n\}$.

Simplifying (2.5), we obtain that

$$\begin{aligned} p|x|^{qj}|y|^{pk} + q|x|^{pk}|y|^{qj} &\geq pq|x|^j|y|^{(q-1)j}|x|^k|y|^{(p-1)k} \\ &= pq\bigl|\bigl(x|y|^{q-1}\bigr)^j\bigl(x|y|^{p-1}\bigr)^k\bigr| \end{aligned} \tag{2.6}$$

for any $j,k \in \{0,1,2,\ldots,n\}$.

Multiplying (2.6) by $|p_j||q_k| \geq 0$, $j,k \in \{0,1,2,\ldots,n\}$ and summing over j and k from 0 to n, we get

$$\begin{aligned} &p\sum_{j=0}^{n} |p_j||x|^{qj} \sum_{k=0}^{n} |q_k||y|^{pk} + q\sum_{k=0}^{n} |q_k||x|^{pk} \sum_{j=0}^{n} |p_j||y|^{qj} \\ &\quad \geq pq\left|\sum_{j=0}^{n} p_j\bigl(x|y|^{q-1}\bigr)^j \sum_{k=0}^{n} q_k\bigl(x|y|^{p-1}\bigr)^k\right|. \end{aligned} \tag{2.7}$$

Since all the series whose partial sums are involved in inequality (2.7) are convergent on the disk $D(0,R)$, letting $n \to \infty$ in (2.7), we deduce the desired inequality (2.2). □

The following particular case is of interest.

Corollary 1 *If* $g(z) = f(z)$ *in* (2.1) *and* (2.2), *then*

$$\frac{1}{p} f_A(|x|^p) f_A(|y|^p) + \frac{1}{q} f_A(|x|^q) f_A(|y|^q) \geq |f(xy)|^2 \tag{2.8}$$

and

$$\frac{1}{p} f_A(|x|^p) f_A(|y|^q) + \frac{1}{q} f_A(|x|^q) f_A(|y|^p) \geq |f(x|y|^{q-1}) f(x|y|^{p-1})|, \tag{2.9}$$

respectively, where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ *and* $x, y \neq 0$ *with* $xy, |x|^p, |x|^q, |y|^p, |y|^q \in D(0,R)$. *In particular, if* $y = x$ *in* (2.8) *and* (2.9), *then we have*

$$\frac{1}{p} f_A^2(|x|^p) + \frac{1}{q} f_A^2(|x|^q) \geq |f(x^2)|^2$$

and

$$f_A(|x|^p) f_A(|x|^q) \geq |f(\operatorname{sgn}(x)|x|^q) f(\operatorname{sgn}(x)|x|^p)|$$

for any $x \in \mathbb{C}$, $x \neq 0$ *with* $x^2, |x|^p, |x|^q \in D(0,R)$ *and* $\operatorname{sgn}(x)$ *is the complex signum function defined to be* $\frac{x}{|x|}$ *if* $x \neq 0$ *and* 0 *if* $x = 0$.

Remark 1 *In the particular case* $p = q = 2$ *in* (2.8) *and* (2.9), *we get the inequalities*

$$f_A(|x|^2) f_A(|y|^2) \geq |f(xy)|^2$$

and

$$f_A(|x|^2) f_A(|y|^2) \geq |f(x|y|)|^2,$$

respectively, for any $x, y \in \mathbb{C}$ *with* $xy, |x|^2, |y|^2 \in D(0,R)$.

Some applications to particular functions of interest are as follows.

(1) If we apply inequalities (2.8) and (2.9) to the function $f(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$, $z \in D(0,1)$, then we get

$$\frac{1}{p(1-|x|^p)(1-|y|^p)} + \frac{1}{q(1-|x|^q)(1-|y|^q)} \geq \frac{1}{|1-xy|^2}$$

and

$$\begin{aligned}&\frac{1}{p(1-|x|^p)(1-|y|^q)} + \frac{1}{q(1-|x|^q)(1-|y|^p)} \\ &\quad \geq \frac{1}{|1-x|y|^{q-1}||1-x|y|^{p-1}|},\end{aligned}$$

respectively, for any $x, y \in \mathbb{C}$ with $x, y \neq 0$, $xy, |x|^p, |x|^q, |y|^p, |y|^q \in D(0,1)$ and $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

(2) If we apply inequalities (2.8) and (2.9) to the function $f(z)=\exp(z)=\sum_{n=0}^{\infty}\frac{1}{n!}z^n$, $z\in\mathbb{C}$, then we can state that

$$\frac{1}{p}\exp\bigl(|x|^p+|y|^p\bigr)+\frac{1}{q}\exp\bigl(|x|^q+|y|^q\bigr)\ge\bigl|\exp(xy)\bigr|^2$$

and

$$\frac{1}{p}\exp\bigl(|x|^p+|y|^q\bigr)+\frac{1}{q}\exp\bigl(|x|^q+|y|^p\bigr)$$
$$\ge\bigl|\exp\bigl(x|y|^{q-1}+x|y|^{p-1}\bigr)\bigr|,$$

respectively, for any $x,y\in\mathbb{C}$ and $p>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

(3) If we apply the function $f(z)=\ln(\frac{1}{1-z})=\sum_{n=0}^{\infty}\frac{1}{n}z^n$, $z\in D(0,1)$, then from (2.8) and (2.9) we have

$$\frac{1}{p}\ln\bigl(1-|x|^p\bigr)\ln\bigl(1-|y|^p\bigr)+\frac{1}{q}\ln\bigl(1-|x|^q\bigr)\ln\bigl(1-|y|^q\bigr)$$
$$\ge\bigl|\ln(1-xy)\bigr|^2$$

and

$$\frac{1}{p}\ln\bigl(1-|x|^p\bigr)\ln\bigl(1-|y|^q\bigr)+\frac{1}{q}\ln\bigl(1-|x|^q\bigr)\ln\bigl(1-|y|^p\bigr)$$
$$\ge\bigl|\ln\bigl(1-x|y|^{q-1}\bigr)\ln\bigl(1-x|y|^{p-1}\bigr)\bigr|,$$

respectively, for any $x,y\in\mathbb{C}$ with $x,y\neq 0$, $|x|^p,|x|^q,|y|^p,|y|^q\in D(0,1)$ and $p>1$, $\frac{1}{p}+\frac{1}{q}=1$.

(IV) Also, if we consider the function $f(z)=\sin(z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n+1}$, $z\in\mathbb{C}$, then, obviously, we have $f_A(z)=\sinh(z)$, $z\in\mathbb{C}$. Applying inequalities (2.8) and (2.9) to this function, we get

$$\frac{1}{p}\sinh\bigl(|x|^p\bigr)\sinh\bigl(|y|^p\bigr)+\frac{1}{q}\sinh\bigl(|x|^q\bigr)\sinh\bigl(|y|^q\bigr)\ge\bigl|\sin(xy)\bigr|^2$$

and

$$\frac{1}{p}\sinh\bigl(|x|^p\bigr)\sinh\bigl(|y|^q\bigr)+\frac{1}{q}\sinh\bigl(|x|^q\bigr)\sinh\bigl(|y|^p\bigr)$$
$$\ge\bigl|\sin\bigl(x|y|^{q-1}\bigr)\sin\bigl(x|y|^{p-1}\bigr)\bigr|,$$

respectively, for $x,y\in\mathbb{C}$ and $p>1$, $\frac{1}{p}+\frac{1}{q}=1$.

Similar results can be obtained for $\cosh(x)$ as well.

The following result also holds.

Theorem 2 *Let $f(z)$ and $g(z)$ be as in Theorem* 1. *Then one has the inequalities*

$$\frac{1}{p}g_A\bigl(|x|^p\bigr)f_A\bigl(|y|^q\bigr)+\frac{1}{q}f_A\bigl(|x|^p\bigr)g_A\bigl(|y|^q\bigr)\ge\bigl|f\bigl(|x|^{p-1}|y|^{q-1}\bigr)g(xy)\bigr|\tag{2.10}$$

and

$$\frac{1}{p}f_A\big(|x|^p\big)g_A\big(|y|^2\big)+\frac{1}{q}g_A\big(|x|^2\big)f_A\big(|y|^q\big)\geq\big|f(xy)g\big(|x|^{\frac{2}{q}}|y|^{\frac{2}{p}}\big)\big|. \tag{2.11}$$

Proof If we choose in (1.3), $x=\frac{|y|^k}{|y|^j}$, $y=\frac{|x|^k}{|x|^j}$, $|x|^j,|y|^j\neq 0$, $j,k\in\{0,1,2,\dots,n\}$, we have

$$\begin{aligned}p|y|^{qk}|x|^{pj}+q|x|^{pk}|y|^{qj}&\geq pq|x|^{(p-1)j}|y|^{(q-1)j}|xy|^k\\&=pq\big|\big(|x|^{p-1}|y|^{q-1}\big)^j(xy)^k\big|\end{aligned} \tag{2.12}$$

for any $j,k\in\{0,1,2,\dots,n\}$.

Multiplying (2.12) with $|p_j||q_k|\geq 0$ and summing over j and k from 0 to n, we obtain that

$$\begin{aligned}&p\sum_{k=0}^n|q_k||y|^{qk}\sum_{j=0}^n|p_j||x|^{pj}+q\sum_{k=0}^n|q_k||x|^{pk}\sum_{j=0}^n|p_j||y|^{qj}\\&\quad\geq pq\left|\sum_{j=0}^n p_j\big(|x|^{p-1}|y|^{q-1}\big)^j\sum_{k=0}^n q_k(xy)^k\right|.\end{aligned} \tag{2.13}$$

From (1.3), we also have the inequality

$$\begin{aligned}&p\sum_{k=1}^n|q_k||x|^{2k}\sum_{j=1}^n|p_j||y|^{qj}+q\sum_{j=1}^n|p_j||x|^{pj}\sum_{k=1}^n|q_k||y|^{2k}\\&\quad\geq pq\left|\sum_{j=1}^n p_j(xy)^j\sum_{k=1}^n q_k\big(|x|^{\frac{2}{q}}|y|^{\frac{2}{p}}\big)^k\right|\end{aligned} \tag{2.14}$$

for any $x,y\in C$, $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, which was obtained by choosing $x=|x|^{\frac{2}{q}k}|y|^j$, $y=|x|^j|y|^{\frac{2}{p}k}$ and repeating the same method as above.

Now, since all the series whose partial sums are involved in inequalities (2.13) and (2.14) are convergent on the disk $D(0,R)$, by letting $n\to\infty$ in (2.13) and (2.14), respectively, we deduce the desired inequalities, *i.e.*, (2.10) and (2.11). □

Corollary 2 *If* $g(z)=f(z)$ *in* (2.10) *and* (2.11), *then*

$$f_A\big(|x|^p\big)f_A\big(|y|^q\big)\geq\big|f(xy)f\big(|x|^{p-1}|y|^{q-1}\big)\big| \tag{2.15}$$

and

$$\frac{1}{p}f_A\big(|x|^p\big)f_A\big(|y|^2\big)+\frac{1}{q}f_A\big(|x|^2\big)f_A\big(|y|^q\big)\geq\big|f(xy)f\big(|x|^{\frac{2}{q}}|y|^{\frac{2}{p}}\big)\big|, \tag{2.16}$$

respectively, where $p>1$, $\frac{1}{p}+\frac{1}{q}=1$ *and* $x,y\neq 0$ *with* $xy,|x|^2,|x|^p,|x|^{\frac{2}{q}},|y|^2,|y|^q,|y|^{\frac{2}{p}}\in D(0,R)$. *In particular, if* $y=x$ *in* (2.15) *and* (2.16), *then we have*

$$f_A\big(|x|^p\big)f_A\big(|x|^q\big)\geq\big|f\big(x^2\big)f\big(|x|^{pq-2}\big)\big|$$

and

$$f_A\left(|x|^2\right)\left[\frac{1}{p}f_A\left(|x|^p\right)+\frac{1}{q}f_A\left(|x|^q\right)\right]\geq\left|f\left(x^2\right)f\left(|x|^2\right)\right|$$

for $x\neq 0$ *with* $x^2, |x|^2, |x|^p, |x|^q \in D(0,R)$.

Remark 2 *In the particular case* $p=q=2$ *in* (2.15) *or* (2.16), *we get the inequality*

$$f_A\left(|x|^2\right)f_A\left(|y|^2\right)\geq\left|f(xy)f\left(|xy|\right)\right|$$

for any $x,y\in\mathbb{C}$ *with* $xy, |xy|, |x|^2, |y|^2 \in D(0,R)$.

In what follows, we provide some applications of inequalities (2.15) and (2.16) to particular functions of interest.

(1) If we apply inequalities (2.15) and (2.16) to the function $f(z)=\frac{1}{1-z}$, $z\in D(0,1)$, then we get

$$|1-xy|\left|1-|x|^{p-1}|y|^{q-1}\right|\geq\left(1-|x|^p\right)\left(1-|y|^q\right) \tag{2.17}$$

and

$$\begin{aligned}&\frac{1}{p(1-|x|^p)(1-|y|^2)}+\frac{1}{q(1-|x|^2)(1-|y|^q)}\\&\quad\geq\frac{1}{|1-xy||1-|x|^{\frac{2}{q}}|y|^{\frac{2}{p}}|},\end{aligned}$$

respectively, where $p>1$, $\frac{1}{p}+\frac{1}{q}=1$ and $x,y\neq 0$ with $xy, |x|^2, |x|^p, |x|^{\frac{1}{q}}, |y|^2, |y|^q, |y|^{\frac{1}{p}} \in D(0,1)$.

(2) If we apply inequalities (2.15) and (2.16) to the function $f(z)=\exp(z)$, $z\in\mathbb{C}$, then we can state that

$$\exp\left(|x|^p+|y|^q\right)\geq\left|\exp\left(xy+|x|^{p-1}|y|^{q-1}\right)\right|$$

and

$$\begin{aligned}&\frac{1}{p}\exp\left(|x|^p+|y|^2\right)+\frac{1}{q}\exp\left(|x|^2+|y|^q\right)\\&\quad\geq\left|\exp\left(xy+|x|^{\frac{2}{q}}|y|^{\frac{2}{p}}\right)\right|,\end{aligned}$$

respectively, for any $x,y\in\mathbb{C}$ with $x,y\neq 0$.

(3) If we take the function $f(z)=\ln(\frac{1}{1-z})$, $z\in D(0,1)$, then from (2.15) and (2.16) we have

$$\ln\left(1-|x|^p\right)\ln\left(1-|y|^q\right)\geq\left|\ln(1-xy)\ln\left(1-|x|^{p-1}|y|^{q-1}\right)\right| \tag{2.18}$$

and

$$\frac{1}{p}\ln\left(1-|x|^p\right)\ln\left(1-|y|^2\right)+\frac{1}{q}\ln\left(1-|x|^2\right)\ln\left(1-|y|^q\right)\geq\left|\ln(1-xy)\ln\left(1-|x|^{\frac{2}{q}}|y|^{\frac{2}{p}}\right)\right|,$$

respectively, where $p>1$, $\frac{1}{p}+\frac{1}{q}=1$ and $x,y\neq 0$ with $xy,|x|^2,|x|^p,|x|^{\frac{1}{q}},|y|^2,|y|^q,|y|^{\frac{1}{p}}\in D(0,1)$.

(4) If we consider the function $f(z)=\sin(z)$, $z\in\mathbb{C}$, then we have $f_A(z)=\sinh(z)$, $z\in\mathbb{C}$. Applying inequalities (2.15) and (2.16) to this function, we get

$$\sinh\bigl(|x|^p\bigr)\sinh\bigl(|y|^q\bigr)\geq\bigl|\sin(xy)\sin\bigl(|x|^{p-1}|y|^{q-1}\bigr)\bigr|$$

and

$$\begin{aligned}&\frac{1}{p}\sinh\bigl(|x|^p\bigr)\sinh\bigl(|y|^2\bigr)+\frac{1}{q}\sinh\bigl(|x|^2\bigr)\sinh\bigl(|y|^q\bigr)\\&\quad\geq\bigl|\sin(xy)\sin\bigl(|x|^{\frac{2}{q}}|y|^{\frac{2}{p}}\bigr)\bigr|,\end{aligned}$$

respectively, where $p>1$, $\frac{1}{p}+\frac{1}{q}=1$ and $x,y\in\mathbb{C}$ with $x,y\neq 0$.

A similar result can be obtained for $\cosh(x)$ as well.

Theorem 3 *Let $f(z)$ and $g(z)$ be as in Theorem* 1. *Then one has the inequalities*

$$\begin{aligned}&\frac{1}{p}g_A\bigl(|x|^2\bigr)f_A\bigl(|y|^q\bigr)+\frac{1}{q}f_A\bigl(|x|^p\bigr)g_A\bigl(|y|^2\bigr)\\&\quad\geq\bigl|f\bigl(|x|^{p-1}|y|^{q-1}\bigr)g\bigl(|x|^{\frac{2}{p}}|y|^{\frac{2}{q}}\bigr)\bigr|\end{aligned}\tag{2.19}$$

and

$$\frac{1}{p}g_A\bigl(|x|^2\bigr)f_A\bigl(|y|^p\bigr)+\frac{1}{q}f_A\bigl(|x|^2\bigr)g_A\bigl(|y|^q\bigr)\geq\bigl|f\bigl(|x|^{\frac{2}{q}}y\bigr)g\bigl(|x|^{\frac{2}{p}}y\bigr)\bigr|.\tag{2.20}$$

Proof Follows from inequality (1.3) on choosing $x=\frac{|y|^{\frac{2}{q}k}}{|y|^j}$, $y=\frac{|x|^{\frac{2}{p}k}}{|x|^j}$ and $x=|x|^{\frac{2}{q}j}|y|^k$, $y=|x|^{\frac{2}{p}k}|y|^j$. That is, for any $i,j\in\{0,1,2,\ldots,n\}$, we have the following inequalities:

$$\begin{aligned}p|x|^{pj}|y|^{2k}+q|x|^{2k}|y|^{qj}&\geq pq|x|^{(p-1)j}|y|^{(q-1)j}|x|^{\frac{2}{p}k}|y|^{\frac{2}{q}k}\\&=pq\bigl|\bigl(|x|^{(p-1)}|y|^{(q-1)}\bigr)^j\bigl(|x|^{\frac{2}{p}}|y|^{\frac{2}{q}}\bigr)^k\bigr|\end{aligned}\tag{2.21}$$

and

$$\begin{aligned}p|x|^{2j}|y|^{qk}+q|x|^{2k}|y|^{pj}&\geq pq|x|^{\frac{2}{q}j}|y|^j|x|^{\frac{2}{p}k}|y|^k\\&=pq\bigl|\bigl(|x|^{\frac{2}{q}}y\bigr)^j\bigl(|x|^{\frac{2}{p}}y\bigr)^k\bigr|,\end{aligned}\tag{2.22}$$

respectively.

Repeating the same method as in Theorem 1 for (2.21) and (2.22), we deduce the desired inequalities, *i.e.*, (2.19) and (2.20). □

As a particular case of interest, we can state the following corollary.

Corollary 3 *If $g(z)=f(z)$ in* (2.19) *and* (2.20), *then*

$$\frac{1}{p}f_A\bigl(|x|^2\bigr)f_A\bigl(|y|^q\bigr)+\frac{1}{q}f_A\bigl(|x|^p\bigr)f_A\bigl(|y|^2\bigr)\geq\bigl|f\bigl(|x|^{p-1}|y|^{q-1}\bigr)f\bigl(|x|^{\frac{2}{p}}|y|^{\frac{2}{q}}\bigr)\bigr|\tag{2.23}$$

and

$$f_A\big(|x|^2\big)\left[\frac{1}{p}f_A\big(|y|^p\big)+\frac{1}{q}f_A\big(|y|^q\big)\right]\geq\big|f\big(|x|^{\frac{2}{q}}y\big)f\big(|x|^{\frac{2}{p}}y\big)\big|, \tag{2.24}$$

where $p>1$, $\frac{1}{p}+\frac{1}{q}=1$ *and* $x,y\neq 0$ *with* $|x|^2,|x|^p,|x|^q,|y|^2,|y|^p,|y|^q\in D(0,R)$. *In particular, if* $y=x$ *in* (2.23) *and* (2.24), *then we have*

$$f_A\big(|x|^2\big)\left[\frac{1}{p}f_A\big(|x|^q\big)+\frac{1}{q}f_A\big(|x|^p\big)\right]\geq\big|f\big(|x|^2\big)f\big(|x|^{pq-2}\big)\big|$$

and

$$f_A\big(|x|^2\big)\left[\frac{1}{p}f_A\big(|x|^p\big)+\frac{1}{q}f_A\big(|x|^q\big)\right]\geq\big|f\big(|x|^{\frac{2}{q}}x\big)f\big(|x|^{\frac{2}{p}}x\big)\big|$$

for $x\neq 0$, $|x|^2,|x|^p,|x|^q\in D(0,R)$.

Inequalities (2.23) and (2.24) are also valuable sources of particular inequalities for complex numbers as will be outlined in the following.

(1) If we apply inequalities (2.23) and (2.24) to the function $f(z)=\frac{1}{1-z}$, $z\in D(0,1)$, then we get

$$\begin{aligned}&\frac{1}{p(1-|x|^2)(1-|y|^q)}+\frac{1}{q(1-|x|^p)(1-|y|^2)}\\&\quad\geq\frac{1}{|(1-|x|^{p-1}|y|^{q-1})(1-|x|^{\frac{2}{p}}|y|^{\frac{2}{q}})|}\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{1-|x|^2}\left(\frac{1}{p(1-|y|^p)}+\frac{1}{q(1-|y|^q)}\right)\\&\quad\geq\frac{1}{|(1-|x|^{\frac{2}{q}}y)(1-|x|^{\frac{2}{p}}y)|},\end{aligned}$$

respectively, for any $x,y\in\mathbb{C}$, $x,y\neq 0$ with $|x|^2,|y|^2,|x|^p,|y|^q,|x|^{\frac{1}{p}},|x|^{\frac{1}{q}},|y|^{\frac{1}{p}},|y|^{\frac{1}{q}}\in D(0,1)$.

(2) If we apply inequalities (2.23) and (2.24) to the function $f(z)=\exp(z)$, $z\in\mathbb{C}$, then we can state that

$$\begin{aligned}&\frac{1}{p}\exp\big(|x|^2+|y|^q\big)+\frac{1}{q}\exp\big(|x|^p+|y|^2\big)\\&\quad\geq\big|\exp\big(|x|^{p-1}|y|^{q-1}+|x|^{\frac{2}{p}}|y|^{\frac{2}{q}}\big)\big|\end{aligned}$$

and

$$\begin{aligned}&\exp\big(|x|^2\big)\left[\frac{1}{p}\exp\big(|y|^p\big)+\frac{1}{q}\exp\big(|y|^q\big)\right]\\&\qquad\qquad\geq\big|\exp\big[\big(|x|^{\frac{2}{q}}+|x|^{\frac{2}{p}}\big)y\big]\big|,\end{aligned}$$

respectively, for $x,y\in\mathbb{C}$, $x,y\neq 0$.

(3) If we apply the function $f(z) = \ln(\frac{1}{1-z})$, $z \in D(0,1)$, then from (2.23) and (2.24) we have

$$\frac{1}{p}\ln\bigl(1-|x|^2\bigr)\ln\bigl(1-|y|^q\bigr)+\frac{1}{q}\ln\bigl(1-|x|^p\bigr)\ln\bigl(1-|y|^2\bigr) \\ \geq \bigl|\ln\bigl(1-|x|^{p-1}|y|^{q-1}\bigr)\ln\bigl(1-|x|^{\frac{2}{p}}|y|^{\frac{2}{q}}\bigr)\bigr|$$

and

$$\ln\bigl(1-|x|^2\bigr)\biggl[\frac{1}{p}\ln\bigl(1-|y|^p\bigr)+\frac{1}{q}\ln\bigl(1-|y|^q\bigr)\biggr] \\ \geq \bigl|\ln\bigl(1-|x|^{\frac{2}{q}}y\bigr)\ln\bigl(1-|x|^{\frac{2}{p}}y\bigr)\bigr|,$$

respectively, for any $x, y \in \mathbb{C}$, $x, y \neq 0$ with $|x|^2, |y|^2, |x|^p, |y|^q, |x|^{\frac{1}{p}}, |x|^{\frac{1}{q}}, |y|^{\frac{1}{p}}, |y|^{\frac{1}{q}} \in D(0,1)$.

(4) If we consider the function $f(z) = \sin(z)$, $z \in \mathbb{C}$, then we have $f_A(z) = \sinh(z)$, $z \in \mathbb{C}$. Applying inequalities (2.23) and (2.24) to this function, we get

$$\frac{1}{p}\sinh\bigl(|x|^2\bigr)\sinh\bigl(|y|^q\bigr)+\frac{1}{q}\sinh\bigl(|x|^p\bigr)\sinh\bigl(|y|^2\bigr) \\ \geq \bigl|\sin\bigl(|x|^{p-1}|y|^{q-1}\bigr)\sin\bigl(|x|^{\frac{2}{p}}|y|^{\frac{2}{q}}\bigr)\bigr|$$

and

$$\sinh\bigl(|x|^2\bigr)\biggl[\frac{1}{p}\sinh\bigl(|y|^p\bigr)+\frac{1}{q}\sinh\bigl(|y|^q\bigr)\biggr] \geq \bigl|\sin\bigl(|x|^{\frac{2}{q}}y\bigr)\sin\bigl(|x|^{\frac{2}{p}}y\bigr)\bigr|,$$

respectively, for any $x, y \in \mathbb{C}$, $x, y \neq 0$.

A similar result can be obtained for $\cosh(x)$ as well.

3 Applications to special functions

In this section, we give some inequalities for some special functions such as polylogarithm, hypergeometric, Bessel and modified Bessel functions for the first kind. Before that, we state here some basic concepts and definitions of those functions.

The *polylogarithm* $Li_n(z)$ is a function defined by the power series

$$Li_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n} \tag{3.1}$$

which converges absolutely for all complex values of the order n and the argument z where $|z| < 1$. It is also known in the literature as *Jonquiére's function.* The special cases $z = -1, 1$ reduce to $Li_n(1) = \zeta(n)$ and $Li_n(-1) = -\eta(n)$, where ζ and η are the *Riemann zeta function* and *Dirichlet eta function*, respectively. When $n = 1$, the first polylogarithm involves the ordinary logarithm, *i.e.*, $Li_1(z) = -\ln(1-z)$, while the second

$$Li_2 = \sum_{k=1}^{\infty} \frac{z^k}{k^2} \tag{3.2}$$

is called the *dilogarithm* or *Spence's function.*

For other integer values of order n, the polylogarithm reduces to the ratio of a polynomial in z, for instance,

$$Li_0(z) = \frac{z}{1-z}, \qquad Li_{-1}(z) = \frac{z}{(1-z)^2},$$
$$Li_{-2}(z) = \frac{z(z+1)}{(1-z)^3}, \qquad Li_{-3}(z) = \frac{z(1+4z+z^2)}{(1-z)^4}.$$

The *hypergeometric function* ${}_2F_1(a,b;c;z)$ is defined for all $|z|<1$ by the series

$$ {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{z^n}{n!}, \tag{3.3}$$

where $a,b,c \in \mathbb{R}$ with $c \neq 0,-1,-2,\ldots$ and the $(t)_n$, $n \in \{0,1,2,\ldots\}$ is a *Pochhammer symbol* which is defined by

$$(t)_n = \begin{cases} 1 & \text{if } n = 0, \\ t(t+1)\cdots(t+n-1) & \text{if } n > 0. \end{cases}$$

Hypergeometric function (3.3) with particular arguments of a, b and c reduces to elementary functions. For instance,

$$ {}_2F_1(1,1;1;z) = {}_2F_1(1,2;2;z) = \frac{1}{1-z},$$
$$ {}_2F_1(1,2;1;z) = \frac{1}{(1-z)^2},$$
$$ {}_2F_1(a,b;b;z) = \frac{1}{(1-z)^a},$$
$$ {}_2F_1(1,1;2;z) = \frac{1}{z}\ln\left(\frac{1}{1-z}\right),$$
$$ {}_2F_1(1,1;2;-z) = \frac{1}{z}\ln(1+x).$$

Further, the *Bessel functions* of the first kind, denoted as $J_\alpha(z)$, are defined by the power series

$$J_\alpha(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(\alpha+k)!}\left(\frac{z}{2}\right)^{2k+\alpha} \tag{3.4}$$

for $\alpha, z \in \mathbb{C}$ with $|z|<1$. If z is replaced by arguments $\pm iz$, then from (3.4) we have

$$I_\alpha(z) = i^{-\alpha} J_\alpha(iz) = \sum_{k=0}^{\infty} \frac{1}{k!(\alpha+k)!}\left(\frac{z}{2}\right)^{2k+\alpha} \tag{3.5}$$

for $\alpha, z \in \mathbb{C}$ with $|z|<1$. These functions (3.5) are called the *modified Bessel functions* of the first kind.

It is clearly seen that from (3.1), (3.3), (3.4) and (3.5), that is, $Li_n(z)$, ${}_2F_1(a,b;c;z)$, $J_\alpha(z)$ and $I_\alpha(z)$ are power series with real coefficients and convergent on the open disk $D(0,1)$.

Therefore, all the results in the above section hold true. For instance, from (2.15) we have the following corollaries.

Corollary 4 *If $Li_n(z)$ is the polylogarithm function, then we have*

$$Li_n\bigl(|x|^p\bigr)Li_n\bigl(|y|^q\bigr) \geq \bigl|Li_n(xy)Li_n\bigl(|x|^{p-1}|y|^{q-1}\bigr)\bigr| \tag{3.6}$$

for any $x, y \in \mathbb{C}$, $x, y \neq 0$ with $xy, |x|^p, |y|^q \in D(0,1)$ and $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

In particular, if $n = 0$ in (3.6), then we have the following inequality:

$$|1 - xy|\bigl|1 - |x|^{p-1}|y|^{q-1}\bigr| \geq \bigl(1 - |x|^p\bigr)\bigl(1 - |y|^q\bigr)$$

for all $x, y \neq 0$, $xy, |x|^p, |y|^q \in D(0,1)$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

If we take $n = 1$ in (3.6), then we get inequality (2.18) for all $x, y \neq 0$ with $xy, |x|^p, |y|^q \in D(0,1)$ and $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

Also, if we choose in (3.6) $n = 2$, then we obtain

$$Li_2\bigl(|x|^p\bigr)Li_2\bigl(|y|^q\bigr) \geq \bigl|Li_2(xy)Li_2\bigl(|x|^{p-1}|y|^{q-1}\bigr)\bigr|$$

for any $x, y \neq 0$, $xy, |x|^p, |y|^q \in D(0,1)$ and $p > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. $Li_2(z)$ is the dilogarithm function which is defined in (3.2).

Corollary 5 *If ${}_2F_1(a,b;c;z)$ is a hypergeometric function, then for any $a, b, c \in \mathbb{R}$, we have*

$$\begin{aligned} &{}_2F_1\bigl(a,b;c;|x|^p\bigr){}_2F_1\bigl(a,b;c;|y|^q\bigr) \\ &\quad \geq \bigl|{}_2F_1(a,b;c;xy){}_2F_1\bigl(a,b;c;|x|^{p-1}|y|^{q-1}\bigr)\bigr|, \end{aligned} \tag{3.7}$$

where $x, y \neq 0$ with $xy, |x|^p, |y|^q \in D(0,1)$ and $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

In particular, if we choose $a = 1$, $c = b$ in (3.7), then we get inequality (2.17). Also, if we choose $a = b = 1$, $c = 2$, then inequality (3.7) reduces to (2.18).

Corollary 6 *If $J_\alpha(z)$ and $I_\alpha(z)$ are the Bessel and modified Bessel functions of the first kind, respectively, then for any $\alpha, x, y \in \mathbb{C}$, we have*

$$I_\alpha\bigl(|x|^p\bigr)I_\alpha\bigl(|y|^q\bigr) \geq \bigl|J_\alpha(xy)J_\alpha\bigl(|x|^{p-1}|y|^{q-1}\bigr)\bigr|, \tag{3.8}$$

where $x, y \neq 0$, $xy, |x|^p, |y|^q \in D(0,1)$ and $p > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

In particular, if $\alpha = 0$ in (3.8), then for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we get

$$J_0\bigl(i|x|^p\bigr)J_0\bigl(i|y|^q\bigr) \geq \bigl|J_0(xy)J_0\bigl(|x|^{p-1}|y|^{q-1}\bigr)\bigr|,$$

where $J_0(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2}(\frac{z}{2})^{2k}$.

Other inequalities involving the polylogarithm, hypergeometric, Bessel and modified Bessel functions can be found in the literature (see [23–28] and references therein).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The first author AI is currently a PhD student under supervision of the second author SSD and the third author MD is the co-supervisor. They jointly worked on deriving the results. All authors read and approved the final manuscript.

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