academicJournals

Vol. 8(44), pp. 2163-2170, 25 November, 2013
DOI: 10.5897/SRE2013.5640
ISSN 1992-2248 © 2013 Academic Journals
http://www.academicjournals.org/SRE

Scientific Research and Essays

Full Length Research Paper

Numerical solutions for the time fractional variant Bussinesq equation by homotopy analysis method

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Accepted 6 November, 2013

In this article, the fractional derivatives in the sense of Caputo, and the homotopy analysis method (HAM) are used to construct the approximate solutions for the nonlinear fractional variant Bussinesq equations with respect to time fractional derivative. The HAM contains a certain auxiliary h parameter which provides us a simple way to adjust and control the convergence region and rate of convergence of the series solution. This method is efficient and powerful in solving wide classes of nonlinear evolution fractional order equation.

Key words: Fractional calculus, homotopy analysis method, Caputo's fractional derivative, fractional variant Bussinesq equation.

INTRODUCTION

In recent years, there has been a great deal of interest in fractional differential equations. First there were almost no practical applications of fractional calculus, and it was considered by many as an abstract area containing only mathematical manipulations of little or no use. Nearly 30 years ago, the paradigm began to shift from pure mathematical formulations to applications in various fields. During the last decade fractional calculus has been applied to almost every field of science, engineering, and mathematics. Several fields of application of fractional differentiation and fractional integration are already well established, some others have just started.

Fractional differential equations (FDEs) have found applications in many problems in physics and engineering (Oldham and Spanier, 1974; Miller and Ross, 1993; Luchko and Gorenflo 1998; Podlubny, 1999; Angelina, 2012; El-Ajou et al., 2010). Since most of the nonlinear FDEs cannot be solved exactly, approximate and numerical methods must be used. Some of the recent analytical approximate methods for solving nonlinear problems include the Adomian decomposition method ADM (Daftardar-Gejji and Bhalekar, 2008; Daftardar-Gejji and Jafari, 2007; Herzallah and Gepreel, 2012), variational iteration method VIM (Sweilam et al., 2007), homotopy-perturbation method (HPM) (Golbabai and Sayevand, 2011; Golbabai and Sayevand, 2010; Gepreel, 2011) and homotopy analysis method homotopy analysis method (HAM) (Liao, 1992, 1995; Gepreel and Saad, 2014; Saad, 2011, 2012; Saad and Ghany, 2011; Ghany and Saad, 2012; Abu and El-Ajou, 2013; Abu Arqub et al., 2013; El-Ajou et al., 2013). The HAM, first proposed in 1992 by Liao (1992), has been successfully applied to solve many problems in physics and science. This method is applied to solve linear and

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Abbreviations: FDEs, Fractional differential equations; **HPM,** homotopy-perturbation method; **HAM,** homotopy analysis method.

nonlinear fractional order systems. Recently, many system have been identified that display fractional-order dynamics, such as viscoelastic systems, quantum evolution of complex system, and the control of fractional-order dynamic systems (Ahmed et al., 2006, 2007). Gepreel and Omran (2012) and Khaled and Mohamed (2013) have used the complex transformation to calculate the exact solution for some nonlinear fractional differential equations. In this work, a new algorithm for solving the fractional order variant Bussinesq equation (Lu, 2005) is proposed based on HAM

$$\begin{aligned} D_t^{\alpha}u+(uv)_x+v_{xxx}&=0,\\ D_t^{\alpha}v+u_x+vv_x&=0. \end{aligned} \tag{1}$$

Where, $D_t^{\alpha}u=\frac{\partial^{\alpha}u}{\partial t^{\alpha}}$, α is the parameter standing for the order of the fractional time derivative, and satisfy $0<\alpha<1$. We use the Caputo fractional derivative on the half axis R^+ (that is, $t\in R^+$) ${}_t^C D_{0_+}^{\alpha}$ for time. Subject to the initial conditions

$$\begin{aligned} u(x,0)&=f(x)=k^2+k^2m^2-2k^2m^2\,sn^2(kx,m),\\ v(x,0)&=g(x)=\lambda+2mk\,sn(kx,m) \end{aligned} \tag{2}$$

Where, $sn(kx,m)$ is a Jacobi elliptic Sine functions with the modulus m and λ,k are arbitrary constants ($0<m<1$).

In this article, we present an alternative approach based on HAM to the system of FDEs (Oldham and Spanier, 1974) with initial conditions (Miller and Ross, 1993), to obtain the analytic approximations solutions for fractional order variant Bussinesq equation and demonstrate the effectiveness of the HAM algorithm. We use the h-curve to control the convergence of series solutions.

BASIC DEFINITIONS

Here, we give some definitions and properties of the fractional calculus. Several definitions of fractional calculus have been proposed in the last two centuries. The fractional calculus is a name for the theory of integrals and derivatives of arbitrary order. There are many books (Oldham and Spanier, 1974; Miller and Ross, 1993; Luchko and Gorenflo, 1998; Podlubny, 1999) that develop fractional calculus and various definitions of fractional integration and differentiation, such as Grunwald-Letnikov's definition, Riemann-Liouville definition, Caputo's definition and generalized function approach.

Definition 1

A real function $h(t),\ t\circledast 0$, is said to be in the space $C_{\mu},\ \mu\in\mathbb{R}$ if there exists a real number $p>\mu,$ such that $h(t)=t^p h_1(t),$ where $h_1(t)\in C[0,\infty),$ and it is said to be in the space C_{μ}^n if and only if $h^{(n)}\in C_{\mu},$ $n\in \mathrm{N}.$

Definition 2

The Riemann-Liouville fractional integral operator J^{α} of order $\alpha\geq 0,$ of a function $h\in C_{\mu},\ \mu\geq -1,$ is defined as

$$\begin{aligned} J^{\alpha}h(t)&=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}h(\tau)d\tau\quad(\alpha>0),\\ J^{0}h(t)&=h(t), \end{aligned} \tag{3}$$

$\Gamma(\alpha)$ is the well- known Gamma function. Some of the properties of the operator J^{α} , which we will need here, are as follows:

$$J^{\alpha}J^{\beta}h(t)=J^{\alpha+\beta}h(t),$$
$$J^{\alpha}J^{\beta}h(t)=J^{\beta}J^{\alpha}h(t),$$
$J^{\alpha}t^{\gamma}=\frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}t^{\alpha+\gamma}$, Where $\beta\geq 0$ and $\gamma\geqslant -1.$

Definition 3

The fractional derivative (D^{α}) of $h(t)$ in the Caputo's sense is defined by

$$D^{\alpha}h(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t(t-\tau)^{n-\alpha-1}h^{(n)}(\tau)d\tau\ ,$$
$$\text{for } n-1<\alpha\leq n,\ n\in\mathbb{N},\ t>0,\ h\in C_{-1}^n. \tag{4}$$

The following are two basic properties of Caputo's fractional Derivative [4]:

Let $h\in C_{-1}^n,\ n\in\mathbb{N}.$ Then $D^{\alpha}h,\ 0\leq\alpha\leq n$ is well defined and $D^{\alpha}h\in C_{-1}.$

Let $n-1<\alpha\leq n,\ n\in\mathbb{N}$ and $h\in C_{\mu}^n,\ \mu\geq -1$ Then

$$(J^{\alpha}D^{\alpha})h(t)=h(t)-\sum_{k=0}^{n-1}h^{(k)}(0^+)\frac{t^k}{k!}. \tag{5}$$

HOMOTOPY ANALYSIS METHOD FOR SYSTEM OF FDEs

Let us consider the following system of differential equation

$$N_i(u_1(t),.....,u_n(t))=0,\quad i=1,2,....n, \tag{6}$$

Subject to the following initial conditions at initial value:

$$u_k(t)=c_k,\quad k=1,...,n,$$

Where, N_i are nonlinear operators, t denotes an independent operator and $u_i(t)$ are the unknown functions. We can construct the following Zeroth-order deformation for $i=1,2,....n,$

$$(1-q)\ell_i(\phi_i(t;q)-u_{i0}(t))=qh_iH_i(t)N_i[\phi_1(t;q),.....,\phi_n(t;q)], \tag{7}$$

Where, $q\in[0,1]$ is an embedding parameter, $h_i\neq 0$ are auxiliary parameters, $H_i(t)\neq 0$ are auxiliary functions, $\ell_i=D_t^{\alpha_i}(n-1<\alpha_i\leq n)$ are auxiliary linear operators such that

$$\ell_i[\phi_i(t)]=0 \text{ when } \phi_i(t)=0. \tag{8}$$

Generally, $u_{i0}(t)$ are initial guesses, which satisfy the initial conditions and $\phi_i(t\ ;q)$ are unknown functions where

$$\phi_i(t;0)=u_{i0}(t),\ \ \phi_i(t;1)=u_i(t),\quad i=1,2,....n. \tag{9}$$

Expanding $\phi_i(t\ ;\ q)$ in Taylor series we have

$$\phi_i(t;q)=u_{i0}(t)+\sum_{m=1}^{\infty}u_{im}(t)q^m,\quad i=1,2,....n, \tag{10}$$

Where,

$$u_{im}(t)=\frac{1}{m!}\frac{\partial^m\phi_i(t;q)}{\partial q^m}|_{q=0},\quad i=1,2,....n. \tag{11}$$

If the auxiliary parameters h_i , the auxiliary functions $H_i(t),$ the initial approximations $u_{i0}(t)$ and the auxiliary linear operators $\mathcal{L}_i$ are so properly chosen the series (Equation 10) converges at $q=1.$ Then, using Equation 9, the series (Equation 10) becomes

$$u_i(t)=u_{i0}(t)+\sum_{m=1}^{\infty}u_{im}(t),\qquad i=1,2,....n. \tag{12}$$

Let us, we define the following vectors

$$\vec{u_i}(t)=\{u_{i0}(t),u_{i1}(t),u_{i2}(t),.....,u_{in}(t)\},\quad i=1,2,....n. \tag{13}$$

Then differentiating Equation 7 m times with respect to q , setting $q=0$ and dividing by $m!$, we have the mth -order deformation equation

$$\ell_i[u_{im}(t)-x_mu_{im-1}(t)]=h_iH_i(t)\mathbf{R}_{im}(\vec{u_{1m-1}},\vec{u_{2m-1}},.....,\vec{u_{nm-1}}), \tag{14}$$

Where,

$$R_{im}(\vec{u_{1m-1}},\vec{u_{2m-1}},.....,\vec{u_{nm-1}})=\frac{1}{(m-1)!}\frac{\partial^{m-1}N_i[\phi_1(t;q),.....,\phi_n(t;q)]}{\partial q^{m-1}}|_{q=0}, \tag{15}$$

And

$$x_m=\begin{cases}0 & m\leq 1,\\ 1 & m>1.\end{cases} \tag{16}$$

Applying the Riemann-Liouville integral operator J^{α_i} on both side of Equation 14, we get

$$u_{im}(t)=x_mu_{im-1}(t)-x_m\sum_{j=0}^{n-1}u_{im-1}^{(j)}(0^+)\frac{t^j}{j!}+h_iH_i(t)J^{\alpha_i}[\mathbf{R}_{im}(\vec{u_{1m-1}},\vec{u_{2m-1}},.....,\vec{u_{nm-1}})], \tag{17}$$

The mth -order deformation Equations 14 are linear and they can be easily solved, especially by means of symbolic computation software such as Mathematic, Maple, Mat Lab.

APPLICATION

To demonstrate the effectiveness of the method, we consider the system of nonlinear partial fractional (Oldham and Spanier, 1974) with the initial conditions (Miller and Ross, 1993). We let

$$\begin{aligned}\ell_1[\phi_1(t;q)]&=D_t^{\alpha}[\phi_1(t;q)],\\ \ell_2[\phi_2(t;q)]&=D_t^{\alpha}[\phi_2(t;q)],\end{aligned} \tag{18}$$

with the property $\ell_i[c_i]=0, i=1,2$ where c_i are the integral constant and the nonlinear operators are defined as

$$N_1[\phi_1,\phi_2] = D_t^{\alpha}\phi_1 + (\phi_1\phi_1)_x + (\phi_2)_{xxx},$$
$$N_2[\phi_1,\phi_2] = D_t^{\alpha}\phi_2 + (\phi_1)_x + \phi_2(\phi_2)_x.$$

Choosing $H_i(t) = 1$ for $i = 1,2$ the zeroth-order deformation equations are

$$\begin{aligned}(1-q)\ell_1[\phi_1(t;q) - u_0(t)] &= qhN_1[\phi_1,\phi_2],\\ (1-q)\ell_2[\phi_2(t;q) - v_0(t)] &= qhN_2[\phi_1,\phi_2],\end{aligned} \tag{19}$$

Where,

$$\begin{aligned}\phi_1(t;0) &= u_0(t), \quad \phi_1(t;1) = u(t),\\ \phi_2(t;0) &= v_0(t), \quad \phi_2(t;1) = v(t).\end{aligned} \tag{20}$$

Then, the mth -order deformation equations become

$$\begin{aligned}\ell_1[u_m(t) - x_m u_{m-1}(t)] &= h\mathbf{R}_{1m}(\vec{u_{m-1}}, \vec{v_{m-1}}),\\ \ell_2[v_m(t) - x_m v_{m-1}(t)] &= h\mathbf{R}_{2m}(\vec{u_{m-1}}, \vec{v_{m-1}}),\end{aligned} \tag{21}$$

Where,

$$\mathcal{R}_{1m}(\vec{u_{m-1}}, \vec{v_{m-1}}) = D_t^{\alpha}u_{m-1} + \sum_{j=0}^{m-1} v_j(u_{m-1-j})_x + \sum_{j=0}^{m-1} u_j(v_{m-1-j})_x + (v_{m-1})_{xxx},$$

$$\mathcal{R}_{2m}(\vec{u_{m-1}}, \vec{v_{m-1}}) = D_t^{\alpha}v_{m-1} + \sum_{j=0}^{m-1} v_j(v_{m-1-j})_x + (u_{m-1})_x.$$

The system (Equation 21) has the following general solutions

$$\begin{aligned}u_m(x,t) &= x_m u_{m-1}(t) + h\mathcal{F}^{\alpha}[D_t^{\alpha}u_{m-1} + \sum_{j=0}^{m-1} v_j(u_{m-1-j})_x + \sum_{j=0}^{m-1} u_j(v_{m-1-j})_x + (v_{m-1})_{xxx}],\\ v_m(x,t) &= x_m v_{m-1}(t) + h\mathcal{F}^{\alpha}[D_t^{\alpha}v_{m-1} + \sum_{j=0}^{m-1} v_j(v_{m-1-j})_x + (u_{m-1})_x].\end{aligned} \tag{22}$$

Where, u_0 and v_0 are the initial conditions, Equations (22) leads to get:

$$\begin{aligned}u_m(x,t) &= (x_m + h)u_{m-1}(t) + (x_m - 1)hu_{m-1}(t) + h\mathcal{F}^{\alpha}[\sum_{j=0}^{m-1} v_j(u_{m-1-j})_x + \sum_{j=0}^{m-1} u_j(v_{m-1-j})_x + (v_{m-1})_{xxx}]\\ v_m(x,t) &= (x_m + h)v_{m-1}(t) + (x_m - 1)hv_{m-1}(t) + h\mathcal{F}^{\alpha}[\sum_{j=0}^{m-1} v_j(v_{m-1-j})_x + (u_{m-1})_x].\end{aligned} \tag{23}$$

Finally, we have

$$\begin{aligned}u(x,t) &= u_0 + \sum_{m=1}^{\infty} u_m(x,t),\\ v(x,t) &= v_0 + \sum_{m=1}^{\infty} v_m(x,t).\end{aligned} \tag{24}$$

Substituting Equation (2) into Equation (24) we have

$$\begin{aligned}u_1(x,t) &= h f_1(x)\frac{t^{\alpha}}{\Gamma(\alpha+1)},\\ v_1(x,t) &= h g_1(x)\frac{t^{\alpha}}{\Gamma(\alpha+1)},\end{aligned} \tag{25}$$

$$\begin{aligned}u_2(x,t) &= (1+h)u_1(x,t) + h^2 f_2(x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},\\ v_2(x,t) &= (1+h)v_1(x,t) + h^2 g_2(x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\end{aligned} \tag{26}$$

$$\begin{aligned}u_3(x,t) &= (1+h)u_2(x,t) + h^3 f_3(x)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + h^2(1+h)f_2(x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},\\ v_3(x,t) &= (1+h)v_2(x,t) + h^3 g_3(x)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + h^2(1+h)g_2(x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}.\end{aligned} \tag{27}$$

and so on. After some calculation, we get

$$\begin{aligned}f(x) &= k^2 + k^2 m^2 - 2k^2 m^2\, sn^2(kx,m),\\ f_1(x) &= g_{xxx} + f(g)_x + g(f)_x,\\ f_2(x) &= (g_1)_{xxx} + g_1 f_x + f_1(g)_x + f(g_1)_x + g(f_1)_x\\ f_3(x) &= \frac{1}{3}(g_2)_{xxx} + f(g_2)_x + f_2(g)_x + g_1(f_1)_x\frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} + f_1(g_1)_x\\ &\quad \frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} + g_2(f)_x + g(f_2)_x,\\ &\quad \cdots\cdots\end{aligned} \tag{28}$$

and

$$\begin{aligned}g(x) &= \lambda + 2mk\, sn(kx,m),\\ g_1(x) &= (f)_x + g(g)_x,\\ g_2(x) &= (f_1)_x + g_1 g_x + g(g_1)_x,\\ g_3(x) &= (f_2)_x + g(g_2)_x + g_1(g_1)_x\frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} + g_2(g)_x,\\ &\quad \cdots\cdots\end{aligned} \tag{29}$$

In this case the approximate solution of time-fractional Equation (1) according to the HAM takes the following form

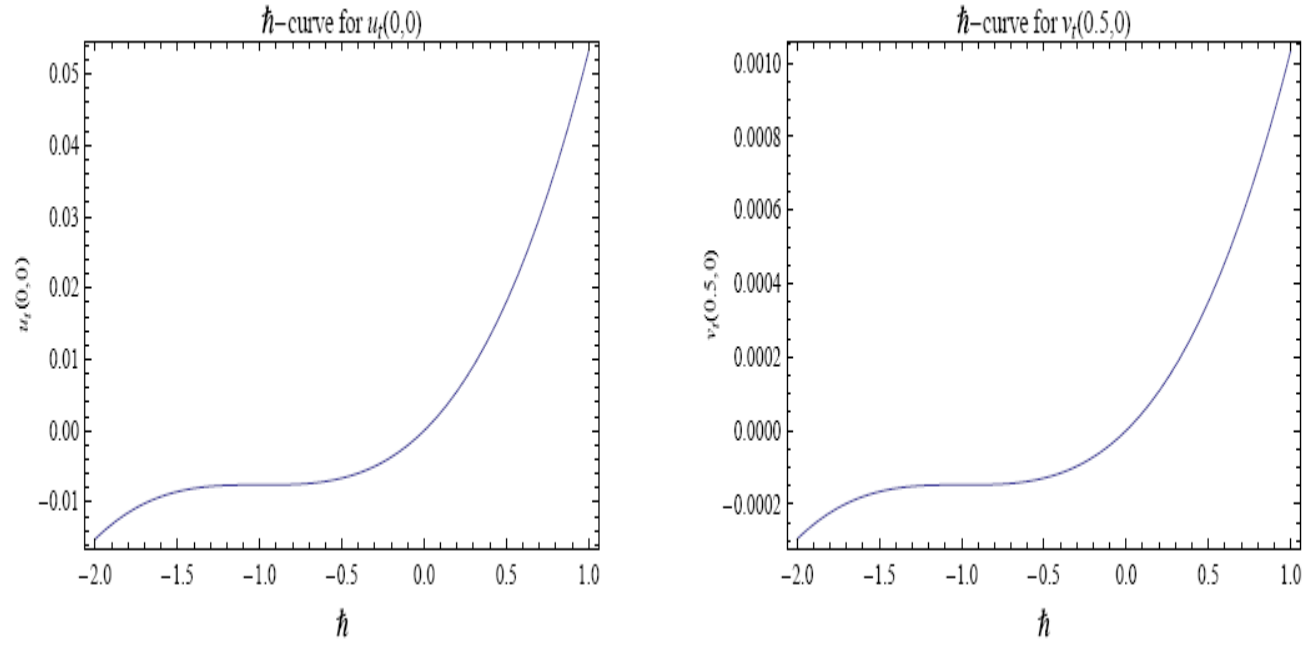


Figure 1. The h-curve of the 4th order approximate solution (5.13) when α = 0.75.

$$u_{app} = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \ldots\ldots,$$
$$v_{app} = v_0(x,t) + v_1(x,t) + v_2(x,t) + v_3(x,t) + \ldots\ldots,$$

Then,

$$u_{app} = f(x) + h f_1(x)\frac{t^{\alpha}}{\Gamma(\alpha+1)} + h(1+h) f_1(x)\frac{t^{\alpha}}{\Gamma(\alpha+1)} + h^2 f_2(x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + (1+h)(h(1+h) f_1\frac{t^{\alpha}}{\Gamma(\alpha+1)} + h^2 f_2(x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}) + h^3 f_3\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + h^2(1+h) f_2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \ldots, \quad (30)$$

$$v_{app} = g(x) + h g_1(x)\frac{t^{\alpha}}{\Gamma(\alpha+1)} + h(1+h) g_1(x)\frac{t^{\alpha}}{\Gamma(\alpha+1)} + h^2 g_2(x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + (1+h)(h(1+h) g_1(x)\frac{t^{\alpha}}{\Gamma(\alpha+1)} + h^2 g_2(x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}) + h^3 g_3(x)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + h^2(1+h) g_2(x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \ldots \quad (31)$$

and so on. In the special case when $h=-1$, the homotopy analysis method equivalent the homotopy perturbation method and the Adomain decomposition method and take the following form:-

$$u_{app}\,|_{\text{ADM}} = u_{app}\,|_{\text{HAM}} = f(x) - f_1(x)\frac{t^{\alpha}}{\Gamma(\alpha+1)} + f_2(x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - f_3(x)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \ldots$$
$$= k^2 + k^2m^2 - 2k^2m^2\, sn^2(kx,m) + 4tk^3m^2\, cn(kx,m)dn(kx,m)sn(kx,m)\lambda - 2k^4m^2t^2\lambda^2[1 - 2sn^2(kx,m) + 3m^2sn^4(kx,m) - 2m^2sn^2(kx,m)] + \frac{4}{3}k^5m^2\lambda^2 cn(kx,m)dn(kx,m)t^3[-2\lambda sn(kx,m) + 6m^2\lambda sn^3(kx,m) - 2m^2\lambda sn(kx,m)] + \ldots \quad (32)$$

and

$$v_{app}\,|_{\text{ADM}} = v_{app}\,|_{\text{HAM}} = g(x) - g_1(x)\frac{t^{\alpha}}{\Gamma(\alpha+1)} + g_2(x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - g_3(x)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \ldots$$
$$= \lambda + 2mk\, sn(kx,m) + 2tk^2mcn(kx,m)dn(kx,m) - k^3mt^2\lambda^2 sn(kx,m)[-1 + 2m^2\, sn^2(kx,m) - m^2] + \frac{1}{3}k^4m\lambda^2 cn(kx,m)dn(kx,m)t^3[-\lambda + 6m^2\lambda sn^2(kx,m) - m^2\lambda] + \ldots \quad (33)$$

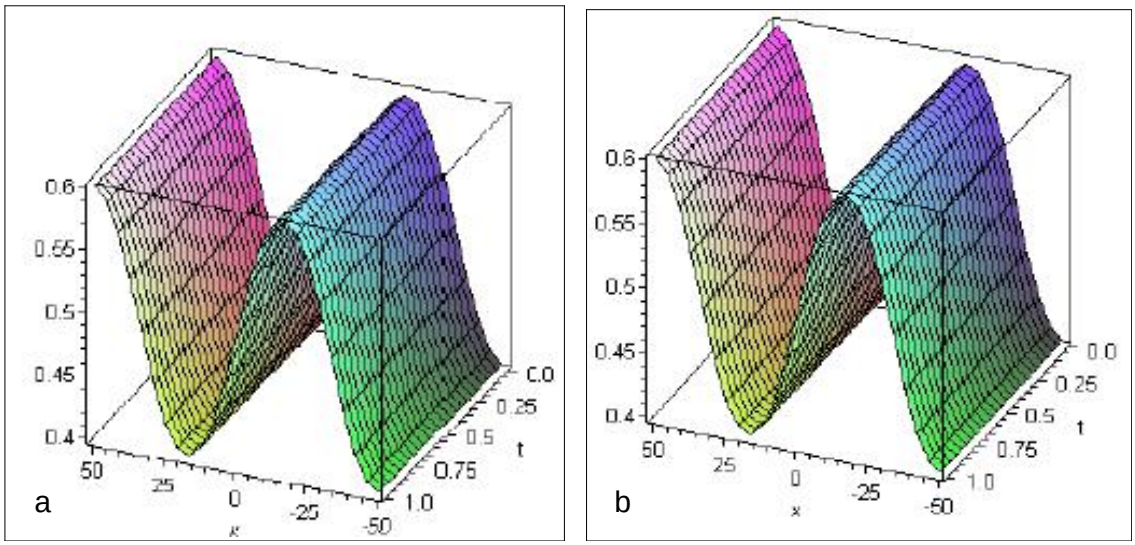
Using the Taylor series expansion near $t=0$, we have

$$u(x,t)\,|_{\text{Exact}} = k^2 + k^2m^2 - 2k^2m^2\, sn^2(k(x-\lambda t),m), \quad (34)$$

and

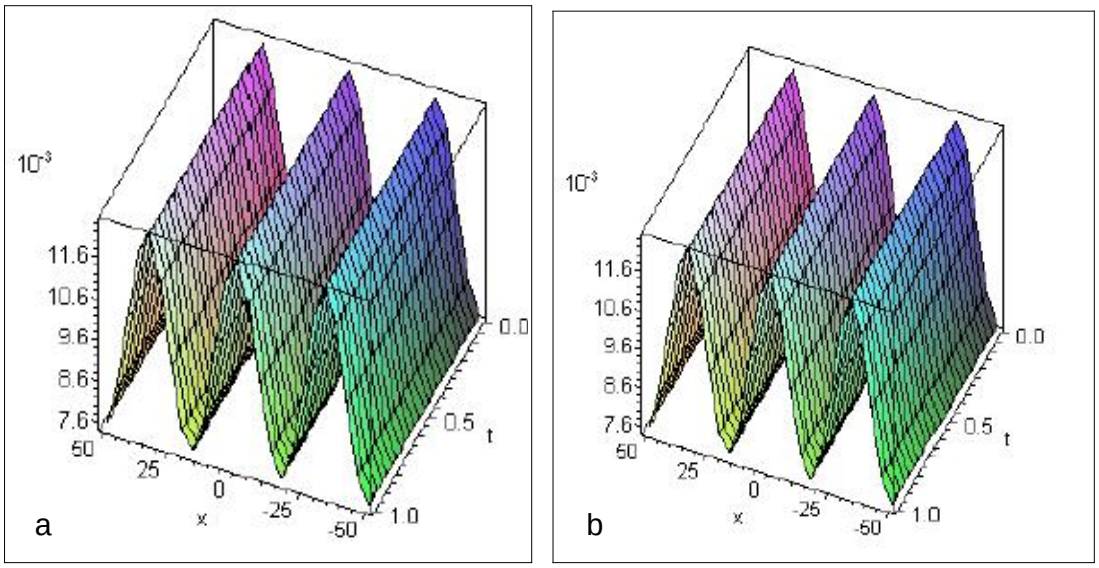
$$v(x,t)\,|_{\text{Exact}} = \lambda + 2mk\, sn(k(x-\lambda t),m). \quad (35)$$

These are the exact solutions of the variant Bussinesq equations partial differential Equation (1) obtained when replacing $\alpha = 1$ and $h=-1$ in Equations 31 and 32. These solutions (Equations 35 and 36) are exactly the same solutions obtained in Equation 28. The h - curves have been plotted in Figure1 for $u_t(0,0)$ and $v_t(0,0)$ as we have expressed, it is easy to check that the HAM solution is convergent to the exact solutions when -1.4<h<-0.6.



V_{Exact} V_{app}

Figure 2. The approximate solution (Equation 34) shown in the figure b) in comparison with the exact solution (36) shown in (a) when $\alpha = 1, m = 0.5, k = 0.1, \lambda = 0.5, -50 < x < 50$ and $0 < t < 1.$

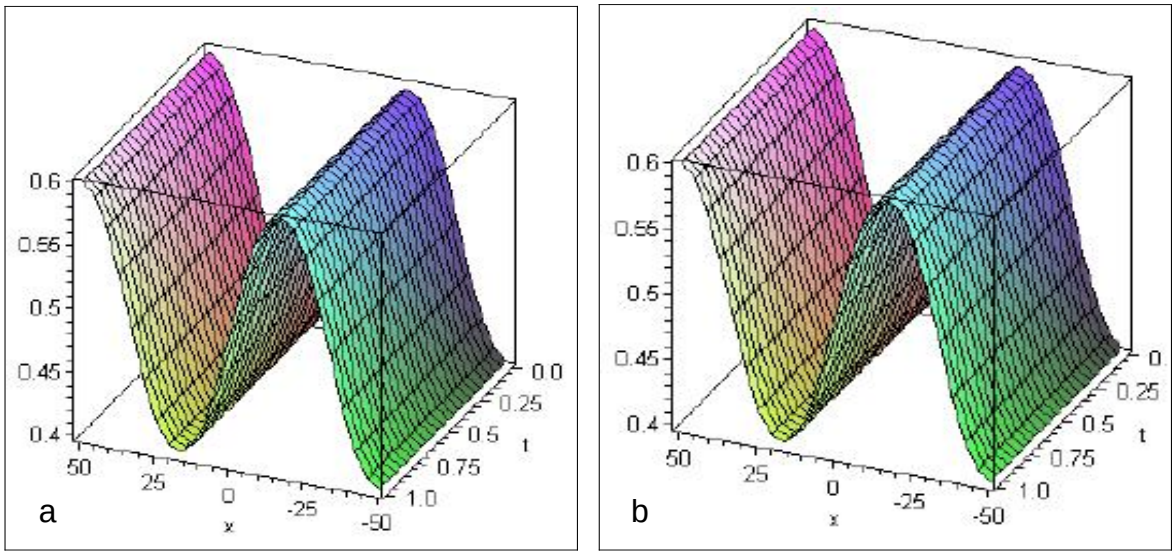


U_{Exact} U_{app}

Figure 3. The approximate solution (33) shown in the figure (b) in comparison with the exact solution (35) shown in (a) when $\alpha = 1, m = 0.5, k = 0.1, \lambda = 0.5, -50 < x < 50$ and $0 < t < 1.$

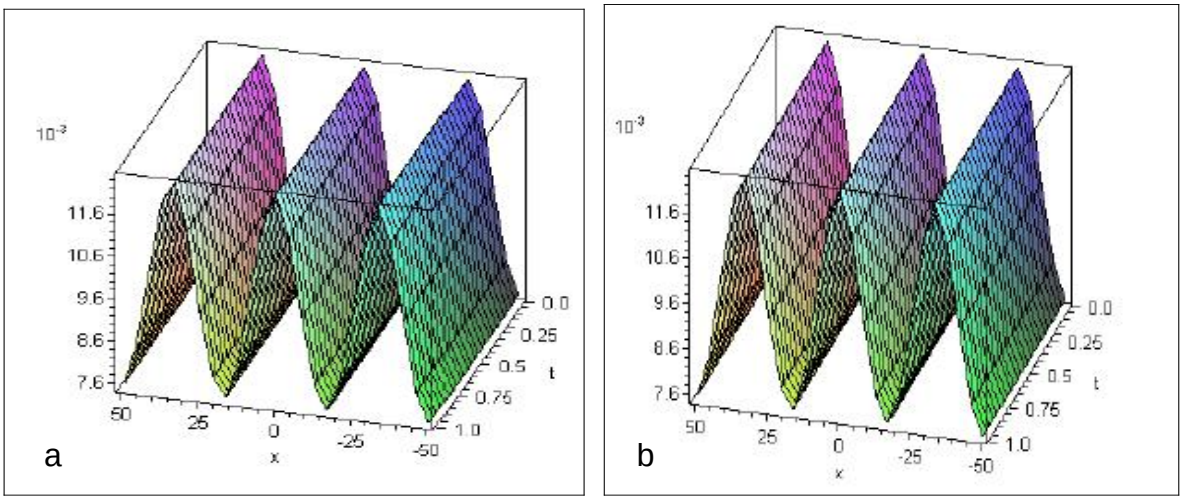
CONCLUSION

In this paper, the HAM has been successfully applied to obtain the numerical solutions of the time fractional variant Bussinesq equations. From Figures 1-5, we deduce the behavior of the approximate solutions is the same behavior of the exact solutions at some different values $\alpha.$ consequently, we deduce the approximate solutions are rapidly convergent series as the exact solutions. The HAM contains a certain auxiliary parameter which provides us with a simple way to adjust and control the convergence region and rate of convergence of the series solution. The obtained results show that the HAM is a very accurate and effective.



V_{Exact} V_{app}

Figure 4. The approximate solution (34) shown in the figure (b) in comparison with the exact solution (36) shown in (a) when $\alpha = .9999, m = 0.5, k = 0.1, \lambda = 0.5, -50 < x < 50$ and $0 < t < 1.$



U_{Exact} U_{app}

Figure 5. The approximate solution (33) shown in the figure (b) in comparison with the exact solution (35) shown in (a) when $\alpha = 0.999, m = 0.5, k = 0.1, \lambda = 0.5, -50 < x < 50$ and $0 < t < 1.$

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