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An algorithm for solving a multi-valued variational inequality

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Abstract

We propose a new extragradient method for solving a multi-valued variational inequality. It is showed that the method converges globally to a solution of the multi-valued variational inequality, provided the multi-valued mapping is continuous with nonempty compact convex values. Preliminary computational experience is also reported.

MSC: 47H04; 47H10; 47J20; 47J25

Keywords: variational inequality; multi-valued mapping; extragradient method

1 Introduction

We consider the following multi-valued variational inequality, denoted by MVI(F, C): to find $x^* \in C$ and $\xi \in F(x^*)$ such that

$$\langle \xi, y - x^* \rangle \geq 0, \quad \forall y \in C, \tag{1.1}$$

where C is a nonempty closed convex set in $\mathbb{R}^n$, F is a multi-valued mapping from C into $\mathbb{R}^n$ with nonempty values, and $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ denote the inner product and the norm in $\mathbb{R}^n$, respectively.

Extragradient-type algorithms have been extensively studied in the literature; see [1–3]. Various algorithms for solving the multi-valued variational inequality have been extensively studied in the literature [4–15]. The well-known proximal point algorithm [12] requires the multi-valued mapping F to be monotone. [11] proposes a projection algorithm for solving the multi-valued variational inequality with a pseudomonotone mapping. In [11], choosing $u_i \in F(x_i)$ needs solving a single-valued variational inequality; see the expression (2.1) in [11]. [6] presents a double projection algorithm, which is an improvement of [11], so that $u_i \in F(x_i)$ can be taken arbitrarily. In [6], however, choosing the hyperplane needs computing the supremum and hence is computationally expensive. To overcome this difficulty, [7] introduces an extragradient algorithm for solving the multi-valued variational inequality in which computing the supremum is avoided. In this paper, we present a new extragradient method for solving the multi-valued variational inequality. In our method, $u_i \in F(x_i)$ can be taken arbitrarily. Moreover, the main difference of our method from those of [6, 7, 11] is the procedure of Armijo-type linesearch. We also present numerical tests to compare our Algorithm 2.2 with those in [6, 11].

This paper is organized as follows. In Section 2, we present the algorithm details. We prove the preliminary results for convergence analysis in Section 3. Numerical results are reported in the last section.

2 Algorithms

Let us recall the definition of a continuous multi-valued mapping. F is said to be upper semicontinuous at $x \in C$ if for every open set V containing $F(x)$, there is an open set U containing x such that $F(y) \subset V$ for all $y \in C \cap U$. F is said to be lower semicontinuous at $x \in C$ if given any sequence x_k converging to x and any $y \in F(x)$, there exists a sequence $y_k \in F(x_k)$ that converges to y. F is said to be continuous at $x \in C$ if it is both upper semicontinuous and lower semicontinuous at x. If F is single-valued, then both upper semicontinuity and lower semicontinuity reduce to the continuity of F.

F is called pseudomonotone on C in the sense of Karamardian [16] if for any $x, y \in C$,

$$\langle v, x-y\rangle \geq 0 \quad \text{for some } v \in F(y) \Rightarrow \langle u, x-y\rangle \geq 0 \text{ for all } u \in F(x). \tag{2.1}$$

Let S be the solution set of (1.1), that is, those points $x^* \in C$ satisfying (1.1). Throughout this paper, we assume that the solution set S of problem (1.1) is nonempty and F is continuous on C with nonempty compact convex values satisfying the following property:

$$\langle \zeta, y-x\rangle \geq 0, \quad \forall y \in C, \forall \zeta \in F(y), \forall x \in S. \tag{2.2}$$

The property (2.2) holds if F is pseudomonotone on C.

Let P_C denote the projector onto C, and let $\mu > 0$ be a parameter.

Proposition 2.1 *$x \in C$ and $\xi \in F(x)$ solve problem* (1.1) *if and only if*

$$r_\mu(x, \xi) := x - P_C(x - \mu\xi) = 0.$$

Algorithm 2.2 Choose $x_0 \in C$ and two parameters $\gamma, \sigma \in (0,1)$. Set $i = 0$.

Step 1. Choose $u_i \in F(x_i)$ and let k_i be the smallest nonnegative integer satisfying

$$v_i \in F\bigl(P_C\bigl(x_i - \gamma^{k_i} u_i\bigr)\bigr), \tag{2.3}$$

$$\gamma^{k_i}\bigl\langle u_i - v_i, r_{\gamma^{k_i}}(x_i, u_i)\bigr\rangle \leq \sigma \bigl\| r_{\gamma^{k_i}}(x_i, u_i)\bigr\|^2. \tag{2.4}$$

Set $\eta_i = \gamma^{k_i}$. If $r_{\eta_i}(x_i, u_i) = 0$, stop.

Step 2. Compute $x_{i+1} := P_C(x_i - \alpha_i d_i)$, where

$$d_i = r_{\eta_i}(x_i, u_i) + \eta_i v_i, \tag{2.5}$$

$$\alpha_i = \frac{(1-\sigma)\|r_{\eta_i}(x_i, u_i)\|^2}{\|d_i\|^2}. \tag{2.6}$$

Let $i := i + 1$ and go to Step 1.

Remark 2.3 Let us compare the above algorithm with those in [6, 7, 11]. First, Aimijo-type linesearch procedures in the four algorithms are different. [6, 7, 11] use different procedures which replace (2.4) by the following ones:

$$\langle v_i, r_\mu(x_i,u_i)\rangle \geq \sigma \left\| r_\mu(x_i,u_i)\right\|^2 \quad \text{or} \quad \langle u_i - v_i, r_\mu(x_i,u_i)\rangle \leq \sigma \left\| r_\mu(x_i,u_i)\right\|^2,$$

where μ is required to be strictly less than 1 or $1/\sigma$, and $v_i \in F(x_i - \gamma^{k_i} r_\mu(x_i,u_i))$. In our algorithm, μ can change according to the value of η_i in each iteration and $v_i \in F(P_C(x_i - \gamma^{k_i} u_i))$. Secondly, the way to generate the next iterate is different. In [6, 11], the next iterate is a projection of the current iterate onto the intersection of the feasible set C and a hyperplane, while in our algorithm as well as in [7] the next iterate is a projection onto the feasible set C. In addition, the searching directions in [7] and our algorithm are also different.

Lemma 2.4 *Let C be a closed convex subset of $\mathbb{R}^n$. For any $x, y \in \mathbb{R}^n$ and $z \in C$, the following statements hold.*

(i) $\langle x - P_C(x), z - P_C(x)\rangle \leq 0$.
(ii) $\|P_C(x) - P_C(y)\|^2 \leq \|x-y\|^2 - \|P_C(x) - x + y - P_C(y)\|^2$.

Proof See [17]. □

The proof of the following lemma is easy and we omit it (see Lemma 3.1 in [18] for example).

Lemma 2.5 *For any $x \in \mathbb{R}^n$, $\xi \in F(x)$ and $\mu > 0$,*

$$\min\{1,\mu\}\left\| r_1(x,\xi)\right\| \leq \left\| r_\mu(x,\xi)\right\| \leq \max\{1,\mu\}\left\| r_1(x,\xi)\right\|.$$

We first show that Algorithm 2.2 is well defined.

Proposition 2.6 *If x_i is not a solution of problem* (1.1), *then there exists a nonnegative integer k_i satisfying* (2.3) *and* (2.4).

Proof Suppose that for all k and all $v \in F(P_C(x_i - \gamma^k u_i))$, we have

$$\gamma^k \langle u_i - v, r_{\gamma^k}(x_i,u_i)\rangle > \sigma \left\| r_{\gamma^k}(x_i,u_i)\right\|^2,$$

and hence,

$$\gamma^k \|u_i - v\| > \sigma \left\| r_{\gamma^k}(x_i,u_i)\right\|.$$

Therefore,

$$\begin{aligned}\|u_i - v\| &> \frac{\sigma}{\gamma^k}\left\| r_{\gamma^k}(x_i,u_i)\right\| \\ &\geq \frac{\sigma}{\gamma^k}\min\{1,\gamma^k\}\left\| r_1(x_i,u_i)\right\| \\ &= \sigma \left\| r_1(x_i,u_i)\right\|, \end{aligned} \tag{2.7}$$

where the second inequality follows from Lemma 2.5 and the equality follows from $\gamma \in (0,1)$ and $k \geq 0$. Since $P_C(\cdot)$ is continuous and $x_i \in C$, $P_C(x_i - \gamma^k u_i) \to x_i (k \to \infty)$. Since F is lower semicontinuous, $u_i \in F(x_i)$ and $P_C(x_i - \gamma^k u_i) \to x_i (k \to \infty)$, there is $v_k \in F(P_C(x_i - \gamma^k u_i))$ such that $v_k \to u_i (k \to \infty)$. Therefore,

$$\|u_i - v_k\| > \sigma \left\| r_1(x_i, u_i) \right\|, \quad \forall k. \tag{2.8}$$

Let $k \to \infty$ in (2.8), we have

$$0 = \|u_i - u_i\| \geq \sigma \left\| r_1(x_i, u_i) \right\| > 0.$$

This contradiction completes the proof. □

3 Main results

Now we obtain the following auxiliary result that will be used for proving the convergence of Algorithm 2.2.

Theorem 3.1 *If the assumption* (2.2) *holds and* $x_i \notin S$, *then for any* $x^* \in S$,

$$\langle d_i, x_i - x^* \rangle \geq (1 - \sigma) \left\| r_{\eta_i}(x_i, \xi_i) \right\|^2 > 0. \tag{3.1}$$

Proof Let $x^* \in S$. Since $u_i \in F(x_i)$ and $\eta_i > 0$, it follows from (2.2) that

$$\langle \eta_i u_i, x_i - x^* \rangle \geq 0. \tag{3.2}$$

Similarly, we have

$$\langle \eta_i v_i, P_C(x_i - \eta_i u_i) - x^* \rangle \geq 0, \tag{3.3}$$

because $v_i \in F(P_C(x_i - \eta_i u_i))$. Since $x^* \in C$, from Lemma 2.4(i) we have

$$\langle x_i - \eta_i u_i - P_C(x_i - \eta_i u_i), P_C(x_i - \eta_i u_i) - x^* \rangle \geq 0. \tag{3.4}$$

It follows from (3.2), (3.3) and (3.4) that

$$\begin{aligned}
\langle d_i, x_i - x^* \rangle &= \langle r_{\eta_i}(x_i, u_i) + \eta_i v_i, x_i - x^* \rangle \\
&= \langle r_{\eta_i}(x_i, u_i) + \eta_i (v_i - u_i), x_i - x^* \rangle + \langle \eta_i u_i, x_i - x^* \rangle \\
&\geq \langle r_{\eta_i}(x_i, u_i) + \eta_i (v_i - u_i), x_i - x^* \rangle \\
&= \langle r_{\eta_i}(x_i, u_i) + \eta_i (v_i - u_i), r_{\eta_i}(x_i, u_i) \rangle \\
&\quad + \langle x_i - \eta_i u_i - P_C(x_i - \eta_i u_i), P_C(x_i - \eta_i u_i) - x^* \rangle \\
&\quad + \langle \eta_i v_i, P_C(x_i - \eta_i u_i) - x^* \rangle \\
&\geq \langle r_{\eta_i}(x_i, u_i) + \eta_i (v_i - u_i), r_{\eta_i}(x_i, u_i) \rangle.
\end{aligned} \tag{3.5}$$

Therefore,

$$\begin{aligned}\langle d_i, x_i - x^* \rangle &\ge \langle r_{\eta_i}(x_i,u_i) + \eta_i(v_i - u_i), r_{\eta_i}(x_i,u_i)\rangle \\ &= \|r_{\eta_i}(x_i,u_i)\|^2 - \eta_i\langle u_i - v_i, r_{\eta_i}(x_i,u_i)\rangle \\ &\ge \|r_{\eta_i}(x_i,u_i)\|^2 - \sigma\|r_{\eta_i}(x_i,u_i)\|^2 \\ &= (1-\sigma)\|r_{\eta_i}(x_i,u_i)\|^2, \end{aligned} \tag{3.6}$$

where the second inequality follows from (2.4). This completes the proof. □

Theorem 3.2 *If $F : C \to 2^{\mathbb{R}^n}$ is continuous with nonempty compact convex values on C and the assumption* (2.2) *holds, then the sequence $\{x_i\}$ generated by Algorithm* 2.2 *converges to a solution $\bar{x}$ of* (1.1).

Proof Let $x^* \in S$. It follows from Lemma 2.4(ii), Lemma 2.5, (2.5), (2.6) and (3.6) that

$$\begin{aligned}\|x_{i+1} - x^*\|^2 &\le \|x_i - x^* - \alpha_i d_i\|^2 \\ &= \|x_i - x^*\|^2 - 2\alpha_i\langle d_i, x_i - x^*\rangle + \alpha_i^2\|d_i\|^2 \\ &\le \|x_i - x^*\|^2 - \frac{(1-\sigma)^2\|r_{\eta_i}(x_i,u_i)\|^4}{\|d_i\|^2} \\ &\le \|x_i - x^*\|^2 - \frac{(1-\sigma)^2(\min\{\eta_i,1\}\|r_1(x_i,u_i)\|)^4}{\|d_i\|^2} \\ &= \|x_i - x^*\|^2 - \frac{(1-\sigma)^2\eta_i^4\|r_1(x_i,u_i)\|^4}{\|r_{\eta_i}(x_i,u_i) + \eta_i v_i\|^2}. \end{aligned} \tag{3.7}$$

It follows that the sequence $\{\|x_{i+1} - x^*\|^2\}$ is nonincreasing, and hence is a convergent sequence. Therefore, $\{x_i\}$ is bounded. Since F is continuous with compact values, Proposition 3.11 in [19] implies that $\{F(x_i) : i \in N\}$ is a bounded set, and so are $\{u_i\}$, $\{r_{\eta_i}(x_i,u_i)\}$ and $\{v_i\}$. Thus, $\{r_{\eta_i}(x_i,u_i) + \eta_i v_i\}$ is bounded. Then there exists a positive number M such that

$$\|r_{\eta_i}(x_i,u_i) + \eta_i v_i\| \le M.$$

It follows from (3.7) that

$$\|x_{i+1} - x^*\|^2 \le \|x_i - x^*\|^2 - (1-\sigma)^2 M^{-2}\eta_i^4\|r_1(x_i,u_i)\|^4. \tag{3.8}$$

Therefore,

$$\lim_{i\to\infty} \eta_i\|r_1(x_i,u_i)\| = 0. \tag{3.9}$$

By the boundedness of $\{x_i\}$, there exists a convergent subsequence $\{x_{i_j}\}$ converging to $\bar{x}$.

If $\bar{x}$ is a solution of problem (1.1), we show next that the whole sequence $\{x_i\}$ converges to $\bar{x}$. Replacing x^* by $\bar{x}$ in the preceding argument, we obtain that the sequence $\{\|x_i - \bar{x}\|\}$ is nonincreasing and hence converges. Since $\bar{x}$ is an accumulation point of $\{x_i\}$, some subsequence of $\{\|x_i - \overline{x}\|\}$ converges to zero. This shows that the whole sequence $\{\|x_i - \overline{x}\|\}$ converges to zero, hence $\lim_{i\to\infty} x_i = \overline{x}$.

Suppose now that $\overline{x}$ is not a solution of problem (1.1). We show first that k_i in Algorithm 2.2 cannot tend to ∞. Since F is continuous with compact values, Proposition 3.11 in [19] implies that $\{F(x_i) : i \in N\}$ is a bounded set, and so the sequence $\{u_i\}$ is bounded. Therefore, there exists a subsequence $\{u_{i_j}\}$ converging to $\overline{u}$. Since F is upper semicontinuous with compact values, Proposition 3.7 in [19] implies that F is closed, and so $\overline{u} \in F(\overline{x})$. By the definition of k_i, we have

$$\gamma^{k_i-1}\langle u_i - v, r_{\gamma^{k_i-1}}(x_i, u_i)\rangle > \sigma \left\| r_{\gamma^{k_i-1}}(x_i, u_i)\right\|^2, \quad \forall v \in F\left(P_C\left(x_i - \gamma^{k_i-1}u_i\right)\right), \tag{3.10}$$

and hence

$$\gamma^{k_i-1}\|u_i - v\| > \sigma \left\| r_{\gamma^{k_i-1}}(x_i, u_i)\right\|, \quad \forall v \in F\left(P_C\left(x_i - \gamma^{k_i-1}u_i\right)\right). \tag{3.11}$$

Therefore,

$$\begin{aligned}\|u_i - v\| &> \frac{\sigma}{\gamma^{k_i-1}}\left\| r_{\gamma^{k_i-1}}(x_i, u_i)\right\| \\ &\geq \frac{\sigma}{\gamma^{k_i-1}}\min\left\{1, \gamma^{k_i-1}\right\}\left\| r_1(x_i, u_i)\right\| \\ &= \sigma\left\| r_1(x_i, u_i)\right\|, \quad \forall v \in F\left(P_C\left(x_i - \gamma^{k_i-1}u_i\right)\right), \forall k_i \geq 1,\end{aligned} \tag{3.12}$$

where the second inequality follows from Lemma 2.5 and the equality follows from $\gamma \in (0,1)$.

If $k_{i_j} \to \infty$, then $P_C(x_{i_j} - \gamma^{k_{i_j}-1}u_{i_j}) \to \overline{x}$. The lower continuity of F, in turn, implies the existence of $\overline{u}_{i_j} \in F(P_C(x_{i_j} - \gamma^{k_{i_j}-1}u_{i_j}))$ such that $\overline{u}_{i_j}$ converges to $\overline{u}$. Therefore,

$$\|u_{i_j} - \overline{u}_{i_j}\| > \sigma\left\| r_1(x_{i_j}, u_{i_j})\right\|. \tag{3.13}$$

Letting $j \to \infty$, we obtain the contradiction

$$0 \geq \sigma\left\| r_1(\overline{x}, \overline{u})\right\|^2 > 0, \tag{3.14}$$

being $r_1(\cdot,\cdot)$ continuous. Therefore, $\{k_i\}$ is bounded and so is $\{\eta_i\}$.

By the boundedness of $\{\eta_i\}$, it follows from (3.9) that $\lim_{i\to\infty}\|r_1(x_i, u_i)\| = 0$. Since $r_1(\cdot,\cdot)$ is continuous and the sequences $\{x_i\}$ and $\{u_i\}$ are bounded, there exists an accumulation point $(\overline{x}, \overline{u})$ of $\{(x_i, u_i)\}$ such that $r_1(\overline{x}, \overline{u}) = 0$. This implies that $\overline{x}$ solves the variational inequality (1.1). Similar to the preceding proof, we obtain that $\lim_{i\to\infty} x_i = \overline{x}$.

Now we provide a result on the convergence rate of the iterative sequence generated by Algorithm 2.2. To establish this result, we need a certain error bound to hold locally (see (3.15) below). The research on an error bound is a large topic in mathematical programming. One can refer to the survey [20] for the roles played by error bounds in the convergence analysis of iterative algorithms; more recent developments on this topic are included in Chapter 6 in [21]. A condition similar to (3.15) has also been used in [22] (see expression (10) therein) to analyze the convergence rate in a very general framework.

For any $\delta > 0$, define

$$P(\delta) := \{(x,\xi) \in C \times \mathbb{R}^n : \xi \in F(x), \|r_1(x,\xi)\| \le \delta\}.$$

We say that F is Lipschitz continuous on C if there exists a constant $L > 0$ such that, for all $x, y \in C$, $H(F(x), F(y)) \le L\|x - y\|$, where H denotes the Hausdorff metric. □

Theorem 3.3 *In addition to the assumptions in Theorem* 3.2, *if F is Lipschitz continuous with modulus $L > 0$ and if there exist positive constants c and δ such that*

$$\operatorname{dist}(x,S) \le c\|r_1(x,\xi)\|, \quad \forall (x,\xi) \in P(\delta), \tag{3.15}$$

then there is a constant $\alpha > 0$ such that for sufficiently large i,

$$\operatorname{dist}(x_i,S) \le \frac{1}{\sqrt{\alpha i + \operatorname{dist}^{-2}(x_0,S)}}.$$

Proof Put $\eta := \min\{1/2, L^{-1}\gamma\sigma\}$. We first prove that $\eta_i > \eta$ for all i. By the construction of η_i, we have $\eta_i \in (0,1]$. If $\eta_i = 1$, then clearly $\eta_i > \frac{1}{2} \ge \eta$. Now we assume that $\eta_i < 1$. Since $\eta_i = \gamma^{k_i}$, it follows that the nonnegative integer $k_i \ge 1$. Thus the construction of k_i implies that

$$\gamma^{k_i-1}\langle u_i - v, r_{\gamma^{k_i-1}}(x_i,u_i)\rangle > \sigma \|r_{\gamma^{k_i-1}}(x_i,u_i)\|^2, \quad \forall v \in F(P_C(x_i - \gamma^{k_i-1}u_i)), \tag{3.16}$$

and hence, as $k_i \ge 1$,

$$\|u_i - v\| > \frac{\sigma}{\gamma^{k_i-1}}\|r_{\gamma^{k_i-1}}(x_i,u_i)\|, \quad \forall v \in F(P_C(x_i - \gamma^{k_i-1}u_i)).$$

Since $u_i \in F(x_i)$ and F is compact-valued, the definition of the Hausdorff metric implies the existence of $v_i \in F(P_C(x_i - \gamma^{k_i-1}u_i))$ such that

$$\frac{\sigma}{\gamma^{k_i-1}}\|r_{\gamma^{k_i-1}}(x_i,u_i)\| < \|u_i - v_i\| \le H(F(x_i), F(P_C(x_i - \gamma^{k_i-1}u_i))) \le L\|r_{\gamma^{k_i-1}}(x_i,u_i)\|.$$

Therefore $\eta_i > L^{-1}\gamma\sigma \ge \eta$.

Let $x^* \in P_S(x_i)$. By (3.8) and (3.15), we obtain that for sufficiently large i,

$$\begin{aligned}\operatorname{dist}^2(x_{i+1},S) &\le \|x_{i+1} - x^*\|^2 \\ &\le \|x_i - x^*\|^2 - (1-\sigma)^2 M^{-2}\eta_i^4 \|r_1(x_i,u_i)\|^4 \\ &\le \|x_i - x^*\|^2 - (1-\sigma)^2 M^{-2}\eta^4 \|r_1(x_i,u_i)\|^4 \\ &\le \operatorname{dist}^2(x_i,S) - (1-\sigma)^2 M^{-2}\eta^4 c^{-4} \operatorname{dist}^4(x_i,S),\end{aligned}$$

where the second inequality follows from $\eta_i > \eta$.

Write α for $(1-\sigma)^2 M^{-2}\eta^4 c^{-4}$. Applying Lemma 6 in Chapter 2 of [23], we have

$$\operatorname{dist}(x_i,S) \le \operatorname{dist}(x_0,S)/\sqrt{\alpha i \operatorname{dist}^2(x_0,S) + 1} = 1/\sqrt{\alpha i + \operatorname{dist}^{-2}(x_0,S)}.$$

This completes the proof. □

Table 1 Example 4.1

ε	Algorithm 2.2		[6, Algorithm 1]	
	It. (Num.)	CPU (Sec.)	It. (Num.)	CPU (Sec.)
10^{-3}	38	0.640625	67	0.546875
10^{-5}	66	0.8125	120	0.828125
10^{-7}	96	1.10938	173	1.15625

Table 2 Example 4.1

ε	Algorithm 2.2		[11, Algorithm 1]	
	It. (Num.)	CPU (Sec.)	It. (Num.)	CPU (Sec.)
10^{-3}	38	0.640625	71	0.96875
10^{-5}	66	0.8125	126	1.53125
10^{-7}	96	1.10938	181	2.14063

4 Numerical experiments

In this section, we present some numerical experiments for the proposed algorithm. The MATLAB codes are run on a PC (with CPU Intel P-T2390) under MATLAB Version 7.0.1.24704(R14) Service Pack 1. We compare the performance of our Algorithm 2.2, [6, Algorithm 1] and [11, Algorithm 1]. In Tables 1 and 2, 'It.' denotes the number of iteration and 'CPU' denotes the CPU time in seconds. The tolerance ε means when $\|r_\mu(x,\xi)\| \le \varepsilon$, the procedure stops.

Example 4.1 *Let* $n = 3$,

$$C := \left\{ x \in \mathbb{R}^n_+ : \sum_{i=1}^n x_i = 1 \right\}$$

and $F : C \to 2^{\mathbb{R}^n}$ *be defined by*

$$F(x) := \bigl\{ (t, t - x_1, t - x_2) : t \in [0,1] \bigr\}.$$

Then the set C and the mapping F satisfy the assumptions of Theorem 3.2 *and* $(0,0,1)$ *is a solution of the multi-valued variational inequality. Example* 4.1 *is tested in* [6, 11]. *We choose* $\sigma = 0.5$, $\gamma = 0.9$ *for our algorithm*; $\sigma = 0.1$, $\gamma = 0.8$, $\mu = 1$ *for Algorithm* 1 *in* [6]; $\sigma = 0.9$, $\gamma = 0.4$, $\mu = 1$ *for Algorithm* 1 *in* [11]. *We use* $x_0 = (0, 0.5, 0.5)$ *as the initial point* (*Table* 1 *and Table* 2).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All the authors contributed equally to the writing of the present article. And they also read and approved the final manuscript.

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References

1. Iusem, AN, Svaiter, BF: A variant of Korpelevich' method for variational inequalities with a new search strategy. Optimization **42**, 309-321 (1997). doi:10.1080/02331939708844365
2. Korpelevich, GM: The extragradient method for finding saddle points and other problems. Èkon. Mat. Metody **12**, 747-756 (1976)
3. Wang, YJ, Xiu, NH, Zhang, JZ: Modified extragradient method for variational inequalities and verification of solution existence. J. Optim. Theory Appl. **119**(1), 167-183 (2003). doi:10.1023/B:JOTA.0000005047.30026.b8
4. Auslender, A, Teboulle, M: Lagrangian duality and related multiplier methods for variational inequality problems. SIAM J. Optim. **10**(4), 1097-1115 (2000). doi:10.1137/S1052623499352656
5. Bao, TQ, Khanh, PQ: A projection-type algorithm for pseudomonotone nonlipschitzian multivalued variational inequalities. In: Generalized Convexity, Generalized Monotonicity and Applications, pp. 113-129. Springer, New York (2005)
6. Fang, CJ, He, YR: A double projection algorithm for multi-valued variational inequalities and a unified framework of the method. Appl. Math. Comput. **217**, 9543-9551 (2011). doi:10.1016/j.amc.2011.04.009
7. Fang, CJ, He, YR: An extragradient method for generalized variational inequality. Pac. J. Optim. **9**(1), 47-59 (2013)
8. Fang, SC, Peterson, EL: Generalized variational inequalities. J. Optim. Theory Appl. **38**(3), 363-383 (1982). doi:10.1007/BF00935344
9. Fukushima, M: The primal Douglas-Rachford splitting algorithm for a class of monotone mappings with application to the traffic equilibrium problem. Math. Program. **72**(1, Ser. A), 1-15 (1996). doi:10.1007/BF02592328
10. Konnov, IV: Combined Relaxation Methods for Variational Inequalities. Springer, Berlin (2001)
11. Li, FL, He, YR: An algorithm for generalized variational inequality with pseudomonotone mapping. J. Comput. Appl. Math. **228**, 212-218 (2009). doi:10.1016/j.cam.2008.09.014
12. Rockafellar, RT: Monotone operators and the proximal point algorithm. SIAM J. Control Optim. **14**(5), 877-898 (1976). doi:10.1137/0314056
13. Saigal, R: Extension of the generalized complementarity problem. Math. Oper. Res. **1**(3), 260-266 (1976). doi:10.1287/moor.1.3.260
14. Salmon, G, Strodiot, JJ, Nguyen, VH: A bundle method for solving variational inequalities. SIAM J. Optim. **14**(3), 869-893 (2003). doi:10.1137/S1052623401384096
15. Xia, FQ, Huang, NJ: A projection-proximal point algorithm for solving generalized variational inequalities. J. Optim. Theory Appl. **150**, 98-117 (2011). doi:10.1007/s10957-011-9825-3
16. Karamardian, S: Complementarity problems over cones with monotone and pseudomonotone maps. J. Optim. Theory Appl. **18**(4), 445-454 (1976). doi:10.1007/BF00932654
17. Zarantonello, EH: Projections on convex sets in Hilbert space and spectral theory. In: Contributions to Nonlinear Functional Analysis. Academic Press, New York (1971)
18. Bnouhachem, A: A self-adaptive method for solving general mixed variational inequalities. J. Math. Anal. Appl. **309**, 136-150 (2005). doi:10.1016/j.jmaa.2004.12.023
19. Aubin, JP, Ekeland, I: Applied Nonlinear Analysis. Wiley, New York (1984)
20. Pang, JS: Error bounds in mathematical programming. Math. Program. **79**, 299-332 (1997). doi:10.1007/BF02614322
21. Facchinei, F, Pang, JS: Finite-Dimensional Variational Inequalities and Complementary Problems. Springer, New York (2003)
22. Solodov, MV: Convergence rate analysis of iterative algorithms for solving variational inequality problems. Math. Program. **96**, 513-528 (2003). doi:10.1007/s10107-002-0369-z
23. Polyak, BT: Introduction to Optimization. Optimization Software Inc., Publications Division, New York (1987)

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