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Global existence and boundedness of solutions of a certain nonlinear integro-differential equation of second order with multiple deviating arguments

Cemil Tunç[1] and Timur Ayhan[2*]

*Correspondence:
tayhan002@gmail.com
[2]Department of Primary School-Mathematics, Faculty of Education, Siirt University, Siirt, 56100, Turkey
Full list of author information is available at the end of the article

Abstract
In this paper, we consider the global existence and boundedness of solutions for a certain nonlinear integro-differential equation of second order with multiple constant delays. We obtain some new sufficient conditions which guarantee the global existence and boundedness of solutions to the considered equation. The obtained result complements some recent ones in the literature. An example is given of the applicability of the obtained result. The main tool employed is an appropriate Lyapunov-Krasovskii type functional.

Keywords: integro-differential equation; second order; global existence; boundedness; Lyapunov-Krasovskii type functional

1 Introduction

In recent years, there have been several papers written on the global existence and boundedness of solutions for certain nonlinear differential and integro-diffrential equations of second order with and without delays; see, for example, Ahmad and Rama Mohana Rao [1], Baxley [2], Burton [3], Constantin [4], Driver [5], Fujimoto and Yamaoka [6], Grace and Lalli [7], Graef and Tunç [8], Kalmanovskii and Myshkis [9], Krasovskii [10], Miller [11, 12], Mustafa and Rogovchenko [13, 14], Napoles Valdes [15], Ogundare *et al.* [16], Reissig *et al.* [17], Tidke [18], Tiryaki and Zafer [19], Tunç [20–25], Tunç and Tunç [26], Yoshizawa [27], Wu *et al.* [28], Yin [29] and the references therein.

It should be noted that Napoles Valdes [15] dealt with the ordinary integro-differential equation of second order:

$$x'' + a(t)f\bigl(t,x,x'\bigr)x' + g\bigl(t,x'\bigr) + h(x) = \int_0^t C(t,\tau)x'(\tau)\,d\tau.$$

The author investigated extendibility, boundedness, stability, and square integrability of solutions to the considered equation. The method of proof consists of the use of a suitable Lyapunov function.

In a recent paper, Graef and Tunç [8] discussed the continuability, boundedness, and square integrability of solutions to the second order functional integro-differential equation with multiple delays:

$$x'' + a(t)f\big(t,x,x'\big)x' + g\big(t,x,x'\big) + \sum_{i=1}^{n} h_i\big(x(t-\tau_i)\big)$$
$$= \int_0^t C(t,s)x'(s)\,ds.$$

The proof of the results in [8] involves the definition of a Lyapunov-Krasovskii type functional.

In this paper, instead of the mentioned integro-differential equations discussed in [8, 15], we consider the following nonlinear and non-autonomous integro-differential equation of second order with multiple constant delays:

$$\big(p(x)x'\big)' + a(t)f\big(t,x,x'\big)x' + b(t)g\big(t,x'\big) + \sum_{i=1}^{n} c_i(t)h_i\big(x(t-\tau_i)\big)$$
$$= \int_0^t C(t,s)x'(s)\,ds, \tag{1}$$

which can be written in the system form as

$$\begin{aligned} x' &= \frac{y}{p(x)}, \\ y' &= \int_0^t C(t,s)\frac{y(s)}{p(x(s))}\,ds - a(t)f\left(t,x,\frac{y}{p(x)}\right)\frac{y}{p(x)} - b(t)g\left(t,\frac{y}{p(x)}\right) \\ &\quad - \sum_{i=1}^{n} c_i(t)h_i\big(x(t)\big) + \sum_{i=1}^{n} c_i(t)\int_{t-\tau_i}^{t} h_i'\big(x(s)\big)\frac{y(s)}{p(x(s))}\,ds, \end{aligned} \tag{2}$$

where τ_i $(i = 1,2,\ldots,n)$ are positive constants, $a, b, c_i : \Re^+ \to \Re^+$, $\Re^+ = (0,\infty)$, $f : \Re^+ \times \Re^2 \to \Re^+$, and $g : \Re^+ \times \Re \to \Re^+$ are continuous functions, $h_i \in C^1(\Re,\Re)$, $p \in C^1(\Re,(0,\infty))$, and $C(t,s)$ is a continuous function for $0 \le t \le s \le \infty$.

It is worth mentioning that the global existence and boundedness of solutions of equation (1) have not yet been discussed in the literature. The aim of this paper is to give some sufficient conditions to guarantee the global existence and boundedness of solutions of equation (1). This case shows the novelty and originality of the present paper. The result to be obtained complements and improves the results in the literature (Graef and Tunç [8], Napoles Valdes [15]). This paper may also be useful for researchers working on the qualitative behavior of solutions of functional integro-differential equations.

We assume that there are positive constants δ_i, β_i, γ_i, λ, p_1, m, M, g_0, g_1, c_i, C_i, R, and τ^* such that the following conditions hold:

(A1) $1 \le p(x) \le p_1$, $\int_{-\infty}^{+\infty} |p'(u)|\,du < \infty$,
(A2) $0 < m \le b(t) \le a(t) \le M$, $0 < c_i \le c_i(t) \le C_i$, $c_i'(t) \le 0$,
(A3) $g(t,0) = 0$ and $0 < g_0 \le \frac{g(t,y)}{y} \le g_1$ $(y \neq 0)$,
(A4) $h_i(0) = 0$, $0 < \delta_i \le \frac{h_i(x)}{x} \le \beta_i$ $(x \neq 0)$, $|h_i'(x)| \le \gamma_i$,
(A5) $\max(\int_0^t |C(t,s)|\,ds + \int_t^{\infty} |C(u,t)|\,du) \le R$,
(A6) $R + 2\lambda\tau^* \le \frac{m}{p_1}(f(t,x,y) + g_0)$ for all t, x and y.

2 Main result

What follows is our main theorem.

Theorem *Suppose that conditions* (A1)-(A6) *hold. Then all solutions of system* (2) *are continuable and bounded.*

Proof We define a Lyapunov-Krasovskii functional by

$$W(t) = W\bigl(t,x(t),y(t)\bigr) = e^{-\frac{\gamma(t)}{\mu}} V_0\bigl(t,x(t),y(t)\bigr), \tag{3}$$

where μ a positive constant,

$$\begin{aligned}\gamma(t) &= \int_0^t \bigl|\theta(s)\bigr|\,ds = \int_0^t \bigl|x'(s)p'\bigl(x(s)\bigr)\bigr|\,ds \\ &= \int_{\alpha_1(t)}^{\alpha_2(t)} \bigl|p'\bigl(x(u)\bigr)\bigr|\,du \le \int_{-\infty}^{+\infty} \bigl|p'\bigl(x(u)\bigr)\bigr|\,du < \infty,\end{aligned}$$

for $\theta(t) = x'(t)p'(x(t))$, $\alpha_1(t) = \min\{x(0),x(t)\}$, $\alpha_2(t) = \max\{x(0),x(t)\}$, and

$$\begin{aligned}V_0(t) = V_0\bigl(t,x(t),y(t)\bigr) &= \frac{1}{2}y^2 + p(x)\sum_{i=1}^n c_i(t)\int_0^x h_i(s)\,ds \\ &\quad + \sum_{i=1}^n \lambda_i \int_{-\tau_i}^0 \int_{t+s}^t y^2(u)\,du\,ds + \int_0^t \int_t^\infty \bigl|C(u,s)\bigr|y^2(s)\,du\,ds.\end{aligned} \tag{4}$$

From the assumptions (A1), (A2), and (A4) it follows that

$$\begin{aligned}V_0(t) &\ge \frac{1}{2}y^2 + p(x)\sum_{i=1}^n c_i(t)\int_0^x h_i(s)\,ds \\ &\ge \frac{1}{2}y^2 + \frac{1}{2}\sum_{i=1}^n c_i\delta_i x^2 \\ &\ge k\bigl(x^2+y^2\bigr),\end{aligned} \tag{5}$$

where $k = \frac{1}{2}\min\{1, \sum_{i=1}^n c_i\delta_i\}$.

Let $(x(t),y(t))$ be a solution of (2). Calculating the time derivative of the functional $V_0(t)$, we obtain

$$\begin{aligned}V_0'(t) &= y\int_0^t C(t,s)\frac{y(s)}{p(x(s))}\,ds - a(t)f\left(t,x,\frac{y}{p(x)}\right)\frac{y^2}{p(x)} - b(t)g\left(t,\frac{y}{p(x)}\right)y \\ &\quad + y\sum_{i=1}^n c_i(t)\int_{t-\tau_i}^t h_i'\bigl(x(s)\bigr)\frac{y(s)}{p(x(s))}\,ds + \theta(t)\sum_{i=1}^n c_i(t)\int_0^x h_i(s)\,ds \\ &\quad + p(x)\sum_{i=1}^n c_i'(t)\int_0^x h_i(s)\,ds + \sum_{i=1}^n (\lambda_i\tau_i)y^2 - \sum_{i=1}^n \lambda_i\int_{t-\tau_i}^t y^2(s)\,ds \\ &\quad + y^2\int_t^\infty \bigl|C(u,t)\bigr|\,du - \int_0^t \bigl|C(t,s)\bigr|y^2(s)\,ds.\end{aligned}$$

By the assumptions (A1)-(A6) and the inequality $2|ab| \le (a^2 + b^2)$, the following estimates can be verified:

$$y\int_0^t C(t,s)\frac{y(s)}{p(x(s))}\,ds \le \int_0^t \left|C(t,s)\right|\left|y(t)\right|\left|y(s)\right|ds$$
$$\le y^2\int_0^t \left|C(t,s)\right|ds + \int_0^t \left|C(t,s)\right|y^2(s)\,ds,$$
$$-a(t)f\left(t,x,\frac{y}{p(x)}\right)\frac{y^2}{p(x)} - b(t)g\left(t,\frac{y}{p(x)}\right)y \le -\frac{m}{p_1}\left(f\left(t,x,\frac{y}{p(x)}\right)+g_0\right)y^2,$$
$$y\sum_{i=1}^n c_i(t)\int_{t-\tau_i}^t h_i'(x(s))\frac{y(s)}{p(x(s))}\,ds \le \sum_{i=1}^n C_i\gamma_i\int_{t-\tau_i}^t \left|y(t)y(s)\right|ds$$
$$\le \sum_{i=1}^n (C_i\gamma_i\tau_i)y^2 + \sum_{i=1}^n C_i\gamma_i\int_{t-\tau_i}^t y^2(s)\,ds,$$
$$\theta(t)\sum_{i=1}^n c_i(t)\int_0^x h_i(s)\,ds \le \frac{|\theta(t)|}{2}\sum_{i=1}^n (C_i\beta_i)x^2,$$
$$p(x)\sum_{i=1}^n c_i'(t)\int_0^x h_i(s)\,ds \le 0.$$

From these estimates we obtain, quite readily,

$$\begin{aligned} V_0'(t) \le{}& \left(R - \frac{m}{p_1}\left(f\left(t,x,\frac{y}{p(x)}\right)+g_0\right)\right)y^2 \\ &+ \sum_{i=1}^n (C_i\gamma_i\tau_i)y^2 + \sum_{i=1}^n C_i\gamma_i\int_{t-\tau_i}^t y^2(s)\,ds \\ &+ \sum_{i=1}^n (\lambda_i\tau_i)y^2 - \sum_{i=1}^n \lambda_i\int_{t-\tau_i}^t y^2(s)\,ds \\ &+ \frac{|\theta(t)|}{2}\sum_{i=1}^n (C_i\beta_i)x^2. \end{aligned}$$

Let

$$\tau^* = \max\{\tau_1,\tau_2,\ldots,\tau_n\}$$

and

$$\lambda = \sum_{i=1}^n \lambda_i = \sum_{i=1}^n C_i\gamma_i.$$

Hence, in view of the discussion and (A6), we can conclude that

$$\begin{aligned} V_0'(t) &\le \left(R + 2\lambda\tau^* - \frac{m}{p_1}\left(f\left(t,x,\frac{y}{p(x)}\right)+g_0\right)\right)y^2 + \frac{|\theta(t)|}{2}\sum_{i=1}^n (C_i\beta_i)x^2 \\ &\le \frac{|\theta(t)|}{2}\sum_{i=1}^n (C_i\beta_i)x^2. \end{aligned} \tag{6}$$

It is now clear that the time derivative of the functional $W(t)$ defined by (3) along any solution of system (2) leads that

$$W'(t) = e^{-\frac{\gamma(t)}{\mu}}\left(-\frac{|\theta(t)|}{\mu}V_0\bigl(t,x(t),y(t)\bigr) + \frac{d}{dt}V_0\bigl(t,x(t),y(t)\bigr)\right).$$

Therefore, using (5), (6), and taking $\mu = \frac{2k}{\sum_{i=1}^{n} C_i\beta_i}$, we obtain

$$W'(t) \le e^{-\frac{\gamma(t)}{\mu}}\left(-\frac{|\theta(t)|}{2}\sum_{i=1}^{n}(C_i\beta_i)x^2 + \frac{|\theta(t)|}{2}\sum_{i=1}^{n}(C_i\beta_i)x^2\right) = 0.$$

This implies that $W'(t) \le 0$. Since all the functions appearing in equation (1) are continuous, it is obvious that there exists at least a solution of equation (1) defined on $[t_0, t_0 + \delta)$ for some $\delta > 0$. We need to show that the solution can be extended to the entire interval $[t_0, \infty)$. We assume on the contrary that there is a first time $T < \infty$ such that the solution exists on $[t_0, T)$ and

$$\lim_{t\to T^-}\bigl(\bigl|x(t)\bigr| + \bigl|y(t)\bigr|\bigr) = \infty.$$

Let $(x(t), y(t))$ be such a solution of system (2) with initial condition (x_0, y_0). Since the Lyapunov-Krasovskii type functional $W(t)$ is positive definite and decreasing, $W'(t) \le 0$, along the trajectories of system (2), we can say that $W(t)$ is bounded $[t_0, T]$. We have

$$W\bigl(T, x(T), y(T)\bigr) \le W(t_0, x_0, y_0) = W_0.$$

Hence, it follows from (3) and (5) that

$$x^2(T) + y^2(T) \le \frac{W_0}{D},$$

where $D = k\exp(-\gamma(T)\mu^{-1})$. This inequality implies that $|x(t)|$ and $|y(t)|$ are bounded as $t \to T^-$. Thus, we can conclude that $T < \infty$ is not possible, we must have $T = \infty$. This completes the proof of the theorem. □

Example We consider the following nonlinear integro-differential equation of second order with two constants delays, $\tau_1 > 0$ and $\tau_2 > 0$:

$$\begin{aligned}
&\left(\left(2 + \frac{\sin x}{1+x^2}\right)x'\right)' + \left(1 + \frac{2}{1+t^2}\right)\bigl(e^{-t} + \sin x + \cos x' + 6\bigr)x' \\
&\quad + \left(1 + \frac{1}{1+t^2}\right)\left(3x' + \frac{x'}{1+x'^2}\right) + 2\bigl(e + e^{-t}\bigr)x(t-\tau_1) \\
&\quad + 2\bigl(e^2 + e^{-t}\bigr)x(t-\tau_2) = \int_0^t \frac{s}{(1+2t)^2}x'(s)\,ds. \qquad (7)
\end{aligned}$$

When we compare equation (7) with equation (1), the existence can be seen of the following estimates:

$$p(x) = 2 + \frac{\sin x}{1+x^2}, \quad 1 \le p(x) \le 3,$$

$$
\begin{aligned}
&\int_{-\infty}^{\infty} \left|p'(u)\right| du \le \int_{-\infty}^{\infty} \left(\left| \frac{\cos u}{1+u^2} \right| + \left| \frac{2u \sin u}{(1+u^2)^2} \right| \right) du \le \pi, \\
&a(t) = 1 + \frac{2}{1+t^2}, \qquad b(t) = 1 + \frac{1}{1+t^2}, \\
&m = 1 \le b(t) \le a(t) \le 3 = M, \\
&c_1(t) = e + e^{-t}, \qquad c_2(t) = e^2 + e^{-t}, \\
&c_1 = e \le c_1(t) \le e + 1 = C_1, \\
&c_2 = e^2 \le c_2(t) \le e^2 + 1 = C_2, \\
&c_1'(t) \le 0, \qquad c_2'(t) \le 0, \quad t \ge 0, \\
&f\bigl(t,x,x'\bigr) = e^{-t} + \sin x + \cos x' + 6, \qquad 4 \le f\bigl(t,x,x'\bigr) \le 9, \\
&g\bigl(t,x'\bigr) = 3x' + \frac{x'}{1+x'^2}, \\
&g(t,0) = 0, \qquad g_0 = 3 \le \frac{g(t,x')}{x'} \le 4 = g_1 \quad (y \ne 0), \\
&h_1(x) = h_2(x) = 2x, \\
&h_1(0) = h_2(0) = 0, \qquad \frac{h_1(x)}{x} = \frac{h_2(x)}{x} = 2 \quad (x \ne 0), \\
&\left|h_1'(x)\right| = \left|h_2'(x)\right| = 2, \\
&\int_0^t \left|C(t,s)\right| ds + \int_t^{\infty} \left|C(u,t)\right| du \\
&\quad = \int_0^t \left| \frac{s}{(1+2t)^2} \right| ds + \int_t^{\infty} \left| \frac{t}{(1+2u)^2} \right| du \le \frac{3}{8} = R.
\end{aligned}
$$

Thus, all the assumptions of the theorem hold. So we can conclude that all solutions of (7) are continuable and bounded.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally in drafting this manuscript and giving the main proofs. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Faculty of Sciences, Yüzüncü Yıl University, Van, 65080, Turkey. [2]Department of Primary School-Mathematics, Faculty of Education, Siirt University, Siirt, 56100, Turkey.

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