Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Iterative process for a strictly pseudo-contractive mapping in uniformly convex Banach spaces

Yu Zhou[1*], Haiyun Zhou[1,2] and Peiyuan Wang[1]

Dedicated to the 80th birthday of Professor Shih-Sen Chang

*Correspondence:
zhouyu871013@gmail.com
[1]Department of Mathematics, Shijiazhuang Mechanical Engineering College, Shijiazhuang, 050003, China
Full list of author information is available at the end of the article

Abstract
This paper is concerned with a new method to prove the weak convergence of a strictly pseudo-contractive mapping in a p-uniformly convex Banach space with more relaxed restrictions on the parameters. Our results extend and improve the corresponding earlier results.
MSC: Primary 41A65; 47H17; secondary 47J20

Keywords: strictly pseudo-contractive mapping; nonexpansive mapping; Mann iteration; uniformly convex Banach space

1 Introduction and preliminaries

In 1967, Browder and Petryshyn [1] gave the classical definition for strictly pseudo-contractive mappings in Hilbert spaces for the first time.

Definition 1.1 Let C be a nonempty closed convex subset of a real Hilbert space H. $T : C \to H$ is called a Browder-Petryshyn-type k-strictly pseudo-contractive mapping. Then there exists $k \in [0,1)$ such that for every $x, y \in C$

$$\langle Tx - Ty, j(x-y)\rangle \le \|x-y\|^2 - k\big\|(I-T)x-(I-T)y\big\|^2. \tag{1.1}$$

In 2010, Zhou [2] gave a new definition for k-strictly pseudo-contractive mappings in q-uniformly smooth Banach spaces.

Definition 1.2 Let C be a nonempty closed convex subset of a q-uniformly smooth Banach space X. $T : C \to C$ is called a Zhou-type k-strictly pseudo-contractive mapping, if there exists $k \in [0,1)$ such that for every $x, y \in C$

$$\langle Tx - Ty, j_q(x-y)\rangle \le \|x-y\|^q - \frac{1-k}{2}\big\|(I-T)x-(I-T)y\big\|^q. \tag{1.2}$$

In 2009, Hu and Wang [3] gave another definition for k-strictly pseudo-contractive mappings in p-uniformly convex Banach spaces.

Definition 1.3 Let C be a nonempty closed convex subset of a p-uniformly convex Banach space X. $T : C \to C$ is called a Hu-type k-strictly pseudo-contractive mapping, if there exists $k \in [0,1)$ such that for every $x, y \in C$

$$\|Tx - Ty\|^p \leq \|x - y\|^p + k\big\|(I - T)x - (I - T)y\big\|^p. \tag{1.3}$$

Remark 1.1 The mappings defined by (1.1) and (1.2) are pseudo-contractive mappings, but the mapping defined by (1.3) may not be pseudo-contractive in general Banach spaces.

Remark 1.2 If and only if $q = 2$, the mappings defined by (1.1) and (1.2) are equivalent.

Remark 1.3 If $p = q = 2$, the mappings defined by (1.1), (1.2), and (1.3) are equivalent in Hilbert space.

In 1979, Reich [4] established a weak convergence theorem via a Mann-type iterative process for nonexpansive mapping in a uniformly convex Banach space with Fréchet differentiable norm.

Theorem R *Let C be a closed convex subset of a uniformly convex Banach space X with a Fréchet differentiable norm and $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$. For any $x_1 \in C$, the iterative sequence $\{x_n\}$ is defined by $x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n$, where the real sequence $\{\alpha_n\} \subset [0,1]$ and $\sum_{n=1}^{\infty}(1 - \alpha_n)\alpha_n = \infty$. Then the sequence $\{x_n\}$ converges weakly to a fixed point of T.*

In 2007, Marino and Xu [5] improved Reich's [4] result and gave several weak convergence theorems via the normal Mann iterative algorithm for strictly pseudo-contractive mappings in Hilbert spaces. Further, they proposed an open problem: *Do the main results of* [5] *still hold true in the framework of Banach spaces which are uniformly convex and have a Fréchet differentiable norm?*

In 2009, Hu and Wang [3] considered above problem in a p-uniformly convex Banach space and established the following theorem.

Theorem H *Let C be a closed convex subset of a p-uniformly convex Banach space X with a Fréchet differentiable norm and $T : C \to C$ be a k-strictly pseudo-contractive mapping in the light of* (1.3) *with coefficients $p, k < \min\{1, 2^{-(p-2)}c_p\}$ and $F(T) \neq \emptyset$. For any $x_1 \in C$ and $n > 1$, the iterative sequence $\{x_n\}$ is defined by $x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n$, where the real sequence $\{\alpha_n\} \subset [0,1]$ and $0 < \varepsilon \leq \alpha_n \leq 1 - \varepsilon < 1 - \frac{2^{p-2}k}{c_p}$. Then the sequence $\{x_n\}$ converges weakly to a fixed point of T.*

Question Can one relax the restriction on the parameters α_n in Theorem H and simplify its proof?

The purpose of this paper is to solve the question mentioned above. To prove our results, we need the following lemmas.

Lemma 1.1 (see [3]) *Let C be a nonempty closed convex subset of a p-uniformly convex Banach space X and $T : C \to C$ be a Hu-type strictly pseudo-contractive mapping in the* *light of* (1.3). *For* $\alpha \in (0,1)$, *define* $T_\alpha : C \to C$ *by* $T_\alpha = (1-\alpha)x + \alpha Tx$, *for* $x \in C$. *If* $\alpha \in (0, 1-(k2^{p-2})/c_p)$, *then* T_α *is a nonexpansive mapping and* $F(T_\alpha) = F(T)$.

Lemma 1.2 (see [3]) *Let* C *be a nonempty closed convex subset of a* p*-uniformly convex Banach space* X *and* $T : C \to C$ *be a Hu-type strictly pseudo-contractive mapping in the light of* (1.3). *For* $\mu \in (0,1)$, $T_\mu : C \to C$ *is defined by* $T_\mu = (1-\mu)x + \mu Tx$, *for* $x \in C$. *Then the following inequality holds*:

$$\|T_\mu x - T_\mu y\|^p \leq \|x-y\|^p - \bigl(W_p(\mu)c_p - \mu\lambda\bigr)\bigl\|(I-T)x-(I-T)y\bigr\|^p, \quad \forall x,y \in C,$$

where $W_p(\mu) = \mu^p(1-\mu) + \mu(1-\mu)^p$.

Lemma 1.3 (see [6]) *Let* C *be a nonempty closed convex subset of a* p*-uniformly convex Banach space* X *and* $T : C \to C$ *be a nonexpansive mapping, then* $I - T$ *is demiclosed at zero*.

Lemma 1.4 (see [7]) *Let* C *be a nonempty closed convex subset of a* p*-uniformly convex Banach space* X *which satisfies the Opial condition and* $T : C \to C$ *be a quasi-nonexpansive mapping with* $F(T) \neq \emptyset$. *If* $I - T$ *is demiclosed at zero, then for any* $x_0 \in C$, *the normal Mann iteration* $\{x_n\}$ *defined by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n, \quad \forall n \geq 0,$$

converges weakly to a fixed point of T, *where* $\{\alpha_n\} \subset [0,1]$ *and* $\sum_{n=0}^{\infty} \min\{\alpha_n, (1-\alpha_n)\} = \infty$.

Lemma 1.5 (see [7]) *Let* C *be a nonempty closed convex subset of a* p*-uniformly convex Banach space* X *whose dual space* X^* *satisfies Kadec-Klee property and* $T : C \to C$ *be a nonexpansive mapping with* $F(T) \neq \emptyset$. *Then, for any* $x_0 \in C$, *the normal Mann iteration* $\{x_n\}$ *defined by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n, \quad \forall n \geq 0,$$

converges weakly to a fixed point of T, *where* $\{\alpha_n\} \subset [0,1]$ *and* $\sum_{n=0}^{\infty} \min\{\alpha_n, (1-\alpha_n)\} = \infty$.

Now we are in a position to state and prove the main results in this paper.

2 Main results

Theorem 2.1 *Let* C *be a nonempty closed convex subset of a* p*-uniformly convex Banach space* X *with Fréchet differential norm. Let* $T : C \to C$ *be a Hu-type* k*-strictly pseudo-contractive mapping in the light of* (1.3) *with coefficients* $p, k < \min\{1, 2^{-(p-2)}c_n\}$ *and* $F(T) \neq \emptyset$. *Assume that a real sequence* $\{\alpha_n\}$ *in* $[0,1]$ *satisfies the conditions*:

(i) $0 \leq \alpha_n \leq \alpha = 1 - (k2^{p-2}/c_p)$, $n \geq 0$;

(ii) $\sum_{n=0}^{\infty} \alpha_n[(1-\alpha_n)2^{2-p}c_p - k] = \infty$.

For any $x_0 \in C$, *the normal Mann iterative sequence* $\{x_n\}$ *is defined by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n, \quad n \geq 0. \tag{2.1}$$

Then the sequence $\{x_n\}$ *defined by* (2.1) *converges weakly to a fixed point of* T.

Proof Let T_α be given as in Lemma 1.1. Then $T_\alpha : C \to C$ is a nonexpansive mapping with $F(T_\alpha) = F(T)$. Set $\beta_n = \frac{\alpha-\alpha_n}{\alpha}$. Then (2.1) reduces to $x_{n+1} = \beta_n x_n + (1-\beta_n)T_\alpha x_n$.

We note that

$$\begin{aligned}\sum_{n=0}^{\infty}\beta_n(1-\beta_n) &= \frac{1}{\alpha^2}\sum_{n=0}^{\infty}\alpha_n(\alpha-\alpha_n)\\ &= \frac{1}{\alpha^2}\sum_{n=0}^{\infty}\alpha_n\left(1-\alpha_n-\frac{k2^{p-2}}{c_p}\right)\\ &= \frac{2^{p-2}}{\alpha^2 c_p}\sum_{n=0}^{\infty}\alpha_n\bigl[(1-\alpha_n)2^{p-2}c_p-k\bigr]\\ &= \infty.\end{aligned}$$

By using Theorem R, we conclude that $\{x_n\}$ converges weakly to a fixed point of T_α, and of T. The proof is complete. □

Remark 2.2 Theorem 2.1 relaxes the iterative parameters in Theorem H and our proof method is also quite concise.

Theorem 2.3 *Let C be a nonempty closed convex subset of a p-uniformly convex Banach space X which satisfies the Opial condition. Let $T : C \to C$ be a Hu-type k-strictly pseudo-contractive mapping in the light of* (1.3) *with coefficients $p, k < \min\{1, 2^{-(p-2)}c_p\}$ and $F(T) \neq \emptyset$. Assume that the real sequence $\{\alpha_n\}$ in $[0,1]$ satisfies the conditions*:

(i) $0 \le \alpha_n \le \alpha = 1-(k2^{p-2}/c_p)$, $n \ge 0$;

(ii) $\sum_{n=0}^{\infty}\alpha_n[(1-\alpha_n)2^{2-p}c_p - k] = \infty$.

For any $x_0 \in C$, the normal Mann iteration $\{x_n\}$ is defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \quad n \ge 0. \tag{2.2}$$

Then the sequence $\{x_n\}$ defined by (2.2) *converges weakly to the fixed point of T.*

Proof Let T_α be given as in Lemma 1.1. Then $T_\alpha : C \to C$ is a nonexpansive mapping with $F(T_\alpha) = F(T)$. Set $\beta_n = \frac{\alpha-\alpha_n}{\alpha}$. Then (2.2) reduces to $x_{n+1} = \beta_n x_n + (1-\beta_n)T_\alpha x_n$. As shown in Theorem 2.1, $\sum_{n=0}^{\infty}\beta_n(1-\beta_n) = \infty$. By Lemma 1.3, $I - T_\alpha$ is demiclosed at zero. By Lemma 1.4, we conclude that $\{x_n\}$ converges weakly to a fixed point of T_α, and of T. The proof is complete. □

Theorem 2.4 *Let C be a nonempty closed convex subset of a p-uniformly convex Banach space X with the dual space X^* satisfying the Kadec-Klee property. Let $T : C \to C$ be a Hu-type k-strictly pseudo-contractive mapping in the light of* (1.3) *with coefficients $p, k <$ $\min\{1, 2^{-(p-2)}c_p\}$ and $F(T) \neq \emptyset$. Assume that the real sequence $\{\alpha_n\}$ in $[0,1]$ satisfies the conditions*:

(i) $0 \le \alpha_n \le \alpha = 1-(k2^{p-2}/c_p)$, $n \ge 0$;

(ii) $\sum_{n=0}^{\infty}\alpha_n[(1-\alpha_n)2^{2-p}c_p - k] = \infty$.

For any $x_0 \in C$, the normal Mann iteration $\{x_n\}$ is defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, \quad n \geq 0. \tag{2.3}$$

Then the sequence $\{x_n\}$ defined by (2.3) *converges weakly to a fixed point of T.*

Proof Let T_α be given as in Lemma 1.1. Then $T_\alpha : C \to C$ is a nonexpansive mapping with $F(T_\alpha) = F(T)$. Set $\beta_n = \frac{\alpha - \alpha_n}{\alpha}$. Then (2.3) reduces to $x_{n+1} = \beta_n x_n + (1 - \beta_n)T_\alpha x_n$. As shown in Theorem 2.1, $\sum_{n=0}^{\infty} \beta_n(1 - \beta_n) = \infty$. By using Lemma 1.5, $\{x_n\}$ defined by (2.3) converges weakly to a fixed point of T_α, and of T. The proof is complete. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All the authors contributed equally.

Author details
[1]Department of Mathematics, Shijiazhuang Mechanical Engineering College, Shijiazhuang, 050003, China. [2]Department of Mathematics and Information, Hebei Normal University, Shijiazhuang, 050016, China.

Acknowledgements
This research was supported by the National Natural Science Foundation of China (11071053).

Received: 14 February 2014 Accepted: 7 August 2014 Published: 29 Sep 2014

References
1. Browder, FE, Petryshyn, WV: Contraction of fixed points of nonlinear mappings in Hilbert spaces. J. Math. Anal. Appl. **20**, 82-90 (1967)
2. Zhou, HY: Convergence theorem for strict pseudo-contractions in uniformly smooth Banach spaces. Acta Math. Sin. Engl. Ser. **26**(4), 743-758 (2010)
3. Hu, LG, Wang, JP: Mann iteration of weak convergence theorem in Banach space. Acta Math. Sin. Engl. Ser. **25**(2), 217-224 (2009)
4. Reich, S: Weak convergence theorem for nonexpansive mappings in Banach spaces. J. Math. Anal. Appl. **67**, 274-276 (1979)
5. Marino, G, Xu, HK: Weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces. J. Math. Anal. Appl. **279**, 336-349 (2003)
6. Browder, FE: Semicontractive and semiaccretive nonlinear mappings in Banach spaces. Bull. Am. Math. Soc. **74**, 660-665 (1968)
7. Agarwal, RP, O'Regan, D, Sahu, DR: Fixed Point Theory for Lipschitzian-Type Mappings with Applications, pp. 299-302. Springer, Berlin (2008)

10.1186/1029-242X-2014-377
Cite this article as: Zhou et al.: **Iterative process for a strictly pseudo-contractive mapping in uniformly convex Banach spaces.** *Journal of Inequalities and Applications* 2014, 2014:377