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On the Korovkin approximation theorem and Volkov-type theorems

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Abstract

In this short paper, we give a generalization of the classical Korovkin approximation theorem (Korovkin in Linear Operators and Approximation Theory, 1960), Volkov-type theorems (Volkov in Dokl. Akad. Nauk SSSR 115:17-19, 1957), and a recent result of (Taşdelen and Erençin in J. Math. Anal. Appl. 331(1):727-735, 2007).

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1 Introduction

In this paper, the classical Korovkin theorem (see [1]) and one of the key results (Theorem 1) of [2] will be generalized to arbitrary compact Hausdorff spaces. For a topological space X, the space of real-valued continuous functions on X, as usual, will be denoted by $C(X)$. We note that if X is a compact Hausdorff space, then $C(X)$ is a Banach space under pointwise algebraic operations and under the norm

$$\|f\| = \sup_{x\in X}\bigl|f(x)\bigr|.$$

Let X be a compact Hausdorff space and E be a subspace of $C(X)$. Then a linear map $A : E \to C(X)$ is called *positive* if $A(f) \geq \mathbf{0}$ in $C(X)$ whenever $f \geq 0$ in E. Here $f \geq \mathbf{0}$ means that $f(x) \geq 0$ in $\mathbb{R}$ for all $x \in X$.

For more details on abstract Korovkin approximations theory, we refer to [3] and [4].

Constant-one function on a topological space X will be denoted by f_0, that is, $f_0(x) = 1$ for all $x \in X$. If $A = (a,b)$ and $B = (c,d)$ are elements of $\mathbb{R}^2$, then the Euclidean distance between A and B, given by

$$\bigl|(a,b) - (c,d)\bigr| = \sqrt{(a-c)^2 + (b-d)^2},$$

is denoted by $|A - B|$.

Definition 1.1 Let X and Y be compact Hausdorff spaces, Z be the product space of X and Y, and let $h \in C(Z \times Z)$ and $f \in C(Z)$ be given. The *module of continuity* of f with respect to h is a function $w_h(f) : [0,\infty) \to \mathbb{R}$ defined by $w(f)(0) = 0$, and

$$w_h(f)(\delta) = \sup\Bigl\{\bigl|f(u,v) - f(x,y)\bigr| : (u,v),(x,y) \in Z \text{ and } \bigl|h\bigl((u,v),(x,y)\bigr)\bigr| < \delta\Bigr\}$$

whenever $\delta > 0$, with the following additional properties:

(i) $w(f)$ is increasing;

(ii) $\lim_{\delta\to 0} = 0$.

We note that the above definition is motivated from [2, p.729] and generalizes the definition which is given there.

Definition 1.2 Let X, Y, and Z be as in Definition 1.1. Let $h \in C(Z\times Z)$ be given. We define $H_{w,h}$ as the set of all continuous functions $f \in C(X\times Y)$ such that for all $(u,v),(x,y) \in X\times Y$, one has

$$\left|f(u,v) - f(x,y)\right| \le w_h(f)\bigl(\left|h\bigl((u,v),(x,y)\bigr)\right|\bigr).$$

When $H_{w,h}$ is mentioned, we always suppose that h satisfies the property for $H_{w,h}$ being a vector subspace of $C(X\times X)$. We note that $H_{w,h}$ has been considered in [2] by taking $X = [0,A]$, $Y = [0,B]$ $(A,B > 0)$,

$$h\bigl((u,v),(x,y)\bigr) = \bigl\|\bigl(f_1(u,v),f_2(u,v)\bigr) - \bigl(f_1(x,y),f_2(x,y)\bigr)\bigr\|,$$

where

$$f_1(u,v) = \frac{u}{1-u} \quad \text{and} \quad f_2(u,v) = \frac{v}{1-v}.$$

The main result of this paper will be obtained via the following lemma.

2 Main result

Lemma 2.1 *Let X and Y be compact Hausdorff spaces and Z be a product space of X and Y. Let $f_1,f_2 \in C(Z)$ and $h \in C(Z\times Z)$ be defined by*

$$h\bigl((u,v),(x,y)\bigr) = \bigl|\bigl(f_1(u,v),f_2(u,v)\bigr) - \bigl(f_1(x,y),f_2(x,y)\bigr)\bigr|$$

so that $H_{w,h}$ is a subspace $C(X\times Y)$ and $f_1,f_2 \in H_{w,h}(Z)$. Let $A : H_{w,h} \to C(Z)$ be a positive linear map. Let $(u,v) \in Z$ be given, and define $\varphi_{u,v}$, $\Phi_{u,v} \in C(Z)$ by

$$\varphi_{u,v} = \bigl(f_1(u,v)f_0 - f_1\bigr)^2 \quad \textit{and} \quad \Phi_{u,v} = \bigl(f_2(u,v)f_0 - f_2\bigr)^2.$$

Then, for all $(u,v) \in Z$, one has

$$\begin{aligned} 0 &\le A(\varphi_{u,v} + \Phi_{u,v}) \\ &\le C_1\bigl[A(f_0) - f_0\bigr](u,v) - C_2\bigl[A(f_1+f_2) - (f_1+f_2)\bigr] + \bigl[A\bigl(f_1^2+f_2^2\bigr) - \bigl(f_1^2+f_2^2\bigr)\bigr], \end{aligned}$$

where

$$C_1 = \bigl(f_1(u,v)^2 + f_2(u,v)^2\bigr) \quad \textit{and} \quad C_2 = -2\bigl(f_1(u,v) + f_2(u,v)\bigr).$$

Proof Note that

$$0 \le \varphi_{u,v} = f_1(u,v)^2 f_0 - 2f_1(u,v)f_1 + f_1^2.$$

Applying the linearity and positivity of A, we have

$$0 \le A(\varphi_{u,v}) = f_1(u,v)^2 A(f_0) - 2f_1(u,v)A(f_1) + A(f_1^2).$$

Then one can have

$$\begin{aligned}
0 &\le A(\varphi_{u,v})(u,v) \\
&= f_1(u,v)^2 A(f_0)(u,v) - 2f_1(u,v)A(f_1)(u,v) + A(f_1^2)(u,v) \\
&= f_1^2(u,v)\bigl[A(f_0)(u,v) - f_0(u,v) + f_0(u,v)\bigr] \\
&\quad - 2f_1(u,v)\bigl[A(f_1)(u,v) - f_1(u,v) + f_1(u,v)\bigr] \\
&\quad + \bigl[A(f_1^2)(u,v) - f_1(u,v)^2 + f_1(u,v)^2\bigr] \\
&= f_1^2(u,v)\bigl[A(f_0) - f_0\bigr](u,v) - 2f_1(u,v)\bigl[A(f_1) - f_1\bigr](u,v) + \bigl[A(f_1^2) - f_1^2\bigr](u,v).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
A(\Phi_{u,v})(u,v) &= f_2^2(u,v)\bigl[A(f_0) - f_0\bigr](u,v) \\
&\quad - 2f_2(u,v)\bigl[A(f_2) - f_2\bigr](u,v) + \bigl[A(f_2^2) - f_2^2\bigr](u,v).
\end{aligned}$$

Now applying A, which is linear, to $\varphi_{u,v} + \Phi_{u,v}$ completes the proof. □

Lemma 2.2 *Let X and Y be compact Hausdorff spaces and f_1, f_2, and h be defined as in Lemma* 2.1. *Let $f \in H_{w,h}$ be given. For each $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\bigl|f(u,v) - f(x,y)\bigr| < \epsilon + \frac{2\|f\|}{\delta^2} h^2\bigl((u,v),(x,y)\bigr).$$

Proof Let $\epsilon > 0$ be given. Since $w(f) : [0,\infty) \to \mathbb{R}$ is continuous, there exists $\delta > 0$ such that $w(f,\delta') = w(f)(\delta') < \epsilon$ for all $0 \le \delta' < \delta$. This implies, since

$$\bigl|f(u,v) - f(x,y)\bigr| \le w\bigl(f, \bigl|h\bigl((u,v),(x,y)\bigr)\bigr|\bigr) \quad \text{for all } (u,v),(x,y) \in Z,$$

that

$$\bigl[(\varphi_{u,v} + \Phi_{u,v})\bigr]^{\frac{1}{2}}(x,y) = \bigl|h\bigl((u,v) - h(x,y)\bigr)\bigr| < \delta \quad \text{implies} \quad \bigl|f(u,v) - f(x,y)\bigr| < \epsilon,$$

where $\varphi_{u,v}$ and $\Phi_{u,v}$ are defined as in Lemma 2.1. If $[(\varphi_{u,v} + \Phi_{u,v})]^{\frac{1}{2}}(x,y) \ge \delta$, then

$$\bigl|f(u,v) - f(x,y)\bigr| \le 2\|f\| \le 2\|f\| \frac{[(\varphi_{u,v} + \Phi_{u,v})](x,y)}{\delta^2}.$$

Hence, for all $(u,v) \in Z$, we have

$$\bigl|f(u,v) - f\bigr| \le \epsilon + 2\|f\| \frac{[(\varphi_{u,v} + \Phi_{u,v})]}{\delta^2}.$$

This completes the proof. □

Lemma 2.3 *Suppose that the hypotheses of Lemma* 2.2 *are satisfied. Let $f \in H_{w,h}$ and $\epsilon > 0$ be given. Then there exists $C > 0$ such that*

$$\|A(f) - f\| < \epsilon + C\bigl(\|A(f_0) - f_0\| + \|A(f_1 + f_2) - (f_1 + f_2)\| + \|A(f_1^2 + f_2^2) - (f_1^2 + f_2^2)\|\bigr).$$

Proof Set $K := \frac{2\|f\|}{\delta^2}$. From Lemma 2.2, there exists $\delta > 0$ such that for each $(u,v) \in Z$ we have

$$\begin{aligned}|f(u,v)f_0 - f| &\le \epsilon + \frac{2\|f\|}{\delta^2}[\varphi_{u,v} + \Phi_{u,v}] \\ &\le \epsilon + \frac{2\|f\|}{\delta^2}\bigl[f_1^2(u,v)f_0 + f_2^2(u,v)f_0 - 2f_1(u,v)f_1 - 2f_2(u,v)f_2 + (f_1^2 + f_2^2)\bigr],\end{aligned}$$

whence

$$\begin{aligned}\bigl|\bigl[A(f) - f(u,v)A(f_0)\bigr](u,v)\bigr| &\le \epsilon A(f_0)(u,v) + K\bigl(A(\varphi_{u,v}) + A(\Phi_{u,v})\bigr) \\ &= \epsilon + \epsilon\bigl[A(f_0) - f_0\bigr](u,v) + KA(\varphi_{u,v} + \Phi_{u,v}).\end{aligned}$$

In particular, we have

$$\begin{aligned}|A(f) - f|(u,v) &\le \bigl|\bigl[A(f) - f(u,v)A(f_0)\bigr](u,v)\bigr| + |f(u,v)|\bigl|(A(f_0) - f_0)(u,v)\bigr| \\ &\le \epsilon + KA(\varphi_{u,v} + \Phi_{u,v})(u,v) + \bigl(\|f\| + \epsilon\bigr)\|A(f_0) - f_0\|.\end{aligned}$$

Now, applying Lemma 2.1 and taking

$$C = 2K + \|f\|,$$

we have what is to be shown. □

We note that in the above theorem C depends only on $\|f\|$ and ϵ, and is independent of the positive linear operator A.

We are now in a position to state the main result of the paper.

Theorem 2.4 *Let X and Y be compact Hausdorff spaces and Z be the product space of X and Y. Let $f_1, f_2 \in C(Z)$, and $h \in C(Z \times Z)$ be defined by*

$$h\bigl((u,v),(x,y)\bigr) = \bigl\|\bigl(f_1(u,v), f_2(u,v)\bigr) - \bigl(f_1(x,y), f_2(x,y)\bigr)\bigr\|$$

so that $H_{w,h}$ is a subspace $C(X \times Y)$ and $f_1, f_2 \in H_{w,h}(Z)$. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of positive operators from $H_{w,h}$ into $C(X \times Y)$ satisfying:

(i) $\|A_n(f_0) - f_0\| \to 0$;
(ii) $\|A_n(f_1) - f_1\| \to 0$;
(iii) $\|A_n(f_2) - f_2\| \to 0$;
(iv) $\|A_n(f_1^2 + f_2^2) - (f_1^2 + f_2^2)\| \to 0$.

Then, for all $f \in H_{w,h}$, we have

$$\|A_n(f) - f\| \to 0.$$

Proof Let $f \in H_{w,h}$ and $\epsilon > 0$ be given. By Lemma 2.3, there exists $C > 0$ (depending only on $\|f\|$ and $\epsilon > 0$) such that for each n,

$$\left\|A_n(f) - f\right\| \le \epsilon + C\left(\left\|A_n(f_0) - f_0\right\| + \left\|A_n(f_1 + f_2) - (f_1 + f_2)\right\| + \left\|A_n\left(f_1^2 + f_2^2\right) - \left(f_1^2 + f_2^2\right)\right\|\right).$$

Since $\epsilon > 0$ is arbitrary and the last three terms of the inequality converge to zero by the assumption, we have

$$A_n(f) \to f.$$

This completes the proof. □

Note also that in Theorem 1 of [2] it is not necessary to take a double sequence of positive operators: as the above result reveals, one can take (A_n) instead of $(A_{n,m})$.

Remarks

(1) If $X = [0,1]$, and $Y = \{y\}$ and $f_1, f_2 \in C(X \times Y)$ are defined by

$$f_{u,v} = u \quad \text{and} \quad f_2 = 0,$$

then Theorem 2.4 becomes the classical Korovkin theorem.

(2) If one takes $X = [0,A]$, $Y = [0,B]$ $(0 < A, B < 1)$, and f_1 and f_2 are defined by

$$f_1(u,v) = \frac{u}{1-u} \quad \text{and} \quad f_2(u,v) = \frac{v}{1-v},$$

then the above theorem becomes Theorem 1 of [2].

(3) For linear positive operators of two variables, Theorem 2.4 generalizes the result of Volkov in [5].

(4) We believe that the above theorem can be generalized to n-fold copies by taking $Z = X_1 \times X_2 \times \cdots \times X_n$ instead of $Z = X \times Y$, where $X_1, X_2, \ldots, X_n$ are compact Hausdorff spaces.

(5) The above theorem is also true if one replaces $C(X)$ by $C_b(X)$, the space of bounded continuous functions, in the case of an arbitrary topological space X.

Competing interests

The author declares that they have no competing interests.

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