Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Further refinements of Gurland's formula for π

Long Lin*

*Correspondence:
linlong1978@sohu.com
School of Mathematics and Informatics, Henan Polytechnic University, Jiaozuo City, Henan Province 454003, People's Republic of China

Abstract

We establish more accurate formulas for approximating π which refine some known results due to Gurland and Mortici.

MSC: Primary 33B15; secondary 26D07; 41A60

Keywords: gamma function; psi function; Wallis ratio; inequality; asymptotic formula

1 Introduction

Gurland [1] proved that for all integers $n \in \mathbb{N} := \{1, 2, 3, \dots\}$,

$$\frac{4n+3}{(2n+1)^2}\left(\frac{(2n)!!}{(2n-1)!!}\right)^2 < \pi < \frac{4}{4n+1}\left(\frac{(2n)!!}{(2n-1)!!}\right)^2. \tag{1.1}$$

Recently, Mortici [2, Theorem 2] improved Gurland's result and obtained the following inequality:

$$\alpha_n < \pi < \beta_n, \tag{1.2}$$

where

$$\alpha_n = \left(\frac{n+\frac{1}{4}}{n^2+\frac{1}{2}n+\frac{3}{32}} + \frac{9}{2{,}048n^5} - \frac{45}{8{,}192n^6}\right)\left(\frac{(2n)!!}{(2n-1)!!}\right)^2 \tag{1.3}$$

and

$$\beta_n = \left(\frac{n+\frac{1}{4}}{n^2+\frac{1}{2}n+\frac{3}{32}} + \frac{9}{2{,}048n^5}\right)\left(\frac{(2n)!!}{(2n-1)!!}\right)^2. \tag{1.4}$$

In this paper, we establish more accurate formulas for approximating π which refine the results due to Gurland and Mortici.

Before stating and proving the main theorems, we first introduce the gamma function and some known results.

The familiar gamma function defined by Euler,

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt \quad (\Re(z) > 0),$$

is one of the most important functions in mathematical analysis and applications in various diverse areas. The logarithmic derivative of $\Gamma(z)$, denoted by $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$, is called the psi (or digamma) function.

The following lemmas are required in the sequel.

Lemma 1.1 ([3, 4]) *If the sequence $(\lambda_n)_{n\in\mathbb{N}}$ converges to zero and if the following limit*:

$$\lim_{n\to\infty} n^k(\lambda_n - \lambda_{n+1}) = l \in \mathbb{R} \quad (k > 1)$$

exists, then

$$\lim_{n\to\infty} n^{k-1}\lambda_n = \frac{l}{k-1} \quad (k > 1),$$

where $\mathbb{R}$ denotes the set of real numbers.

Lemma 1.1 is useful for accelerating some convergences or in constructing some better asymptotic expansions.

Lemma 1.2 *For $x > 0$*,

$$\frac{1}{2x} - \frac{1}{8x^2} + \frac{1}{64x^4} - \frac{1}{128x^6} < \psi(x+1) - \psi\left(x + \frac{1}{2}\right) < \frac{1}{2x} - \frac{1}{8x^2} + \frac{1}{64x^4}. \tag{1.5}$$

Proof The lower bound in (1.5) is obtained by considering the function $F(x)$ defined for $x > 0$ by

$$F(x) = \psi(x+1) - \psi\left(x + \frac{1}{2}\right) - \frac{1}{2x} + \frac{1}{8x^2} - \frac{1}{64x^4} + \frac{1}{128x^6}.$$

Using the following representations:

$$\psi(x) = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-xt}}{1-e^{-t}}\right) \mathrm{d}t \tag{1.6}$$

in [5, p.259, 6.3.21] and

$$\frac{1}{x^r} = \frac{1}{\Gamma(r)} \int_0^\infty t^{r-1} e^{-xt} \,\mathrm{d}t \tag{1.7}$$

in [5, p.255, 6.1.1], we find (for $r > 0$ and $x > 0$) that

$$\begin{aligned} F(x) &= \int_0^\infty \frac{1}{1+e^{t/2}} e^{-xt} \,\mathrm{d}t - \int_0^\infty \left(\frac{1}{2} - \frac{1}{8}t + \frac{1}{384}t^3 - \frac{1}{15{,}360}t^5\right) e^{-xt} \,\mathrm{d}t \\ &= \int_0^\infty \frac{p(t)}{1+e^{t/2}} e^{-xt} \,\mathrm{d}t \end{aligned} \tag{1.8}$$

with

$$\begin{aligned} p(t) &= 1 - \left(\frac{1}{2} - \frac{1}{8}t + \frac{1}{384}t^3 - \frac{1}{15{,}360}t^5\right)\left(1 + e^{t/2}\right) \\ &= 1 - \left(\frac{1}{2} - \frac{1}{8}t + \frac{1}{384}t^3 - \frac{1}{15{,}360}t^5\right)\left(1 + \sum_{n=0}^\infty \frac{1}{2^n n!} t^n\right) \end{aligned}$$

$$\begin{aligned}
&= \sum_{n=7}^{\infty}\left(\frac{17}{2} + \frac{451}{30}(n-7) + \frac{199}{24}(n-7)^2 + \frac{47}{24}(n-7)^3 \right.\\
&\quad \left. + \frac{5}{24}(n-7)^4 + \frac{1}{120}(n-7)^5\right)\frac{t^n}{2^{n+2}\cdot n!}\\
&> 0 \quad \text{for } t > 0,
\end{aligned}$$

so that (1.8) implies $F(x) > 0$ for $x > 0$. Hence, the first inequality in (1.5) holds for $x > 0$.

The upper bound in (1.5) is obtained by considering the function $G(x)$ defined for $x > 0$ by

$$G(x) = \frac{1}{2x} - \frac{1}{8x^2} + \frac{1}{64x^4} - \left[\psi(x+1) - \psi\left(x + \frac{1}{2}\right)\right].$$

Using the above representations (1.6) and (1.7), we find that

$$G(x) = \int_0^{\infty}\left(\frac{1}{2} - \frac{1}{8}t + \frac{1}{384}t^3\right)e^{-xt}\,\mathrm{d}t - \int_0^{\infty}\frac{1}{1+e^{t/2}}e^{-xt}\,\mathrm{d}t = \int_0^{\infty}\frac{q(t)}{1+e^{t/2}}e^{-xt}\,\mathrm{d}t \tag{1.9}$$

with

$$\begin{aligned}
q(t) &= \left(\frac{1}{2} - \frac{1}{8}t + \frac{1}{384}t^3\right)\left(1 + e^{t/2}\right) - 1\\
&= \left(\frac{1}{2} - \frac{1}{8}t + \frac{1}{384}t^3\right)\left(1 + \sum_{n=0}^{\infty}\frac{1}{2^n n!}t^n\right) - 1\\
&= \sum_{n=5}^{\infty}\left(2 + \frac{35}{12}(n-5) + (n-5)^2 + \frac{1}{12}(n-5)^3\right)\frac{t^n}{2^{n+2}\cdot n!}\\
&> 0 \quad \text{for } t > 0,
\end{aligned}$$

so that (1.9) implies $G(x) > 0$ for $x > 0$. Hence, the second inequality in (1.5) holds for $x > 0$. This completes the proof of Lemma 1.2. □

Remark 1.3 A function f is said to be completely monotonic on an interval I if it has derivatives of all orders on I and satisfies the following inequality:

$$(-1)^n f^{(n)}(x) \geq 0 \quad \left(x \in I; n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}\right). \tag{1.10}$$

Dubourdieu [6, p.98] pointed out that if a non-constant function f is completely monotonic on $I = (a, \infty)$, then a strict inequality holds true in (1.10). See also [7] for a simpler proof of this result.

From (1.8) and (1.9), we obtain

$$(-1)^n F^{(n)}(x) = \int_0^{\infty}\frac{t^n p(t)}{1+e^{t/2}}e^{-xt}\,\mathrm{d}t > 0 \quad (x > 0; n \in \mathbb{N}_0)$$

and

$$(-1)^n G^{(n)}(x) = \int_0^{\infty}\frac{t^n q(t)}{1+e^{t/2}}e^{-xt}\,\mathrm{d}t > 0 \quad (x > 0; n \in \mathbb{N}_0).$$

Hence, the functions $F(x)$ and $G(x)$ are both completely monotonic on $(0, \infty)$.

2 Main results

The famous Wallis sequence $(W_n)_{n\geq 1}$ is defined by

$$W_n = \prod_{k=1}^{n} \frac{4k^2}{4k^2-1} \quad \bigl(n \in \mathbb{N} := \{1,2,3,\ldots\}\bigr).$$

Wallis (1655) showed that $W_\infty = \pi/2$.

It is known (see [8–10]) that

$$W_n = \frac{\pi}{2} + O\left(\frac{1}{n}\right) \quad (n \to \infty). \tag{2.1}$$

The convergence of W_n is very slow, so it is not suitable for approximating π. The Wallis sequence can be expressed as (see [11–13])

$$W_n = \frac{1}{2n+1}\left(\frac{(2n)!!}{(2n-1)!!}\right)^2 = \frac{1}{2n+1}\left(\sqrt{\pi}\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})}\right)^2.$$

Now we define the sequence $(u_n)_{n\in\mathbb{N}}$ by

$$u_n = \frac{2}{2n+1}\left(\frac{(2n)!!}{(2n-1)!!}\right)^2\left(1+\frac{a}{n}+\frac{b}{n^2}+\frac{c}{n^3}+\frac{p}{n^4}+\frac{q}{n^5}+\frac{r}{n^6}\right). \tag{2.2}$$

We are interested in finding fixed parameters a, b, c, p, q and r such that $(u_n)_{n\in\mathbb{N}}$ converges as fast as possible to the constant π. Our study is based on Lemma 1.1.

Theorem 2.1 *Let the sequence* $(u_n)_{n\in\mathbb{N}}$ *be defined by* (2.2). *Then for*

$$\begin{aligned} &a = \frac{1}{4}, \qquad b = -\frac{3}{32}, \qquad c = \frac{3}{128}, \\ &p = \frac{3}{2{,}048}, \qquad q = -\frac{33}{8{,}192}, \qquad r = -\frac{39}{65{,}536}, \end{aligned} \tag{2.3}$$

we have

$$\lim_{n\to\infty} n^8(u_n - u_{n+1}) = -\frac{4{,}893\pi}{262{,}144} \quad \text{and} \quad \lim_{n\to\infty} n^7(u_n - \pi) = -\frac{699\pi}{262{,}144}. \tag{2.4}$$

The speed of convergence of the sequence $(u_n)_{n\in\mathbb{N}}$ *is given by the order estimate* $O(n^{-7})$.

Proof We write the difference $u_n - u_{n+1}$ as the following power series in n^{-1}:

$$\begin{aligned} u_n - u_{n+1} = {} & \frac{\pi(4a-1)}{4n^2} + \frac{\pi(32b-24a+9)}{16n^3} + \frac{\pi(384c-480b+284a-125)}{128n^4} \\ & + \frac{\pi(3{,}136b-1{,}680a+795-3{,}584c+2{,}048p)}{512n^5} \\ & + \frac{\pi(40{,}960q-92{,}160p-19{,}523+39{,}932a+108{,}800c-77{,}760b)}{8{,}192n^6} \\ & + \pi(118{,}167-238{,}392a-749{,}056c+808{,}960p-540{,}672q \end{aligned}$$

$$+ 472{,}096b + 196{,}608r)/(32{,}768n^7)$$
$$+ \pi(10{,}838{,}016q - 5{,}963{,}776r - 12{,}486{,}656p + 2{,}854{,}972a + 9{,}779{,}840c$$
$$- 5{,}688{,}480b - 1{,}422{,}745)/(262{,}144n^8) + O\left(\frac{1}{n^9}\right).$$

The fastest sequence $(u_n)_{n\in\mathbb{N}}$ is obtained when the first six coefficients of this power series vanish. In this case, $a = \frac{1}{4}$, $b = -\frac{3}{32}$, $c = \frac{3}{128}$, $p = \frac{3}{2{,}048}$, $q = -\frac{33}{8{,}192}$ and $r = -\frac{39}{65{,}536}$, we have

$$u_n - u_{n+1} = -\frac{4{,}893\pi}{262{,}144n^8} + O\left(\frac{1}{n^9}\right).$$

Finally, by using Lemma 1.1, we obtain assertion (2.4) of Theorem 2.1. □

Solutions (2.3) provide the following approximation for π:

$$\left(1 + \frac{1}{4n} - \frac{3}{32n^2} + \frac{3}{128n^3} + \frac{3}{2{,}048n^4} - \frac{33}{8{,}192n^5} - \frac{39}{65{,}536n^6}\right)\frac{2}{2n+1}\left(\frac{(2n)!!}{(2n-1)!!}\right)^2 = \pi + O\left(\frac{1}{n^7}\right). \tag{2.5}$$

This fact motivated us to observe the following theorem.

Theorem 2.2 *For all $n \in \mathbb{N}$, we have*

$$\lambda_n < \pi < \mu_n, \tag{2.6}$$

where

$$\lambda_n = \left(1 + \frac{1}{4n} - \frac{3}{32n^2} + \frac{3}{128n^3} + \frac{3}{2{,}048n^4} - \frac{33}{8{,}192n^5} - \frac{39}{65{,}536n^6}\right) \cdot \frac{2}{2n+1}\left(\frac{(2n)!!}{(2n-1)!!}\right)^2 \tag{2.7}$$

and

$$\mu_n = \left(1 + \frac{1}{4n} - \frac{3}{32n^2} + \frac{3}{128n^3} + \frac{3}{2{,}048n^4}\right)\frac{2}{2n+1}\left(\frac{(2n)!!}{(2n-1)!!}\right)^2. \tag{2.8}$$

Proof Inequality (2.6) can be rewritten as

$$\alpha(n) < \frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})} < \beta(n), \tag{2.9}$$

where

$$\alpha(x) = \left(\left(1 + \frac{1}{4x} - \frac{3}{32x^2} + \frac{3}{128x^3} + \frac{3}{2{,}048x^4}\right)\frac{2}{2n+1}\right)^{-1/2}$$
$$= \left(\frac{1{,}024x^4(2x+1)}{2{,}048x^4 + 512x^3 - 192x^2 + 48x + 3}\right)^{1/2}$$

and

$$\beta(x) = \left(\left(1 + \frac{1}{4x} - \frac{3}{32x^2} + \frac{3}{128x^3} + \frac{3}{2{,}048x^4} - \frac{33}{8{,}192x^5} - \frac{39}{65{,}536x^6}\right)\frac{2}{2x+1}\right)^{-1/2}$$
$$= \left(\frac{32{,}768x^6(2x+1)}{65{,}536x^6 + 16{,}384x^5 - 6{,}144x^4 + 1{,}536x^3 + 96x^2 - 264x - 39}\right)^{1/2}.$$

The lower bound in (2.9) is obtained by considering the function $f(x)$ defined for $x \geq 1$ by

$$f(x) = \ln\Gamma(x+1) - \ln\Gamma\left(x + \frac{1}{2}\right) - \frac{1}{2}\ln\left(\frac{1{,}024x^4(2x+1)}{2{,}048x^4 + 512x^3 - 192x^2 + 48x + 3}\right).$$

Using the asymptotic expansion [5, p.257, 6.1.41]

$$\ln\Gamma(x) \sim \left(x - \frac{1}{2}\right)\ln x - x + \ln\sqrt{2\pi} + \frac{1}{12x} - \frac{1}{360x^3} + \frac{1}{1{,}620x^5} - \frac{1}{1{,}680x^7} + \cdots \quad (x \to \infty), \tag{2.10}$$

we find

$$\lim_{x\to\infty} f(x) = 0.$$

Differentiating $f(x)$ and applying the second inequality in (1.5), we find that, for $x \geq 1$,

$$f'(x) = \psi(x+1) - \psi\left(x + \frac{1}{2}\right) - \frac{2{,}048x^5 + 1{,}024x^4 - 320x^3 + 87x + 6}{x(2x+1)(2{,}048x^4 + 512x^3 - 192x^2 + 48x + 3)}$$
$$< \frac{1}{2x} - \frac{1}{8x^2} + \frac{1}{64x^4} - \frac{2{,}048x^5 + 1{,}024x^4 - 320x^3 + 87x + 6}{x(2x+1)(2{,}048x^4 + 512x^3 - 192x^2 + 48x + 3)}$$
$$= -\frac{592x^3 + 120x^2 - 54x - 3}{64x^4(2x+1)(2{,}048x^4 + 512x^3 - 192x^2 + 48x + 3)} < 0.$$

Consequently, the sequence $(f(n))_{n\in\mathbb{N}}$ is strictly decreasing. This leads to

$$f(n) > \lim_{n\to\infty} f(n) = 0 \quad (n \in \mathbb{N}),$$

which means that the first inequality in (2.9) is valid for $n \in \mathbb{N}$.

The upper bound in (2.9) is obtained by considering the function $g(x)$ defined for $x \geq 1$ by

$$g(x) = \ln\Gamma(x+1) - \ln\Gamma\left(x + \frac{1}{2}\right) - \frac{1}{2}\ln\left(\frac{32{,}768x^6(2x+1)}{65{,}536x^6 + 16{,}384x^5 - 6{,}144x^4 + 1{,}536x^3 + 96x^2 - 264x - 39}\right).$$

We conclude from the asymptotic expansion (2.10) that

$$\lim_{x\to\infty} g(x) = 0.$$

Differentiating $g(x)$ and applying the first inequality in (1.5) yields, for $x \geq 1$,

$$\begin{aligned} g'(x) &= \psi(x+1) - \psi\left(x+\frac{1}{2}\right) \\ &\quad - \frac{65{,}536x^7 + 32{,}768x^6 - 10{,}240x^5 + 2{,}784x^3 - 1{,}392x^2 - 933x - 117}{x(2x+1)(65{,}536x^6 + 16{,}384x^5 - 6{,}144x^4 + 1{,}536x^3 + 96x^2 - 264x - 39)} \\ &> \frac{1}{2x} - \frac{1}{8x^2} + \frac{1}{64x^4} - \frac{1}{128x^6} \\ &\quad - \frac{65{,}536x^7 + 32{,}768x^6 - 10{,}240x^5 + 2{,}784x^3 - 1{,}392x^2 - 933x - 117}{x(2x+1)(65{,}536x^6 + 16{,}384x^5 - 6{,}144x^4 + 1{,}536x^3 + 96x^2 - 264x - 39)} \\ &= \frac{342x + 2{,}832x^4 + 354x^2 + 17{,}312x^5 - 2{,}412x^3 + 39}{x^6(2x+1)(65{,}536x^6 + 16{,}384x^5 - 6{,}144x^4 + 1{,}536x^3 + 96x^2 - 264x - 39)} > 0. \end{aligned}$$

Consequently, the sequence $(g(n))_{n\in\mathbb{N}}$ is strictly increasing. This leads to

$$g(n) < \lim_{n\to\infty} g(n) = 0 \quad (n \in \mathbb{N}),$$

which means that the second inequality in (2.9) is valid for $n \in \mathbb{N}$. The proof of Theorem 2.2 is complete. □

Remark 2.3 Let α_n, β_n, λ_n and μ_n be defined by (1.3), (1.4), (2.7) and (2.8), respectively. Direct computation would yield

$$\lambda_n - \alpha_n = \frac{3(1{,}056n^2 + 704n + 141)}{32{,}768n^6(2n+1)(32n^2 + 16n + 3)}\left(\frac{(2n)!!}{(2n-1)!!}\right)^2 > 0$$

and

$$\mu_n - \beta_n = -\frac{3(64n^2 + 60n + 9)}{2{,}048n^5(2n+1)(32n^2 + 16n + 3)}\left(\frac{(2n)!!}{(2n-1)!!}\right)^2 < 0,$$

which show that inequality (2.6) is sharper than inequality (1.2).

By using Lemma 1.1, we find that

$$\alpha_n = \pi + O\left(\frac{1}{n^6}\right), \qquad \beta_n = \pi + O\left(\frac{1}{n^5}\right),$$
$$\lambda_n = \pi + O\left(\frac{1}{n^7}\right), \qquad \mu_n = \pi + O\left(\frac{1}{n^5}\right).$$

Among sequences α_n, β_n, λ_n and μ_n, the sequence λ_n is the best in the sense that it is the fastest sequence which would approximate the constant π.

The logarithm of the gamma function has the asymptotic expansion (see [14, p.32]):

$$\begin{aligned} \ln\Gamma(x+t) &\sim \left(x+t-\frac{1}{2}\right)\ln x - x + \frac{1}{2}\ln(2\pi) \\ &\quad + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}B_{n+1}(t)}{n(n+1)}\frac{1}{x^n} \quad (x\to\infty). \end{aligned} \tag{2.11}$$

Here $B_n(t)$ denote the Bernoulli polynomials defined by the following generating function:

$$\frac{xe^{tx}}{e^x-1}=\sum_{n=0}^{\infty}B_n(t)\frac{x^n}{n!}. \tag{2.12}$$

Note that the Bernoulli numbers B_n ($n\in\mathbb{N}_0$) are defined by $B_n:=B_n(0)$ in (2.12).

From (2.11) we easily obtain

$$\frac{\Gamma(x+t)}{\Gamma(x+s)}\sim x^{t-s}\exp\left(\sum_{j=1}^{\infty}\frac{(-1)^{j+1}(B_{j+1}(t)-B_{j+1}(s))}{j(j+1)}\frac{1}{x^j}\right)\quad(x\to\infty). \tag{2.13}$$

Taking $(s,t)=(1,\frac{1}{2})$ in (2.13) and noting that

$$B_n(0)=(-1)^nB_n(1)=B_n\quad\text{and}\quad B_n\left(\frac{1}{2}\right)=\left(2^{1-n}-1\right)B_n\quad(n\in\mathbb{N}_0)$$

(see [5, p.805]), we obtain

$$\left[\frac{\Gamma(x+\frac{1}{2})}{\Gamma(x+1)}\right]^2\sim\frac{1}{x}\exp\left(\sum_{j=1}^{\infty}\frac{2((-1)^j(1-2^{-j})-1)B_{j+1}}{j(j+1)}\frac{1}{x^j}\right)\quad(x\to\infty), \tag{2.14}$$

namely,

$$\begin{aligned}\left[\frac{\Gamma(x+\frac{1}{2})}{\Gamma(x+1)}\right]^2\sim\frac{1}{x}\exp\biggl(&-\frac{1}{4x}+\frac{1}{96x^3}-\frac{1}{320x^5}+\frac{17}{7{,}168x^7}-\frac{31}{9{,}216x^9}\\&+\frac{691}{90{,}112x^{11}}-\frac{5{,}461}{212{,}992x^{13}}+\frac{929{,}569}{7{,}864{,}320x^{15}}-\cdots\biggr)\quad(x\to\infty).\end{aligned} \tag{2.15}$$

From (2.15) we imply

$$\begin{aligned}\left(\frac{(2n-1)!!}{(2n)!!}\right)^2&=\frac{1}{\pi}\left(\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}\right)^2\\&\sim\frac{1}{n\pi}\exp\biggl(-\frac{1}{4n}+\frac{1}{96n^3}-\frac{1}{320n^5}+\frac{17}{7{,}168n^7}-\frac{31}{9{,}216n^9}\\&\qquad+\frac{691}{90{,}112n^{11}}-\frac{5{,}461}{212{,}992n^{13}}+\frac{929{,}569}{7{,}864{,}320n^{15}}-\cdots\biggr),\end{aligned} \tag{2.16}$$

which implies the following asymptotic expansion for π:

$$\begin{aligned}\pi\sim\left(\frac{(2n)!!}{(2n-1)!!}\right)^2\frac{1}{n}\exp\biggl(&-\frac{1}{4n}+\frac{1}{96n^3}-\frac{1}{320n^5}+\frac{17}{7{,}168n^7}\\&-\frac{31}{9{,}216n^9}+\frac{691}{90{,}112n^{11}}-\frac{5{,}461}{212{,}992n^{13}}+\frac{929{,}569}{7{,}864{,}320n^{15}}-\cdots\biggr).\end{aligned} \tag{2.17}$$

The formula (2.17) motivated us to observe the following theorem.

Theorem 2.4 *For all $n\in\mathbb{N}$, we have*

$$\delta_n<\pi<\omega_n, \tag{2.18}$$

where

$$\delta_n = \left(\frac{(2n)!!}{(2n-1)!!}\right)^2 \frac{1}{n} \exp\left(-\frac{1}{4n} + \frac{1}{96n^3} - \frac{1}{320n^5} + \frac{17}{7{,}168n^7} - \frac{31}{9{,}216n^9}\right) \tag{2.19}$$

and

$$\omega_n = \left(\frac{(2n)!!}{(2n-1)!!}\right)^2 \frac{1}{n} \exp\left(-\frac{1}{4n} + \frac{1}{96n^3} - \frac{1}{320n^5} + \frac{17}{7{,}168n^7}\right). \tag{2.20}$$

Proof Inequality (2.18) can be rewritten as

$$a(n) < \frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})} < b(n), \tag{2.21}$$

where

$$a(x) = \sqrt{x} \exp\left(\frac{1}{8x} - \frac{1}{192x^3} + \frac{1}{640n^5} - \frac{17}{14{,}336x^7}\right)$$

and

$$b(x) = \sqrt{x} \exp\left(\frac{1}{8x} - \frac{1}{192x^3} + \frac{1}{640n^5} - \frac{17}{14{,}336x^7} + \frac{31}{18{,}432x^9}\right).$$

The lower bound in (2.21) is obtained by considering the function $F(x)$ defined for $x \geq 1$ by

$$F(x) = \ln\Gamma(x+1) - \ln\Gamma\left(x+\frac{1}{2}\right) - \frac{1}{2}\ln x - \ln\left(\frac{1}{8x} - \frac{1}{192x^3} + \frac{1}{640n^5} - \frac{17}{14{,}336x^7}\right).$$

Differentiating $F(x)$ and applying the first inequality in (1.5) yields, for $x \geq 1$,

$$\begin{aligned} F'(x) &= \psi(x+1) - \psi\left(x+\frac{1}{2}\right) - \frac{1}{2x} \\ &\quad + \frac{105(256x^6 - 32x^4 + 16x^2 - 17)}{x(26{,}880x^6 - 1{,}120x^4 + 336x^2 - 255)} \\ &> -\frac{1}{8x^2} + \frac{1}{64x^4} - \frac{1}{128x^6} + \frac{105(256x^6 - 32x^4 + 16x^2 - 17)}{x(26{,}880x^6 - 1{,}120x^4 + 336x^2 - 255)} \\ &= \frac{c(x)}{128x^6(26{,}880x^6 - 1{,}120x^4 + 336x^2 - 255)} \end{aligned}$$

with

$$\begin{aligned} c(x) &= 2{,}609{,}505 + 30{,}426{,}660(x-1) + 158{,}153{,}746(x-1)^2 \\ &\quad + 488{,}558{,}528(x-1)^3 + 1{,}001{,}794{,}352(x-1)^4 + 1{,}435{,}099{,}904(x-1)^5 \\ &\quad + 1{,}466{,}609{,}984(x-1)^6 + 1{,}069{,}107{,}200(x-1)^7 + 544{,}552{,}960(x-1)^8 \\ &\quad + 184{,}504{,}320(x-1)^9 + 37{,}416{,}960(x-1)^{10} + 3{,}440{,}640(x-1)^{11}. \end{aligned}$$

Hence, $F'(x) > 0$ for $x \geq 1$, and therefore, the sequence $(F(n))_{n\in\mathbb{N}}$ is strictly increasing. This leads to

$$F(n) \geq F(1) = \ln\left(\frac{430{,}080}{25{,}841}\right) - \frac{1}{2}\ln\pi = 2.23964391\ldots > 0 \quad (n \in \mathbb{N}),$$

which means that the first inequality in (2.21) is valid for $n \in \mathbb{N}$.

The upper bound in (2.21) is obtained by considering the function $G(x)$ defined for $x \geq 1$ by

$$\begin{aligned} G(x) &= \ln\Gamma(x+1) - \ln\Gamma\left(x + \frac{1}{2}\right) \\ &\quad - \frac{1}{2}\ln x - \ln\left(\frac{1}{8x} - \frac{1}{192x^3} + \frac{1}{640n^5} - \frac{17}{14{,}336x^7} + \frac{31}{18{,}432x^9}\right). \end{aligned}$$

We conclude from the asymptotic expansion (2.10) that

$$\lim_{x\to\infty} G(x) = 0.$$

Differentiating $G(x)$ and applying the first inequality in (1.5) yields, for $x \geq 1$,

$$\begin{aligned} G'(x) &= \psi(x+1) - \psi\left(x + \frac{1}{2}\right) - \frac{1}{2x} \\ &\quad + \frac{315(256x^8 - 32x^6 + 16x^4 - 17x^2 + 31)}{x(80{,}640x^8 - 3{,}360x^6 + 1{,}008x^4 - 765x^2 + 1{,}085)} \\ &> -\frac{1}{8x^2} + \frac{1}{64x^4} - \frac{1}{128x^6} \\ &\quad + \frac{315(256x^8 - 32x^6 + 16x^4 - 17x^2 + 31)}{x(80{,}640x^8 - 3{,}360x^6 + 1{,}008x^4 - 765x^2 + 1{,}085)} \\ &= \frac{d(x)}{128x^6(80{,}640x^8 - 3{,}360x^6 + 1{,}008x^4 - 765x^2 + 1{,}085)} \end{aligned}$$

with

$$\begin{aligned} d(x) &= 9{,}062{,}160 + 113{,}121{,}510(x-1) + 677{,}246{,}923(x-1)^2 \\ &\quad + 2{,}518{,}307{,}800(x-1)^3 + 6{,}417{,}427{,}702(x-1)^4 \\ &\quad + 11{,}782{,}991{,}328(x-1)^5 + 16{,}015{,}812{,}432(x-1)^6 \\ &\quad + 16{,}312{,}281{,}216(x-1)^7 + 12{,}448{,}132{,}032(x-1)^8 \\ &\quad + 7{,}028{,}152{,}320(x-1)^9 + 2{,}852{,}935{,}680(x-1)^{10} \\ &\quad + 788{,}336{,}640(x-1)^{11} + 132{,}894{,}720(x-1)^{12} + 10{,}321{,}920(x-1)^{13}. \end{aligned}$$

Hence, $G'(x) > 0$ for $x \geq 1$, and therefore, the sequence $(G(n))_{n\in\mathbb{N}}$ is strictly increasing. This leads to

$$G(n) < \lim_{n\to\infty} G(n) = 0 \quad (n \in \mathbb{N}),$$

Table 1 Comparison between inequalities (2.6) and (2.18)

n	$\delta_n - \lambda_n$	$\mu_n - \omega_n$
2	0.00004326	0.00032062
10	$8.22562801 \times 10^{-10}$	$1.24569373 \times 10^{-7}$
100	$8.37118328 \times 10^{-17}$	$1.26417115 \times 10^{-12}$
1,000	$8.64864687 \times 10^{-24}$	$1.26540403 \times 10^{-17}$
10,000	$8.37692558 \times 10^{-31}$	$1.2655212 \times 10^{-22}$

which means that the second inequality in (2.21) is valid for $n \in \mathbb{N}$. The proof of Theorem 2.4 is complete. □

Remark 2.5 The following numerical computations (see Table 1) would show that, for $n \in \mathbb{N} \setminus \{1\}$, inequality (2.18) is sharper than inequality (2.6).

By using Lemma 1.1, we find that

$$\delta_n = \pi + O\left(\frac{1}{n^{11}}\right) \quad \text{and} \quad \omega_n = \pi + O\left(\frac{1}{n^9}\right),$$

which provide the higher-order estimates for the constant π.

Remark 2.6 Some calculations in this work were performed by using the Maple software for symbolic calculations.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors read and approved the final manuscript.

Received: 21 September 2012 Accepted: 27 January 2013 Published: 12 February 2013

References
1. Gurland, J: On Wallis' formula. Am. Math. Mon. **63**, 643-645 (1956)
2. Mortici, C: Refinements of Gurland's formula for pi. Comput. Math. Appl. **62**, 2616-2620 (2011)
3. Mortici, C: New approximations of the gamma function in terms of the digamma function. Appl. Math. Lett. **23**, 97-100 (2010)
4. Mortici, C: Product approximations via asymptotic integration. Am. Math. Mon. **117**, 434-441 (2010)
5. Abramowitz, M, Stegun, IA (eds.): Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing. Applied Mathematics Series, vol. 55. National Bureau of Standards, Washington (1972)
6. Dubourdieu, J: Sur un théorème de M. S. Bernstein relatif à la transformation de Laplace-Stieltjes. Compos. Math. **7**, 96-111 (1939) (in French)
7. van Haeringen, H: Completely monotonic and related functions. J. Math. Anal. Appl. **204**, 389-408 (1996)
8. Hirschhorn, MD: Comments on the paper "Wallis' sequence…" by Lampret. Aust. Math. Soc. Gaz. **32**, 194 (2005)
9. Lampret, V: Wallis sequence estimated through the Euler-Maclaurin formula: even from the Wallis product π could be computed fairly accurately. Aust. Math. Soc. Gaz. **31**, 328-339 (2004)
10. Păltănea, E: On the rate of convergence of Wallis' sequence. Aust. Math. Soc. Gaz. **34**, 34-38 (2007)
11. Chen, C-P, Qi, F: The best bounds in Wallis' inequality. Proc. Am. Math. Soc. **133**, 397-401 (2005)
12. Mortic, C: New approximation formulas for evaluating the ratio of gamma functions. Math. Comput. Model. **52**, 425-433 (2010)
13. Mortic, C: Completely monotone functions and the Wallis ratio. Appl. Math. Lett. **25**, 717-722 (2012)
14. Luke, YL: The Special Functions and Their Approximations, vol. I. Academic Press, New York (1969)

doi:10.1186/1029-242X-2013-48
Cite this article as: Lin: Further refinements of Gurland's formula for π. *Journal of Inequalities and Applications* 2013 **2013**:48.