

A characteristic particle length

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Abstract: It is argued that there are characteristic intervals associated with any particle that can be derived without reference to the speed of light c . Such intervals are inferred from zeros of wavefunctions, which are solutions to the Schrödinger equation. The characteristic length is $\ell = \beta^2 \hbar^2 / (8Gm^3)$, where $\beta = 3.8 \dots$; this length might lead to observational effects on objects the size of a virus.

Key words: Characteristic interval, schrödinger equation, fundamental length

1. Introduction

Consider the spreading of the wave-packet in nonrelativistic quantum mechanics. One can ask at what point does any attraction stop its spreading, the end of spreading being signified by the wavefunction vanishing $\psi = 0$. As it stands, the potential-free wave-packet [1] eq.12.21

$$|\psi(x, t)|^2 = \left\{ 2\pi \left[(\Delta x)^2 + \frac{\hbar^2 t^2}{4m^2 (\Delta x)^2} \right] \right\}^{-\frac{1}{2}} \exp \left\{ \frac{-x^2}{2[(\Delta x)^2 + \hbar^2 t^2 / 4m^2 (\Delta x)^2]} \right\}, \quad (1)$$

is a solution to $V = 0$ Schrödinger's equation [1]eq.6.16

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V\psi. \quad (2)$$

Solutions to Schrödinger's equation obey Ehrenfest's theorem [1]eq.7.10

$$\frac{d}{dt} \langle p_a \rangle = - \left\langle \frac{\partial V}{\partial a} \right\rangle, \quad (3)$$

for the derivation of the case (1) the potential V was taken to vanish: thus gravitational or any other interaction cannot be taken into account via V . To overcome this, one has to start again and derive the wavefunction with a nonvanishing potential V ; in the present case only V a newtonian gravitational potential is considered.

In [2] it was asked whether the deviation of geodesics and wave spreading could cancel. Nonrelativistic quantum mechanics and newtonian gravity are also used to describe the cow experiment [3]. Here throughout charge and spin are ignored, although for most particles this is a big assumption; this is done for reasons of simplicity. The conventions used are: a wavefunction is any solution to the Schrödinger equation, a wave-packet

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is a wavefunction with a distribution which has a well-defined mean and variance, signature $-, +, +, +$. The use of d is avoided as it can both denote distance and dimension; n is used to denote the number of spatial dimensions. It turns out that $n = 1, 2, 4$ are all exceptional cases, so to avoid equation clutter we usually stick to $d = 3$ unless stated otherwise. Some constants, see [4], used are: first zero of the Bessel J function $\beta = 3.831705970$, Euler's constant $\gamma = 0.5772156649$, Planck's constant divided by 2π $\hbar = 1.054571726(47) \times 10^{-34} Js$, gravitational constant $G = 6.67384(80) \times 10^{-11} m^3 kg^{-1} s^{-2}$, and speed of light $c = 2.99792458 \times 10^8 ms^{-1}$. The Planck units are

$$m_p \equiv \sqrt{\frac{\hbar c}{G}}, \quad l_p \equiv \sqrt{\frac{\hbar G}{c^3}}, \quad t_p \equiv \sqrt{\frac{\hbar G}{c^5}}. \quad (4)$$

2. Static Case

Taking the time-independent Schrödinger equation, (2) with vanishing LHS, and with vanishing potential $V = 0$, the spherically symmetric solution is

$$\psi(r) = A + \frac{B}{r}, \quad (5)$$

where A and B are amplitude constants. Now taking the Newtonian gravitational potential

$$V = -\frac{GM\mathbf{m}_t}{r}, \quad (6)$$

equating the Schrödinger mass m_s , the gravitating mass M , and the test mass \mathbf{m}_t and using the notation

$$k \equiv \frac{2Gm_s^3}{\hbar^2}, \quad (7)$$

where k is of dimensions L^{-1} , the time-independent spherically symmetric Schrödinger equations becomes

$$0 = \psi_{rr} + \frac{2}{r}\psi_r + \frac{k}{r}\psi, \quad (8)$$

which is a Bessel equation with solution

$$\psi(r) = \frac{C}{\sqrt{r}} BesselJ(1, 2\sqrt{kr}) + \frac{D}{\sqrt{r}} BesselY(1, 2\sqrt{kr}), \quad (9)$$

where C and D are amplitude constants, Figure 1.

Note $\beta^2/4 = 3.67049266$ is where the intercept is. Briefly, for $n = 1$ (9) is replaced by trigonometric functions, for $n = 2$ (5) and (6) involve logs, for $n = 4$ the Bessel order and argument diverge, for $n \geq 5$ the external $\sqrt{r} \rightarrow r^{1-n/2}$, the argument $\sqrt{r} \rightarrow r^{2-n/2}$ and the order $1 \rightarrow (n-2)/(4-n)$. Expanding (9) to first order in r and choosing no mixing of the terms

$$A = \sqrt{r}C, \quad B = -\frac{D}{\pi\sqrt{k}}, \quad (10)$$

gives the lowest order correction to (5)

$$\psi(r) = A \left[1 - \frac{kr}{2} + \dots \right] + \frac{B}{r} \left[1 + kr(1-2\gamma) - kr \ln(\sqrt{kr}) + \dots \right]. \quad (11)$$

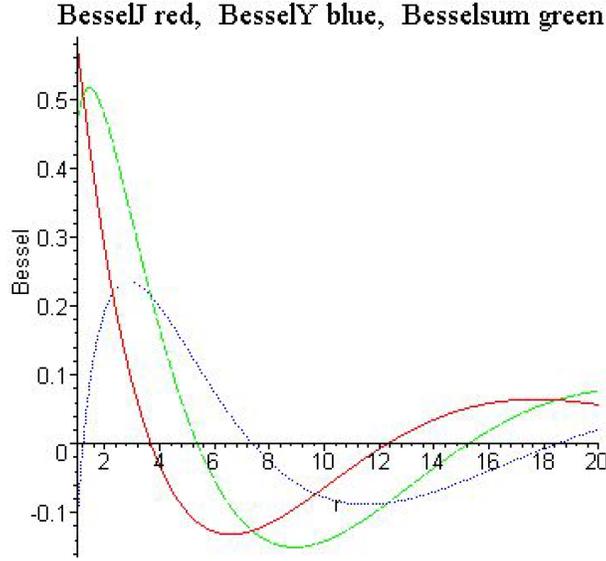


Figure 1. red line= $BesselJ(1, 2\sqrt{r})/\sqrt{r}$, blue dot= $BesselY(1, 2\sqrt{r})/\sqrt{r}$, green dash= $BesselJ(1, 2\sqrt{r})/\sqrt{r} + BesselY(1, 2\sqrt{r})/\sqrt{r}$, see equation (9)

At first sight this is counter intuitive as one would expect the addition of a potential to add short range decaying terms to the wavefunction; however, one should think of the Schrödinger equation as a statement of the conservation of energy and gravitational energy is negative, hence the increasing terms. That (9) sometimes has negative wavefunction is not necessarily unphysical as it is products $\psi\psi^*$ that correspond to measurable quantities. There is the question of what the zeros of ψ correspond to. The solutions (5) and (9) are preinterpretation solutions to the Schrödinger equation in the sense that one cannot construct expectations to the momenta and so forth as there is no time-dependence or overall energy: preinterpretational can be thought of as the Schrödinger equation (2) with the LHS taken to vanish. Another way of thinking of this is that a solution (5) or (9) is a choice of vacuum, so that normally one chooses only $A \neq 0$, but when self-gravitation is taken into account, the simplest choice is only $C \neq 0$. Once this choice has been taken, one has a critical distance where the wavefunction vanishes

$$\ell \equiv r_{crit} = \frac{\beta^2}{4k} = \frac{\beta^2 \hbar^2}{8Gm^3}. \quad (12)$$

A way of looking at this choice is that one takes only the BesselJ term in (9), the curve crosses the axis when $\psi = 0$, that is the critical value ℓ in (12).

3. Nonstatic Case

In the nonstatic case the time-dependent Gaussian wave-packet solution is

$$\psi(r, t) = \left[A + B \left(\frac{r}{f(t)} \right)^{(2-n)} \right] f(t)^{-\frac{n}{2}} \exp\left(-\frac{r^2}{2f(t)\sigma^2} \right) \quad (13)$$

where $f(t) \equiv 1 + i\hbar t/(m\sigma^2)$, A , B are constants as in (5), σ is the raw variance. This was found by trial and error substitutions into the Schrödinger equation (2). The A term corresponds to a flat solution with

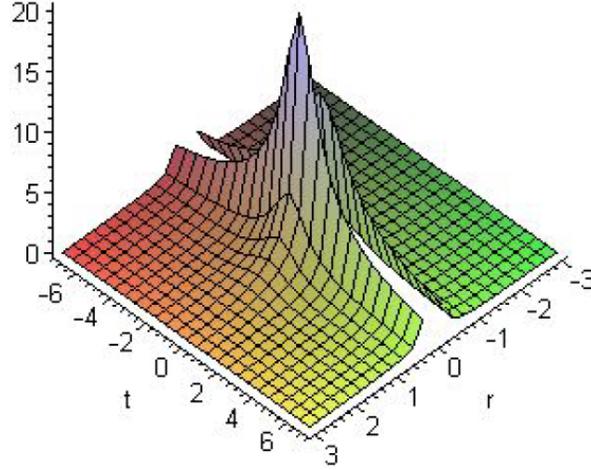


Figure 2. Matterhorn of A and B amplitudes both non-vanishing, see equation (13)

a Gaussian added, the B term corresponds to a reciprocal point particle potential with a Gaussian added, this term diverges as r goes to 0, and the terms can be added as can be checked explicitly and as would be anticipated from the superposition principle. The solution is to a $V = 0$ Schrödinger equation, adding $V = -\alpha/r^2$ is straightforward, time-dependent solutions for other potentials in particular for $V = -k/r$ are unknown. The solution can be expressed in terms of modified Whittaker functions

$$\psi(r, t) = \left[A + B \left(\frac{r}{f(t)} \right)^{(2-n)} \right] f(t)^{-\frac{n}{2}} \left(-\frac{r^2}{\sigma^2 f(t)} \right)^\alpha \text{WhittakerM}\left\{ \alpha, -\alpha - 1/2, -\frac{r^2}{\sigma^2 f(t)} \right\} \quad (14)$$

or hypergeometric functions

$$\psi(r, t) = \left[A + B \left(\frac{r}{f(t)} \right)^{(2-n)} \right] f(t)^{-\frac{n}{2}} \text{hypergeom}\left\{ [-2\alpha], [-2\alpha], -\frac{r^2}{2\sigma^2 f(t)} \right\}, \quad (15)$$

where α is a constant. There does not appear to be a solution with the Gaussian distribution replaced by any Pearson distribution or a similar distribution, however because to the radially dependent terms in front of the gaussian there is now nonvanishing excess kurtosis, and σ is no longer the variance which is why it is called the raw variance above. Setting $n = 3$, $A = B = m = \hbar = \sigma = 1$ then $\psi\psi^*$ gives the Matterhorn shape in the Figure 2, there is a divergence at the origin as would be expected for $B \neq 0$ because of the divergence of the reciprocal potential.

There seems to be no time-dependent version of the Bessel solutions (9) or even an approximation to this. The characteristic delocalization time interval in the above is

$$t_{deloc} = \frac{m\sigma^2}{\hbar}, \quad (16)$$

usually the raw variance σ is taken to be a hand chosen delocalization length; however, taking it to be ℓ given by (12) gives the characteristic time

$$\tau = \frac{\beta^4 \hbar^3}{64G^2 m^5}. \quad (17)$$

4. Discussion and conclusion

Critical lengths ℓ are taken to occur when $\psi = 0$ in a solution to the Schrödinger equation, because the wavefunction vanishes at this length, and so do measurable quantities such as energy density and momentum. The critical lengths (12) and times (17) for typical masses are given in the table below: for elementary particles they are too long to be measured, whereas for astronomical sized particles they are too short to be observed. In both cases any effect would be masked by other factors; however, for objects the size of a virus there is a possibility of a measurable effect. The long range for elementary particles might suggest that they have an effect in the 'next' universe, see [5], however relativistic cosmology has the speed of light built into it from the beginning, so that the critical length (12) is not really applicable.

Table. Table of Characteristic Quantities.

Particle	Mass in Kg.	Distance ℓ in meters	Time τ in seconds
Electron	9×10^{-31}	4×10^{32}	2×10^{69}
Proton	2×10^{-27}	3×10^{22}	3×10^{52}
Lead atom	4×10^{-25}	5×10^{15}	9×10^{40}
Buckyball molecule	1×10^{-24}	3×10^{14}	9×10^{38}
Protein	6×10^{-23}	1×10^9	1×10^{30}
Haemoglobin	1×10^{-22}	3×10^8	9×10^{28}
DNA	2×10^{-21}	4×10^4	3×10^{22}
Small virus	7×10^{-20}	9×10^{-1}	5×10^{14}
Large virus	1×10^{-17}	3×10^{-7}	9×10^3
Bacteria	9×10^{-16}	4×10^{-13}	2×10^{-6}
Yeast	6×10^{-14}	4×10^{-17}	3×10^{-13}
Man	9×10^1	4×10^{-64}	2×10^{-91}
Earth	6×10^{24}	1×10^{-132}	1×10^{-205}
Sun	2×10^{30}	4×10^{-148}	3×10^{-233}
Galaxy	6×10^{42}	1×10^{-186}	1×10^{-295}
Cluster	1×10^{46}	3×10^{-196}	9×10^{-312}

The static generalization of (9) to the Klein-Gordon equation is immediate as the time derivative terms do not enter; however, in the time-dependent case generalization of (13) to the Klein-Gordon equation is unlikely to have a similar form as (13) has single powers of \hbar which do not occur in the Klein-Gordon equation. No method is known to generalize to the Dirac equation.

Comparison can be made between the values in the table and other known radii: the classical electron radius is $r_e = e^2/(mc^2) = 3 \times 10^{-15}m$ and the Bohr radius is $a_0 = (4\pi\epsilon_0\hbar^2)/(m_e e^2) = 5 \times 10^{-11}m$, both of which are of orders of magnitude different from the values of ℓ in the table below. Note that the denominator of the Bohr radius is of similar form to (12) when $e \rightarrow m$. These radii govern lattice spacings and cross sections, but it is not clear what, if anything, is a lattice dependent on ℓ or how cross sections could depend on it; presumably any lattice spacing would be about the size of a virus, which is larger than usual.

Using the Compton wavelength $r_c = \hbar/(mc)$, the Schwarzschild radius $r_s = (2Gm)/c^2$, Planck units (4), the critical length (12), and the critical time (17), it is possible to produce the dimensionless ratios

$$\frac{r_s}{r_c} = 2 \left(\frac{m}{m_p} \right)^2, \quad \frac{r_c}{\ell} = \frac{8}{\beta^2} \left(\frac{m}{m_p} \right)^2, \quad \frac{r_s}{\ell} = \frac{16}{\beta^2} \left(\frac{m}{m_p} \right)^4, \quad \frac{t_p}{\tau} = \frac{64}{\beta^4} \left(\frac{m}{m_p} \right)^5, \quad (18)$$

which shows that there are no new dimensionless quantities except the universal mathematical constant β , the only dimensionless quantity which takes a different value for each particle is the mass in Planck units.

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