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Conditions for starlikeness of the Libera operator

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Abstract
Let $\mathcal{A}$ denote the class of functions f that are analytic in the unit disc $\mathbb{D}$ and normalized by $f(0) = f'(0) - 1 = 0$. In this paper some conditions are determined for starlikeness of the Libera integral operator $F(z) = \frac{2}{z}\int_0^z f(t)\,\mathrm{d}t$.
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1 Introduction

Let $\mathcal{H}$ be the class of functions analytic in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and let us denote by $\mathcal{A}_n$ the class of functions $f \in \mathcal{H}$ with the normalization of the form

$$f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \cdots, \quad z \in \mathbb{D},$$

with $\mathcal{A}_1 = \mathcal{A}$.

Let $\mathcal{SS}^*(\beta)$ denote the class of strongly starlike functions of order β, $0 < \beta \le 1$,

$$\mathcal{SS}^*(\beta) = \left\{f \in \mathcal{A} : \left|\arg \frac{zf'(z)}{f(z)}\right| < \frac{\beta\pi}{2}, z \in \mathbb{D}\right\},$$

which was introduced in [1] and [2], and $\mathcal{SS}^*(1) \equiv \mathcal{S}^*$ is the well-known class of starlike functions in $\mathbb{D}$. Functions in the class

$$\mathcal{R}(\beta) = \left\{f \in \mathcal{A} : \mathfrak{Re}\left\{f'(z)\right\} > \beta, z \in \mathbb{D}\right\},$$

where $\beta < 1$ are called functions with bounded turning. The Libera transform $L : \mathcal{A} \to \mathcal{A}$, $L[f] = F$, where

$$F(z) = \frac{2}{z}\int_0^z f(t)\,\mathrm{d}t, \tag{1.1}$$

is the Libera integral operator, which has been studied by several authors on different counts. In [3] Mocanu considered the problem of starlikeness of F and proved the following result.

Theorem 1.1 [3] *If $f(z)$ is analytic and $\mathfrak{Re}\{f'(z)\} > 0$ in the unit disc $\mathbb{D}$ and if the function F is given in* (1.1), *then $F \in \mathcal{S}^*$.*

This result may be written briefly as follows:

$$L\big[\mathcal{R}(0)\big] \subset \mathcal{S}^* = \mathcal{SS}^*(1), \tag{1.2}$$

where $L[\mathcal{R}(0)] = \{L[f] : f \in \mathcal{R}(0)\}$. In 1995 Mocanu [4] improved (1.2) by showing that

$$L\big[\mathcal{R}(0)\big] \subset \mathcal{SS}^*(8/9). \tag{1.3}$$

In 2002 Miller and Mocanu [5] showed that a subcase of this last result can be sharpened to

$$L\big[\mathcal{R}(0) \cap \mathcal{A}_2\big] \subset \mathcal{SS}^*(2/3).$$

The problem of strongly starlikeness of $L[f]$ for $f \in \mathcal{R}(0)$ was consider also in [6] where it is shown that

$$L\big[\mathcal{R}(0) \cap \mathcal{A}_2\big] \subset \mathcal{SS}^*(3/5).$$

The above inclusion relationship is equivalent to the following differential implication:

$$f \in \mathcal{A}_2 \quad \text{and} \quad \mathfrak{Re}\big\{f'(z)\big\} > 0 \quad \Longrightarrow \quad \left|\arg\left\{\frac{zF'(z)}{F(z)}\right\}\right| < \frac{3\pi}{10}$$

or equivalently

$$F \in \mathcal{A}_2 \quad \text{and} \quad \mathfrak{Re}\left\{F'(z) + \frac{1}{2}zF''(z)\right\} > 0 \quad \Longrightarrow \quad \left|\arg\left\{\frac{zF'(z)}{F(z)}\right\}\right| < \frac{3\pi}{10},$$

where F is given by (1.1).

In [7] Ponnusamy improved (1.2) by showing that

$$L\big[\mathcal{R}(-\varrho)\big] \subset \mathcal{S}^*, \quad \varrho = 0.09032572\ldots. \tag{1.4}$$

On the order of starlikeness of convex functions was considered also in the recent paper [8].

2 Main result

In this paper we go back to the problem of starlikeness of Libera transform. We need the following lemmas.

Lemma 2.1 [9, p.73] *Let n be a positive integer, $\lambda > 0$ and let $\beta_0 = \beta_0(\lambda, n)$ be the positive root of the equation*

$$\beta\pi = 3\pi/2 - \tan^{-1}(n\lambda\beta). \tag{2.1}$$

In addition, let

$$\alpha = \alpha(\beta, \lambda, n) = \beta + (2/\pi)\tan^{-1}(n\lambda\beta) \tag{2.2}$$

for $0 < \beta \leq \beta_0$. *If* $p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \cdots$ *is analytic in* $\mathbb{D}$, *then*

$$p(z) + \lambda z p'(z) \prec \left(\frac{1+z}{1-z}\right)^{\alpha}, \quad z \in \mathbb{D}, \tag{2.3}$$

implies the following subordination:

$$p(z) \prec \left(\frac{1+z}{1-z}\right)^{\beta}, \quad z \in \mathbb{D}. \tag{2.4}$$

If in Lemma 2.1 we put $n = 1$, $\lambda = 1/2$, then the solution β_0 of (2.1) satisfies $\beta_0 > 1$, so we may take $\beta = 1$, which gives $\pi\alpha/2 = \pi/2 + \tan^{-1}(1/2) = 2.03\ldots$.

Corollary 2.2 *Assume that* $f(z) \in \mathcal{A}_1$. *If*

$$\left|\arg\left\{F'(z) + (1/2)zF''(z)\right\}\right| < \pi/2 + \tan^{-1}(1/2) = 2.03\ldots, \quad z \in \mathbb{D},$$

then

$$\mathfrak{Re}\left\{F'(z)\right\} > 0, \quad z \in \mathbb{D}.$$

Note that if $F(z) \in \mathcal{A}_2$, then a sufficient condition for $F \in \mathcal{R}(0)$ is $|\arg\{f'(z)\}| < 3\pi/4 = 2.356\ldots$; see [5, p.96].

Lemma 2.3 [10] *Let* $p(z)$ *be of the form*

$$p(z) = 1 + \sum_{n=m\geq 1}^{\infty} a_n z^n, \quad a_m \neq 0 \ (z \in \mathbb{D}), \tag{2.5}$$

with $p(z) \neq 0$ *in* $\mathbb{D}$. *If there exists a point* z_0, $|z_0| < 1$, *such that*

$$\left|\arg\left\{p(z)\right\}\right| < \pi\alpha/2 \quad \text{in } |z| < |z_0| \quad \text{and} \quad \left|\arg\left\{p(z_0)\right\}\right| = \pi\alpha/2$$

for some $\alpha > 0$, *then we have*

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\alpha,$$

where

$$k \geq m\left(a^2+1\right)/(2a), \quad \text{when } \arg\left\{p(z_0)\right\} = \pi\alpha/2 \tag{2.6}$$

and

$$k \leq -m\left(a^2+1\right)/(2a), \quad \text{when } \arg\left\{p(z_0)\right\} = -\pi\alpha/2, \tag{2.7}$$

where

$$\left\{p(z_0)\right\}^{1/\alpha} = \pm ia, \quad a > 0.$$

Lemma 2.4 [9, p.75], [11] *Let $p(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$ be analytic in the unit disc $\mathbb{D}$. If*

$$p(z) + zp'(z) \prec \frac{1+z}{1-z}, \quad z \in \mathbb{D},$$

then

$$p(z) \prec q(z) = \frac{2}{z} \log \frac{1}{1-z} - 1$$

and

$$\left|\arg\left\{p(z)\right\}\right| < \theta_0 = \max_{|z|=1} \left|\arg\left\{q(z)\right\}\right| = 0.9110\ldots, \quad z \in \mathbb{D}, \tag{2.8}$$

where θ_0 lies between 0.911621904 *and* 0.911621907.

Theorem 2.5 *Let $q(z)$ be analytic in $\mathbb{D}$ and suppose that*

$$\left|\arg\left\{q(z)\right\}\right| < \frac{\beta\pi}{2}, \quad z \in \mathbb{D}$$

for some $\beta \in (0,1]$. If $p(z)$ is analytic and $p(z) \neq 0$ in $\mathbb{D}$ with $p(0) = 1$ and such that

$$\left|\arg\left\{q(z)\left(zp'(z) + p^2(z) + p(z)\right)\right\}\right| < \tan^{-1} \beta, \quad z \in \mathbb{D}, \tag{2.9}$$

then we have

$$\left|\arg\left\{p(z)\right\}\right| < \frac{\beta\pi}{2}, \quad z \in \mathbb{D}.$$

Proof If there exists a point z_0, $|z_0| < 1$, for which

$$\left|\arg\left\{p(z)\right\}\right| < \pi\beta/2 \quad \left(|z| < |z_0|\right)$$

and

$$\left|\arg\left\{p(z_0)\right\}\right| = \pi\beta/2, \quad p(z_0) = (\pm ia)^{\beta},$$

then from Nunokawa's Lemma 2.3, we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\beta,$$

where

$$k \geq \frac{a^2+1}{2a} \geq 1, \quad \text{when } \arg\left\{p(z_0)\right\} = \pi\beta/2$$

and

$$k \leq -\frac{a^2+1}{2a} \leq -1, \quad \text{when } \arg\left\{p(z_0)\right\} = -\pi\beta/2.$$

For the case $\arg\{p(z_0)\} = \beta\pi/2$, we have

$$\begin{aligned}
&\left|\arg\left\{q(z_0)\left[z_0p'(z_0) + p^2(z_0) + p(z_0)\right]\right\}\right| \\
&\quad = \left|\arg\left\{q(z_0)p(z_0)\left[1 + p(z_0) + z_0p'(z_0)/p(z_0)\right]\right\}\right| \\
&\quad = \left|\frac{\pi\beta}{2} + \arg\left\{q(z_0)\right\} + \arg\left\{1 + p(z_0) + z_0p'(z_0)/p(z_0)\right\}\right| \\
&\quad = \left|\frac{\pi\beta}{2} + \arg\left\{q(z_0)\right\} + \tan^{-1}\left\{\frac{\beta k + a^\beta\sin(\pi\beta/2)}{1 + a^\beta\cos(\pi\beta/2)}\right\}\right|,
\end{aligned} \tag{2.10}$$

where $p(z_0) = (ia)^\beta$, $0 < a$ and

$$k \geq \frac{a^2+1}{2a} \geq 1.$$

Let us put

$$g(a) = \frac{k\beta + a^\beta\sin(\pi\beta/2)}{1 + a^\beta\cos(\pi\beta/2)}, \quad 0 < a,$$

then it is easy to see that

$$g(a) \geq \frac{\beta + a^\beta\sin(\pi\beta/2)}{1 + a^\beta\cos(\pi\beta/2)}, \quad 0 < a. \tag{2.11}$$

Putting

$$h(x) = \frac{\beta + x\sin(\pi\beta/2)}{1 + x\cos(\pi\beta/2)}, \quad 0 \leq x,$$

we have

$$h'(x) = \frac{\sin(\pi\beta/2) - \beta\cos(\pi\beta/2)}{(1 + x\cos(\pi\beta/2))^2} > 0, \quad 0 \leq x,$$

because $\tan(\pi\beta/2) > \beta$. Therefore, for $x > 0$ we get $h(x) > h(0) = \beta$, so from (2.11) we have

$$g(a) > \beta,$$

and so

$$\tan^{-1}\left\{\frac{k\beta + a^\beta\sin(\pi\beta/2)}{1 + a^\beta\cos(\pi\beta/2)}\right\} > \tan^{-1}\beta, \quad 0 < a.$$

Therefore, we have the following inequality from (2.10):

$$\begin{aligned}
&\left|\arg\left\{q(z_0)\left[z_0p'(z_0) + p^2(z_0) + p(z_0)\right]\right\}\right| \\
&\quad \geq \frac{\pi\beta}{2} + \tan^{-1}\frac{k + a^\beta\sin(\pi\beta/2)}{1 + a^\beta\cos(\pi\beta/2)} - \left|\arg\left\{q(z_0)\right\}\right| \qquad (2.12) \\
&\quad > \tan^{-1}\beta. \qquad (2.13)
\end{aligned}$$

This contradicts the hypothesis and for the case $\arg\{p(z_0)\} = -\beta\pi/2$, applying the same method as above, we also have (2.12). This is also a contradiction and it completes the proof. □

Corollary 2.6 *Assume that*

$$\left|\arg\{f'(z)\}\right| < \tan^{-1}\beta, \quad z \in \mathbb{D} \tag{2.14}$$

and

$$\left|\arg\{F(z)/z\}\right| < \frac{\beta\pi}{2}, \quad z \in \mathbb{D} \tag{2.15}$$

for some $\beta \in (0,1]$, *where* $F(z)$ *is given in* (1.1). *Then we have*

$$\left|\arg\left\{\frac{zF'(z)}{F(z)}\right\}\right| < \frac{\beta\pi}{2}, \quad z \in \mathbb{D},$$

hence $F(z)$ *is strongly starlike of order* β.

Proof If we set

$$p(z) = \frac{zF'(z)}{F(z)},$$

then

$$f'(z) = F'(z) + \frac{1}{2}zF''(z) = \frac{1}{2}\left(\frac{F(z)}{z}\right)\left(zp'(z) + p^2(z) + p(z)\right).$$

If we let $q(z) = F(z)/z$, then by (2.14) and (2.15) the assumptions of Theorem 2.5 are satisfied. Therefore,

$$\left|\arg\{p(z)\}\right| < \frac{\beta\pi}{2}, \quad z \in \mathbb{D}.$$ □

Theorem 2.7 *Let* $q(z)$ *be analytic in* $\mathbb{D}$, *with* $q(0) = 1$ *and satisfy*

$$\mathfrak{Re}\{zq'(z) + q(z)\} > 0, \quad z \in \mathbb{D}.$$

If $p(z)$ *is analytic in* $\mathbb{D}$, *with* $p(0) = 1$ *and if*

$$\left|\arg\{q(z)\left(zp'(z) + p^2(z) + p(z)\right)\}\right| < \frac{5\pi}{6} - \theta_0 = 1.706\ldots, \quad z \in \mathbb{D},$$

where θ_0 *is given in* (2.8), *then we have*

$$\mathfrak{Re}\{p(z)\} > 0, \quad z \in \mathbb{D}.$$

Proof By Lemma 2.4, we have

$$\left|\arg\{q(z)\}\right| < \theta_0 = 0.911\ldots, \quad z \in \mathbb{D}. \tag{2.16}$$

If there exists a point z_0, $|z_0| < 1$, such that

$$\left|\arg\{p(z)\}\right| < \pi/2 \quad \left(|z| < |z_0|\right)$$

and

$$\left|\arg\{p(z_0)\}\right| = \pi/2, \quad p(z_0) = \pm ia, 0 < a,$$

then from Nunokawa's Lemma 2.3, we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik,$$

where

$$k \geq \frac{a^2+1}{2a} \geq 1, \quad \text{when } \arg\{p(z_0)\} = \pi/2$$

and

$$k \leq -\frac{a^2+1}{2a} \leq -1, \quad \text{when } \arg\{p(z_0)\} = -\pi/2.$$

For the case $\arg\{p(z_0)\} = \pi/2$, we have

$$\begin{aligned}
\arg\{1 + ia + ik\} &\geq \arg\left\{1 + ia + i\frac{a^2+1}{2a}\right\} \\
&= \tan^{-1}\frac{\mathfrak{Im}\{1 + ia + i\frac{a^2+1}{2a}\}}{\mathfrak{Re}\{1 + ia + i\frac{a^2+1}{2a}\}} \\
&= \tan^{-1}\left\{\frac{3a^2+1}{2a}\right\} \\
&\geq \tan^{-1}\{\sqrt{3}\} \\
&= \frac{\pi}{3}.
\end{aligned}$$

Therefore, for the case $\arg\{p(z_0)\} = \pi/2$, we have

$$\frac{\pi}{3} \leq \arg\{1 + ia + ik\} < \frac{\pi}{2}.$$

Moreover, by (2.16)

$$\arg\{q(z_0)\} < \theta_0.$$

Therefore, we can write

$$\begin{aligned}
&\left|\arg\{q(z_0)\left(z_0 p'(z_0) + p^2(z_0) + p(z_0)\right)\}\right| \\
&\quad = \left|\arg\{p(z_0)\left[1 + p(z_0) + z_0 p'(z_0)/p(z_0)\right]q(z_0)\}\right| \\
&\quad \geq \left|\arg\{p(z_0)(1 + ia + ik)\}\right| - \left|\arg\{q(z_0)\}\right|
\end{aligned}$$

$$\begin{aligned} &\geq \frac{\pi}{2} + \frac{\pi}{3} - \left|\arg\left\{q(z_0)\right\}\right| \\ &\geq \frac{5\pi}{6} - \theta_0. \end{aligned} \tag{2.17}$$

This contradicts the hypothesis and for the case $\arg\{p(z_0)\} = -\pi/2$, applying the same method as above, we have

$$\left|\arg\left\{q(z_0)\left(z_0 p'(z_0) + p^2(z_0) + p(z_0)\right)\right\}\right| \geq \frac{5\pi}{6} - \theta_0.$$

This is also a contradiction and it completes the proof. □

Corollary 2.8 *Assume that*

$$\left|\arg\left\{f'(z)\right\}\right| < \frac{5\pi}{6} - \theta_0 = 1.706\ldots, \quad z \in \mathbb{D}, \tag{2.18}$$

then we have

$$\mathfrak{Re}\left\{\frac{zF'(z)}{F(z)}\right\} > 0, \quad z \in \mathbb{D}, \tag{2.19}$$

where $F(z)$ *is Libera integral given in* (1.1).

Proof Because

$$f'(z) = F'(z) + \frac{1}{2}zF''(z),$$

by Corollary 2.2 and by (2.18) we obtain

$$\mathfrak{Re}\left\{F'(z)\right\} > 0, \quad z \in \mathbb{D}.$$

Therefore, if we let $q(z) = F(z)/z$, then

$$\mathfrak{Re}\left\{zq'(z) + q(z)\right\} = \mathfrak{Re}\left\{F'(z)\right\} > 0, \quad z \in \mathbb{D}.$$

If we set

$$p(z) = \frac{zF'(z)}{F(z)},$$

then

$$f'(z) = F'(z) + \frac{1}{2}zF''(z) = \frac{1}{2}\left(\frac{F(z)}{z}\right)\left(zp'(z) + p^2(z) + p(z)\right).$$

The assumptions of Theorem 2.7 are satisfied. Therefore, (2.19) holds. □

Corollary 2.8 is an extension of Mocanu's result (1.2) from the paper [3] because in (2.18) we have $|\arg\{f'(z)\}| < 1.706\ldots$, while in (1.2) we have the stronger assumption that $|\arg\{f'(z)\}| < \pi/2 = 1.57\ldots$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors jointly worked on the results and they read and approved the final manuscript.

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