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Sharp boundedness for vector-valued multilinear integral operators

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Abstract
In this paper, the sharp inequalities for some vector-valued multilinear integral operators are obtained. As applications, we get the weighted L^p ($p > 1$) norm inequalities and an $L\log L$-type estimate for the vector-valued multilinear operators.
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1 Preliminaries and theorems

As the development of singular integral operators and their commutators, multilinear singular integral operators have been well studied (see [1–5]). In this paper, we study some vector-valued multilinear integral operators as follows.

Suppose that m_j are positive integers ($j = 1, \dots, l$), $m_1 + \cdots + m_l = m$ and A_j are functions on R^n ($j = 1, \dots, l$). Let $F_t(x, y)$ be defined on $R^n \times R^n \times [0, +\infty)$. Set

$$F_t(f)(x) = \int_{R^n} F_t(x,y)f(y)\,dy$$

and

$$F_t^A(f)(x) = \int_{R^n} \frac{\prod_{j=1}^l R_{m_j+1}(A_j;x,y)}{|x-y|^m} F_t(x,y)f(y)\,dy$$

for every bounded and compactly supported function f, where

$$R_{m_j+1}(A_j;x,y) = A_j(x) - \sum_{|\alpha|\le m_j} \frac{1}{\alpha!} D^\alpha A_j(y)(x-y)^\alpha.$$

Let H be the Banach space $H = \{h : \|h\| < \infty\}$ such that, for each fixed $x \in R^n$, $F_t(f)(x)$ and $F_t^A(f)(x)$ may be viewed as a mapping from $[0, +\infty)$ to H. For $1 < s < \infty$, the vector-valued multilinear operator related to F_t is defined by

$$\left|T^A(f)(x)\right|_s = \left(\sum_{i=1}^\infty \left(T^A(f_i)(x)\right)^s\right)^{1/s},$$

where

$$T^A(f_i)(x) = \left\| F_t^A(f_i)(x) \right\|,$$

and F_t satisfies: for fixed $\varepsilon > 0$,

$$\left\| F_t(x,y) \right\| \le C|x-y|^{-n}$$

and

$$\left\| F_t(y,x) - F_t(z,x) \right\| \le C|y-z|^{\varepsilon}|x-z|^{-n-\varepsilon}$$

if $2|y-z| \le |x-z|$. Set

$$\left| T(f)(x) \right|_s = \left(\sum_{i=1}^{\infty} \left| T(f_i)(x) \right|^s \right)^{1/s} \quad \text{and} \quad |f|_s = \left(\sum_{i=1}^{\infty} \left| f_i(x) \right|^s \right)^{1/s}.$$

Suppose that $|T|_s$ is bounded on $L^p(R^n)$ for $1 < p < \infty$ and weak (L^1, L^1)-bounded.

Note that when $m = 0$, T^A is just a vector-valued multilinear commutator of T and A (see [6]). While when $m > 0$, T^A is a non-trivial generalization of the commutator. It is well known that multilinear operators are of great interest in harmonic analysis and have been studied by many authors (see [1–5]). In [7], Hu and Yang proved a variant sharp estimate for the multilinear singular integral operators. In [6], Pérez and Trujillo-Gonzalez proved a sharp estimate for some multilinear commutator. The main purpose of this paper is to prove a sharp inequality for the vector-valued multilinear integral operators. As applications, we obtain the weighted L^p $(p > 1)$ norm inequalities and an $L \log L$-type estimate for the vector-valued multilinear operators.

First, let us introduce some notations. Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For any locally integrable function f, the sharp function of f is defined by

$$f^{\#}(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q \left| f(y) - f_Q \right| dy,$$

where, and in what follows, $f_Q = |Q|^{-1} \int_Q f(x)\,dx$. It is well known that (see [8])

$$f^{\#}(x) \approx \sup_{x \in Q} \inf_{c \in C} \frac{1}{|Q|} \int_Q \left| f(y) - c \right| dy.$$

We say that f belongs to $BMO(R^n)$ if $f^{\#}$ belongs to $L^\infty(R^n)$ and $\|f\|_{BMO} = \|f^{\#}\|_{L^\infty}$. For $0 < r < \infty$, we denote $f_r^{\#}$ by

$$f_r^{\#}(x) = \left[\left(|f|^r \right)^{\#}(x) \right]^{1/r}.$$

Let M be the Hardy-Littlewood maximal operator, that is, $M(f)(x) = \sup_{x \in Q} |Q|^{-1} \times \int_Q |f(y)|\,dy$. For $k \in N$, we denote by M^k the operator M iterated k times, *i.e.*, $M^1(f)(x) = M(f)(x)$ and $M^k(f)(x) = M(M^{k-1}(f))(x)$ for $k \ge 2$.

Let Φ be a Young function and $\tilde{\Phi}$ be the complementary associated to Φ, we denote the Φ-average by, for a function f,

$$\|f\|_{\Phi,Q} = \inf\left\{\lambda > 0 : \frac{1}{|Q|}\int_Q \Phi\left(\frac{|f(y)|}{\lambda}\right)dy \le 1\right\}$$

and the maximal function associated to Φ by

$$M_\Phi(f)(x) = \sup_{x\in Q}\|f\|_{\Phi,Q}.$$

The Young functions to be used in this paper are $\Phi(t) = \exp(t^r) - 1$ and $\Psi(t) = t\log^r(t+e)$, the corresponding Φ-average and maximal functions are denoted by $\|\cdot\|_{\exp L^r,Q}$, $M_{\exp L^r}$ and $\|\cdot\|_{L(\log L)^r,Q}$, $M_{L(\log L)^r}$. We have the following inequality, for any $r > 0$ and $m \in N$ (see [6])

$$M(f) \le M_{L(\log L)^r}(f), \qquad M_{L(\log L)^m}(f) \approx M^{m+1}(f).$$

For $r \ge 1$, we denote that

$$\|b\|_{\mathrm{osc}_{\exp L^r}} = \sup_Q \|b - b_Q\|_{\exp L^r,Q},$$

the space $\mathrm{Osc}_{\exp L^r}$ is defined by

$$\mathrm{Osc}_{\exp L^r} = \left\{b \in L^1_{\log}\left(R^n\right) : \|b\|_{\mathrm{osc}_{\exp L^r}} < \infty\right\}.$$

It has been known that (see [6])

$$\|b - b_{2Q}\|_{\exp L^r,2^kQ} \le Ck\|b\|_{\mathrm{Osc}_{\exp L^r}}.$$

It is obvious that $\mathrm{Osc}_{\exp L^r}$ coincides with the BMO space if $r = 1$, and $\mathrm{Osc}_{\exp L^r} \subset BMO$ if $r > 1$. We denote the Muckenhoupt weights by A_p for $1 \le p < \infty$ (see [8]).

Now we state our main results as follows.

Theorem 1 *Let $1 < s < \infty$, $r_j \ge 1$ and $D^\alpha A_j \in \mathrm{Osc}_{\exp L^{r_j}}$ for all α with $|\alpha| = m_j$ and $j = 1,\ldots,l$. Define $1/r = 1/r_1 + \cdots + 1/r_l$. Then, for any $0 < p < 1$, there exists a constant $C > 0$ such that for any $f = \{f_i\} \in C_0^\infty(R^n)$ and $x \in R^n$,*

$$\left(\left|T^A(f)\right|_s\right)_p^\#(x) \le C\prod_{j=1}^{l}\left(\sum_{|\alpha_j|=m_j}\left\|D^{\alpha_j}A_j\right\|_{\mathrm{Osc}_{\exp L^{r_j}}}\right)M_{L(\log L)^{1/r}}\left(|f|_s\right)(x).$$

Theorem 2 *Let $1 < s < \infty$, $r_j \ge 1$ and $D^\alpha A_j \in \mathrm{Osc}_{\exp L^{r_j}}$ for all α with $|\alpha| = m_j$ and $j = 1,\ldots,l$.*

(1) *If $1 < p < \infty$ and $w \in A_p$, then*

$$\left\|\left|T^A(f)\right|_s\right\|_{L^p(w)} \le C\prod_{j=1}^{l}\left(\sum_{|\alpha_j|=m_j}\left\|D^{\alpha_j}A_j\right\|_{\mathrm{Osc}_{\exp L^{r_j}}}\right)\left\||f|_s\right\|_{L^p(w)};$$

(2) *If* $w \in A_1$. *Define* $1/r = 1/r_1 + \cdots + 1/r_l$ *and* $\Phi(t) = t\log^{1/r}(t+e)$. *Then there exists a constant* $C > 0$ *such that for all* $\lambda > 0$,

$$\begin{aligned} &w\left(\left\{x \in R^n : \left|T^A(f)(x)\right|_s > \lambda\right\}\right) \\ &\quad \le C\int_{R^n} \Phi\left[\lambda^{-1}\prod_{j=1}^{l}\left(\sum_{|\alpha_j|=m_j}\left\|D^{\alpha_j}A_j\right\|_{\mathrm{Osc}_{\exp L^{r_j}}}\right)\left|f(x)\right|_s\right]w(x)\,dx. \end{aligned}$$

Remark The conditions in Theorems 1 and 2 are satisfied by many operators.

Now we give some examples.

Example 1 Littlewood-Paley operators.

Fix $\varepsilon > 0$ and $\mu > (3n+2)/n$. Let ψ be a fixed function which satisfies the following properties:

(1) $\int_{R^n} \psi(x)\,dx = 0$,

(2) $|\psi(x)| \le C(1+|x|)^{-(n+1)}$,

(3) $|\psi(x+y) - \psi(x)| \le C|y|^{\varepsilon}(1+|x|)^{-(n+1+\varepsilon)}$ when $2|y| < |x|$.

We denote that $\Gamma(x) = \{(y,t) \in R_+^{n+1} : |x-y| < t\}$ and the characteristic function of $\Gamma(x)$ by $\chi_{\Gamma(x)}$. The Littlewood-Paley multilinear operators are defined by

$$\begin{aligned} g_\psi^A(f)(x) &= \left(\int_0^\infty \left|F_t^A(f)(x)\right|^2 \frac{dt}{t}\right)^{1/2}, \\ S_\psi^A(f)(x) &= \left[\int\int_{\Gamma(x)} \left|F_t^A(f)(x,y)\right|^2 \frac{dy\,dt}{t^{n+1}}\right]^{1/2} \end{aligned}$$

and

$$g_\mu^A(f)(x) = \left[\int\int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\mu} \left|F_t^A(f)(x,y)\right|^2 \frac{dy\,dt}{t^{n+1}}\right]^{1/2},$$

where

$$\begin{aligned} F_t^A(f)(x) &= \int_{R^n} \frac{\prod_{j=1}^{l} R_{m_j+1}(A_j;x,y)}{|x-y|^m}\psi_t(x-y)f(y)\,dy, \\ F_t^A(f)(x,y) &= \int_{R^n} \frac{\prod_{j=1}^{l} R_{m_j+1}(A_j;x,z)}{|x-z|^m}f(z)\psi_t(y-z)\,dz \end{aligned}$$

and $\psi_t(x) = t^{-n}\psi(x/t)$ for $t > 0$. Set $F_t(f)(y) = f * \psi_t(y)$. We also define that

$$\begin{aligned} g_\psi(f)(x) &= \left(\int_0^\infty \left|F_t(f)(x)\right|^2 \frac{dt}{t}\right)^{1/2}, \\ S_\psi(f)(x) &= \left(\int\int_{\Gamma(x)} \left|F_t(f)(y)\right|^2 \frac{dy\,dt}{t^{n+1}}\right)^{1/2} \end{aligned}$$

and

$$g_\mu(f)(x) = \left(\int\int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\mu} \left|F_t(f)(y)\right|^2 \frac{dy\,dt}{t^{n+1}}\right)^{1/2},$$

which are the Littlewood-Paley operators (see [9]). Let H be the space

$$H = \left\{h : \|h\| = \left(\int_0^\infty \left|h(t)\right|^2 dt/t\right)^{1/2} < \infty\right\}$$

or

$$H = \left\{h : \|h\| = \left(\int\int_{R_+^{n+1}} \left|h(y,t)\right|^2 dy\,dt/t^{n+1}\right)^{1/2} < \infty\right\}.$$

Then, for each fixed $x \in R^n$, $F_t^A(f)(x)$ and $F_t^A(f)(x,y)$ may be viewed as the mapping from $[0,+\infty)$ to H, and it is clear that

$$\begin{aligned} g_\psi^A(f)(x) &= \left\|F_t^A(f)(x)\right\|, \qquad g_\psi(f)(x) = \left\|F_t(f)(x)\right\|, \\ S_\psi^A(f)(x) &= \left\|\chi_{\Gamma(x)}F_t^A(f)(x,y)\right\|, \qquad S_\psi(f)(x) = \left\|\chi_{\Gamma(x)}F_t(f)(y)\right\| \end{aligned}$$

and

$$g_\mu^A(f)(x) = \left\|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2} F_t^A(f)(x,y)\right\|, \qquad g_\mu(f)(x) = \left\|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2} F_t(f)(y)\right\|.$$

It is easy to see that g_ψ, S_ψ and g_μ satisfy the conditions of Theorems 1 and 2 (see [10–12]), thus Theorems 1 and 2 hold for g_ψ^A, S_ψ^A and g_μ^A.

Example 2 Marcinkiewicz operators.

Fix $\lambda > \max(1, 2n/(n+2))$ and $0 < \gamma \le 1$. Let Ω be homogeneous of degree zero on R^n with $\int_{S^{n-1}} \Omega(x')\,d\sigma(x') = 0$. Assume that $\Omega \in \mathrm{Lip}_\gamma(S^{n-1})$. The Marcinkiewicz multilinear operators are defined by

$$\begin{aligned} \mu_\Omega^A(f)(x) &= \left(\int_0^\infty \left|F_t^A(f)(x)\right|^2 \frac{dt}{t^3}\right)^{1/2}, \\ \mu_S^A(f)(x) &= \left[\int\int_{\Gamma(x)} \left|F_t^A(f)(x,y)\right|^2 \frac{dy\,dt}{t^{n+3}}\right]^{1/2} \end{aligned}$$

and

$$\mu_\lambda^A(f)(x) = \left[\int\int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\lambda} \left|F_t^A(f)(x,y)\right|^2 \frac{dy\,dt}{t^{n+3}}\right]^{1/2},$$

where

$$F_t^A(f)(x) = \int_{|x-y|\le t} \frac{\prod_{j=1}^l R_{m_j+1}(A_j;x,y)}{|x-y|^m} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)\,dy$$

and

$$F_t^A(f)(x,y) = \int_{|y-z|\le t} \frac{\prod_{j=1}^l R_{m_j+1}(A_j;y,z)}{|y-z|^m} \frac{\Omega(y-z)}{|y-z|^{n-1}} f(z)\,dz.$$

Set

$$F_t(f)(x) = \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)\,dy.$$

We also define that

$$\mu_\Omega(f)(x) = \left(\int_0^\infty \left|F_t(f)(x)\right|^2 \frac{dt}{t^3}\right)^{1/2},$$
$$\mu_S(f)(x) = \left(\int\int_{\Gamma(x)} \left|F_t(f)(y)\right|^2 \frac{dy\,dt}{t^{n+3}}\right)^{1/2}$$

and

$$\mu_\lambda(f)(x) = \left(\int\int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\lambda} \left|F_t(f)(y)\right|^2 \frac{dy\,dt}{t^{n+3}}\right)^{1/2},$$

which are the Marcinkiewicz operators (see [13]). Let H be the space

$$H = \left\{h : \|h\| = \left(\int_0^\infty \left|h(t)\right|^2 dt/t^3\right)^{1/2} < \infty\right\}$$

or

$$H = \left\{h : \|h\| = \left(\int\int_{R_+^{n+1}} \left|h(y,t)\right|^2 dy\,dt/t^{n+3}\right)^{1/2} < \infty\right\}.$$

Then it is clear that

$$\mu_\Omega^A(f)(x) = \left\|F_t^A(f)(x)\right\|, \qquad \mu_\Omega(f)(x) = \left\|F_t(f)(x)\right\|,$$
$$\mu_S^A(f)(x) = \left\|\chi_{\Gamma(x)} F_t^A(f)(x,y)\right\|, \qquad \mu_S(f)(x) = \left\|\chi_{\Gamma(x)} F_t(f)(y)\right\|$$

and

$$\mu_\lambda^A(f)(x) = \left\|\left(\frac{t}{t+|x-y|}\right)^{n\lambda/2} F_t^A(f)(x,y)\right\|, \qquad \mu_\lambda(f)(x) = \left\|\left(\frac{t}{t+|x-y|}\right)^{n\lambda/2} F_t(f)(y)\right\|.$$

It is easy to see that μ_Ω, μ_S and μ_λ satisfy the conditions of Theorems 1 and 2 (see [13, 14]), thus Theorems 1 and 2 hold for μ_Ω^A, μ_S^A and μ_λ^A.

Example 3 Bochner-Riesz operators.

Let $\delta > (n-1)/2$, $B_t^\delta(\hat{f})(\xi) = (1-t^2|\xi|^2)_+^\delta \hat{f}(\xi)$ and $B_t^\delta(z) = t^{-n}B^\delta(z/t)$ for $t > 0$. Set

$$F_{\delta,t}^A(f)(x) = \int_{R^n} \frac{\prod_{j=1}^l R_{m_j+1}(A_j;x,y)}{|x-y|^m} B_t^\delta(x-y) f(y)\,dy.$$

The maximal Bochner-Riesz multilinear operators are defined by

$$B_{\delta,*}^A(f)(x) = \sup_{t>0}\left|B_{\delta,t}^A(f)(x)\right|.$$

We also define that

$$B_{\delta,*}(f)(x) = \sup_{t>0}\left|B_t^\delta(f)(x)\right|,$$

which is the maximal Bochner-Riesz operator (see [15]). Let H be the space $H = \{h : \|h\| = \sup_{t>0}|h(t)| < \infty\}$, then

$$B_{\delta,*}^A(f)(x) = \left\|B_{\delta,t}^A(f)(x)\right\|, \qquad B_*^\delta(f)(x) = \left\|B_t^\delta(f)(x)\right\|.$$

It is easy to see that $B_{\delta,*}^A$ satisfies the conditions of Theorems 1 and 2 (see [16]), thus Theorems 1 and 2 hold for $B_{\delta,*}^A$.

2 Some lemmas

We give some preliminary lemmas.

Lemma 1 ([3]) *Let A be a function on R^n and $D^\alpha A \in L^q(R^n)$ for all α with $|\alpha| = m$ and some $q > n$. Then*

$$\left|R_m(A;x,y)\right| \le C|x-y|^m \sum_{|\alpha|=m}\left(\frac{1}{|\tilde{Q}(x,y)|}\int_{\tilde{Q}(x,y)}\left|D^\alpha A(z)\right|^q dz\right)^{1/q},$$

where $\tilde{Q}$ is the cube centered at x and having side length $5\sqrt{n}|x-y|$.

Lemma 2 ([8, p.485]) *Let $0 < p < q < \infty$ and for any function $f \ge 0$, we define that, for $1/r = 1/p - 1/q$,*

$$\|f\|_{WL^q} = \sup_{\lambda>0}\lambda\left|\left\{x \in R^n : f(x) > \lambda\right\}\right|^{1/q}, \qquad N_{p,q}(f) = \sup_E \|f\chi_E\|_{L^p}/\|\chi_E\|_{L^r},$$

where the sup is taken for all measurable sets E with $0 < |E| < \infty$. Then

$$\|f\|_{WL^q} \le N_{p,q}(f) \le \left(q/(q-p)\right)^{1/p}\|f\|_{WL^q}.$$

Lemma 3 ([6]) *Let $r_j \ge 1$ for $j = 1,\dots,m$, we denote that $1/r = 1/r_1 + \cdots + 1/r_m$. Then*

$$\frac{1}{|Q|}\int_Q\left|f_1(x)\cdots f_m(x)g(x)\right|dx \le \|f\|_{\exp L^{r_1},Q}\cdots\|f\|_{\exp L^{r_m},Q}\|g\|_{L(\log L)^{1/r},Q}.$$

3 Proof of the theorem

It is only to prove Theorem 1.

Proof of Theorem 1 It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant C_0 that the following inequality holds:

$$\left(\frac{1}{|Q|}\int_Q\left|\left|T^A(f)(x)\right|_s - C_0\right|^p dx\right)^{1/p} \le C\prod_{j=1}^l\left(\sum_{|\alpha_j|=m_j}\left\|D^{\alpha_j}A_j\right\|_{\mathrm{Osc}_{\exp L^{r_j}}}\right)M_{L(\log L)^{1/r}}\left(|f|_s\right)(x).$$

Without loss of generality, we may assume $l = 2$. Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Let $\tilde{Q} = 5\sqrt{n}Q$ and $\tilde{A}_j(x) = A_j(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!}(D^\alpha A_j)_{\tilde{Q}} x^\alpha$, then $R_m(A_j; x, y) = R_m(\tilde{A}_j; x, y)$ and $D^\alpha \tilde{A}_j = D^\alpha A_j - (D^\alpha A_j)_{\tilde{Q}}$ for $|\alpha| = m_j$. We split $f = g + h = \{g_i\} + \{h_i\}$ for $g_i = f_i \chi_{\tilde{Q}}$ and $h_i = f_i \chi_{R^n \setminus \tilde{Q}}$. Write

$$\begin{aligned}
F_t^A(f_i)(x) &= \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{A}_j; x, y)}{|x-y|^m} F_t(x,y) f_i(y)\,dy \\
&= \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{A}_j; x, y)}{|x-y|^m} F_t(x,y) h_i(y)\,dy + \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j}(\tilde{A}_j; x, y)}{|x-y|^m} F_t(x,y) g_i(y)\,dy \\
&\quad - \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \int_{R^n} \frac{R_{m_2}(\tilde{A}_2; x, y)(x-y)^{\alpha_1}}{|x-y|^m} D^{\alpha_1}\tilde{A}_1(y) F_t(x,y) g_i(y)\,dy \\
&\quad - \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \int_{R^n} \frac{R_{m_1}(\tilde{A}_1; x, y)(x-y)^{\alpha_2}}{|x-y|^m} D^{\alpha_2}\tilde{A}_2(y) F_t(x,y) g_i(y)\,dy \\
&\quad + \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \frac{1}{\alpha_1!\alpha_2!} \int_{R^n} \frac{(x-y)^{\alpha_1+\alpha_2} D^{\alpha_1}\tilde{A}_1(y) D^{\alpha_2}\tilde{A}_2(y)}{|x-y|^m} F_t(x,y) g_i(y)\,dy.
\end{aligned}$$

Then, by Minkowski's inequality, we have

$$\begin{aligned}
&\left[\frac{1}{|Q|}\int_Q \left| \left|T^A(f)(x)\right|_s - \left|T^{\tilde{A}}(h)(x_0)\right|_s \right|^p dx\right]^{1/p} \\
&\quad \le \left[\frac{1}{|Q|}\int_Q \left| \left\|F_t^A(f)(x)\right\|_s - \left\|F_t^{\tilde{A}}(h)(x_0)\right\|_s \right|^p dx\right]^{1/p} \\
&\quad \le \left[\frac{1}{|Q|}\int_Q \left(\sum_{i=1}^\infty \left\|F_t^A(f_i)(x) - F_t^{\tilde{A}}(h_i)(x_0)\right\|^s\right)^{p/s} dx\right]^{1/p} \\
&\quad \le \left[\frac{C}{|Q|}\int_Q \left(\sum_{i=1}^\infty \left\|\int_{R^n} \frac{\prod_{j=1}^2 R_{m_j}(\tilde{A}_j; x, y)}{|x-y|^m} F_t(x,y) g_i(y)\,dy\right\|^s\right)^{p/s} dx\right]^{1/p} \\
&\qquad + \left[\frac{C}{|Q|}\int_Q \left(\sum_{i=1}^\infty \left\|\sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!}\right.\right.\right. \\
&\qquad \times \left.\left.\left.\int_{R^n} \frac{R_{m_2}(\tilde{A}_2; x, y)(x-y)^{\alpha_1}}{|x-y|^m} D^{\alpha_1}\tilde{A}_1(y) F_t(x,y) g_i(y)\,dy\right\|^s\right)^{p/s} dx\right]^{1/p} \\
&\qquad + \left[\frac{C}{|Q|}\int_Q \left(\sum_{i=1}^\infty \left\|\sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!}\right.\right.\right. \\
&\qquad \times \left.\left.\left.\int_{R^n} \frac{R_{m_1}(\tilde{A}_1; x, y)(x-y)^{\alpha_2}}{|x-y|^m} D^{\alpha_2}\tilde{A}_2(y) F_t(x,y) g_i(y)\,dy\right\|^s\right)^{p/s} dx\right]^{1/p} \\
&\qquad + \left[\frac{C}{|Q|}\int_Q \left(\sum_{i=1}^\infty \left\|\sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \frac{1}{\alpha_1!\alpha_2!}\right.\right.\right. \\
&\qquad \times \left.\left.\left.\int_{R^n} \frac{(x-y)^{\alpha_1+\alpha_2} D^{\alpha_1}\tilde{A}_1(y) D^{\alpha_2}\tilde{A}_2(y)}{|x-y|^m} F_t(x,y) g_i(y)\,dy\right\|^s\right)^{p/s} dx\right]^{1/p}
\end{aligned}$$

$$+\left[\frac{C}{|Q|}\int_Q\left(\sum_{i=1}^{\infty}\left\|F_t^{\tilde{A}}(h_i)(x)-F_t^{\tilde{A}}(h_i)(x_0)\right\|^s\right)^{p/s}dx\right]^{1/p}$$
$$:= I_1 + I_2 + I_3 + I_4 + I_5.$$

Now, let us estimate I_1, I_2, I_3, I_4 and I_5, respectively. First, for $x \in Q$ and $y \in \tilde{Q}$, by Lemma 1, we get

$$R_{m_j}(\tilde{A}_j;x,y) \le C|x-y|^{m_j}\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{Osc}_{\exp L^{r_j}}}.$$

Thus, by Lemma 2 and the weak type (1,1) of $|T|_s$, we obtain

$$\begin{aligned}
I_1 &\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{Osc}_{\exp L^{r_j}}}\right)\left(\frac{1}{|Q|}\int_Q\left|T(g)(x)\right|_s^p dx\right)^{1/p}\\
&= C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{Osc}_{\exp L^{r_j}}}\right)|Q|^{-1}\frac{\||T(g)|_s\chi_Q\|_{L^p}}{|Q|^{1/p-1}}\\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{Osc}_{\exp L^{r_j}}}\right)|Q|^{-1}\left\|\left|T(g)\right|_s\right\|_{WL^1}\\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{Osc}_{\exp L^{r_j}}}\right)|Q|^{-1}\left\||g|_s\right\|_{L^1}\\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{Osc}_{\exp L^{r_j}}}\right)M\left(|f|_s\right)(\tilde{x})\\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{Osc}_{\exp L^{r_j}}}\right)M_{L(\log L)^{1/r}}\left(|f|_s\right)(\tilde{x}).
\end{aligned}$$

For I_2, note that $\|\chi_Q\|_{\exp L^{r_2},Q} \le C$, similar to the proof of I_1 and by using Lemma 3, we get

$$\begin{aligned}
I_2 &\le C\sum_{|\alpha_2|=m_2}\|D^{\alpha_2}A_2\|_{\mathrm{Osc}_{\exp L^{r_2}}}\sum_{|\alpha_1|=m_1}\left(\frac{1}{|Q|}\int_{R^n}\left|T\left(D^{\alpha_1}\tilde{A}_1 g\right)(x)\right|_s^p dx\right)^{1/p}\\
&\le C\sum_{|\alpha_2|=m_2}\|D^{\alpha_2}A_2\|_{\mathrm{Osc}_{\exp L^{r_2}}}\sum_{|\alpha_1|=m_1}|Q|^{-1}\left\|\left|T\left(D^{\alpha_1}\tilde{A}_1 g\right)(x)\right|_s\chi_Q\right\|_{WL^1}\\
&\le C\sum_{|\alpha_2|=m_2}\|D^{\alpha_2}A_2\|_{\mathrm{Osc}_{\exp L^{r_2}}}\sum_{|\alpha_1|=m_1}\frac{1}{|Q|}\int_{R^n}\left|D^{\alpha_1}\tilde{A}_1(x)\right|\left|g(x)\right|_s dx\\
&\le C\sum_{|\alpha_2|=m_2}\|D^{\alpha_2}A_2\|_{\mathrm{Osc}_{\exp L^{r_2}}}\|\chi_Q\|_{\exp L^{r_2},Q}\\
&\quad\times\sum_{|\alpha_1|=m_1}\left\|D^{\alpha_1}A_1-\left(D^{\alpha_1}A_1\right)_{\tilde{Q}}\right\|_{\exp L^{r_1},\tilde{Q}}\left\||f|_s\right\|_{L(\log L)^{1/r},\tilde{Q}}\\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{Osc}_{\exp L^{r_j}}}\right)M_{L(\log L)^{1/r}}\left(|f|_s\right)(\tilde{x}).
\end{aligned}$$

For I_3, similar to the proof of I_2, we get

$$I_3 \le C\prod_{j=1}^{2}\Big(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{Osc}_{\exp L^{r_j}}}\Big)M_{L(\log L)^{1/r}}\big(|f|_s\big)(\tilde{x}).$$

Similarly, for I_4, by using Lemma 3, we get

$$\begin{aligned}
I_4 &\le C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\left(\frac{1}{|Q|}\int_{R^n}\big|T\big(D^{\alpha_1}\tilde{A}_1D^{\alpha_2}\tilde{A}_2f_1\big)(x)\big|_s^p\,dx\right)^{1/p}\\
&\le C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}|Q|^{-1}\big\|\big|T\big(D^{\alpha_1}\tilde{A}_1D^{\alpha_2}\tilde{A}_2g\big)\big|_s\chi_Q\big\|_{WL^1}\\
&\le C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\frac{1}{|Q|}\int_{R^n}\big|D^{\alpha_1}\tilde{A}_1(x)D^{\alpha_2}\tilde{A}_2(x)\big|\big|g(x)\big|_s\,dx\\
&\le C\sum_{|\alpha_1|=m_1}\big\|D^{\alpha_1}A_1-\big(D^{\alpha_1}A_1\big)_{\tilde{Q}}\big\|_{\exp L^{r_1},\tilde{Q}}\\
&\quad\times\sum_{|\alpha_2|=m_2}\big\|D^{\alpha_2}A_2-\big(D^{\alpha_2}A_2\big)_{\tilde{Q}}\big\|_{\exp L^{r_2},\tilde{Q}}\big\||f|_s\big\|_{L(\log L)^{1/r},\tilde{Q}}\\
&\le C\prod_{j=1}^{2}\Big(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{Osc}_{\exp L^{r_j}}}\Big)M_{L(\log L)^{1/r}}\big(|f|_s\big)(\tilde{x}).
\end{aligned}$$

For I_5, we write

$$\begin{aligned}
&F_t^{\tilde{A}}(h_i)(x)-F_t^{\tilde{A}}(h_i)(x_0)\\
&\quad=\int_{R^n}\left(\frac{F_t(x,y)}{|x-y|^m}-\frac{F_t(x_0,y)}{|x_0-y|^m}\right)\prod_{j=1}^{2}R_{m_j}(\tilde{A}_j;x,y)h_i(y)\,dy\\
&\qquad+\int_{R^n}\big(R_{m_1}(\tilde{A}_1;x,y)-R_{m_1}(\tilde{A}_1;x_0,y)\big)\frac{R_{m_2}(\tilde{A}_2;x,y)}{|x_0-y|^m}F_t(x_0,y)h_i(y)\,dy\\
&\qquad+\int_{R^n}\big(R_{m_2}(\tilde{A}_2;x,y)-R_{m_2}(\tilde{A}_2;x_0,y)\big)\frac{R_{m_1}(\tilde{A}_1;x_0,y)}{|x_0-y|^m}F_t(x_0,y)h_i(y)\,dy\\
&\qquad-\sum_{|\alpha_1|=m_1}\frac{1}{\alpha_1!}\int_{R^n}\left[\frac{R_{m_2}(\tilde{A}_2;x,y)(x-y)^{\alpha_1}}{|x-y|^m}F_t(x,y)-\frac{R_{m_2}(\tilde{A}_2;x_0,y)(x_0-y)^{\alpha_1}}{|x_0-y|^m}F_t(x_0,y)\right]\\
&\qquad\times D^{\alpha_1}\tilde{A}_1(y)h_i(y)\,dy\\
&\qquad-\sum_{|\alpha_2|=m_2}\frac{1}{\alpha_2!}\int_{R^n}\left[\frac{R_{m_1}(\tilde{A}_1;x,y)(x-y)^{\alpha_2}}{|x-y|^m}F_t(x,y)-\frac{R_{m_1}(\tilde{A}_1;x_0,y)(x_0-y)^{\alpha_2}}{|x_0-y|^m}F_t(x_0,y)\right]\\
&\qquad\times D^{\alpha_2}\tilde{A}_2(y)h_i(y)\,dy\\
&\qquad+\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\frac{1}{\alpha_1!\alpha_2!}\int_{R^n}\left[\frac{(x-y)^{\alpha_1+\alpha_2}}{|x-y|^m}F_t(x,y)-\frac{(x_0-y)^{\alpha_1+\alpha_2}}{|x_0-y|^m}F_t(x_0,y)\right]\\
&\qquad\times D^{\alpha_1}\tilde{A}_1(y)D^{\alpha_2}\tilde{A}_2(y)h_i(y)\,dy\\
&\quad=I_5^{(1)}+I_5^{(2)}+I_5^{(3)}+I_5^{(4)}+I_5^{(5)}+I_5^{(6)}.
\end{aligned}$$

By Lemma 1, we know that, for $x \in Q$ and $y \in 2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}$,

$$\begin{aligned}\left|R_{m_j}(\tilde{A}_j;x,y)\right| &\le C|x-y|^{m_j} \sum_{|\alpha_j|=m_j} \left(\|D^{\alpha_j}A\|_{\mathrm{Osc}_{\exp L^{r_j}}} + \left|(D^{\alpha_j}A)_{\tilde{Q}(x,y)} - (D^{\alpha_j}A)_{\tilde{Q}}\right|\right) \\ &\le Ck|x-y|^{m_j} \sum_{|\alpha_j|=m_j} \|D^{\alpha_j}A\|_{\mathrm{Osc}_{\exp L^{r_j}}}.\end{aligned}$$

Note that $|x-y| \sim |x_0-y|$ for $x \in Q$ and $y \in R^n \setminus \tilde{Q}$, we obtain, by the condition of F_t,

$$\begin{aligned}\|I_5^{(1)}\| &\le C\int_{R^n}\left(\frac{|x-x_0|}{|x_0-y|^{m+n+1}} + \frac{|x-x_0|^{\varepsilon}}{|x_0-y|^{m+n+\varepsilon}}\right)\prod_{j=1}^{2} R_{m_j}(\tilde{A}_j;x,y)\left|h_i(y)\right|dy \\ &\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{Osc}_{\exp L^{r_j}}}\right) \\ &\quad \times \sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} k^2\left(\frac{|x-x_0|}{|x_0-y|^{n+1}} + \frac{|x-x_0|^{\varepsilon}}{|x_0-y|^{n+\varepsilon}}\right)\left|f_i(y)\right|dy \\ &\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{Osc}_{\exp L^{r_j}}}\right)\sum_{k=1}^{\infty}k^2\left(2^{-k}+2^{-\varepsilon k}\right)\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}\left|f_i(y)\right|dy.\end{aligned}$$

Thus, by Minkowski's inequality, we get

$$\begin{aligned}\left(\sum_{i=1}^{\infty}\|I_5^{(1)}\|^s\right)^{1/s} &\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\|D^{\alpha}A_j\|_{\mathrm{Osc}_{\exp L^{r_j}}}\right) \\ &\quad \times \sum_{k=1}^{\infty}k^2\left(2^{-k}+2^{-\varepsilon k}\right)\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}\left|f(y)\right|_s dy \\ &\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\|D^{\alpha}A_j\|_{\mathrm{Osc}_{\exp L^{r_j}}}\right)M\left(|f|_s\right)(\tilde{x}) \\ &\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{Osc}_{\exp L^{r_j}}}\right)M_{L(\log L)^{1/r}}\left(|f|_s\right)(\tilde{x}).\end{aligned}$$

For $I_5^{(2)}$, by the formula (see [3])

$$R_{m_j}(\tilde{A};x,y) - R_{m_j}(\tilde{A};x_0,y) = \sum_{|\beta|<m_j}\frac{1}{\beta!}R_{m_j-|\beta|}\left(D^{\beta}\tilde{A};x,x_0\right)(x-y)^{\beta}$$

and Lemma 1, we have

$$\left|R_{m_j}(\tilde{A};x,y) - R_{m_j}(\tilde{A};x_0,y)\right| \le C\sum_{|\beta|<m_j}\sum_{|\alpha|=m_j}|x-x_0|^{m_j-|\beta|}|x-y|^{|\beta|}\|D^{\alpha}A\|_{\mathrm{Osc}_{\exp L^{r_j}}},$$

thus

$$\left(\sum_{i=1}^{\infty}\left\|I_5^{(2)}\right\|^s\right)^{1/s} \le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\left\|D^{\alpha_j}A_j\right\|_{\mathrm{Osc}_{\exp L^{r_j}}}\right)\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} k\frac{|x-x_0|}{|x_0-y|^{n+1}}\left|f(y)\right|_s\,dy$$
$$\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\left\|D^{\alpha_j}A_j\right\|_{\mathrm{Osc}_{\exp L^{r_j}}}\right)M_{L(\log L)^{1/r}}\left(|f|_s\right)(\tilde{x}).$$

Similarly,

$$\left(\sum_{i=1}^{\infty}\left\|I_5^{(3)}\right\|^s\right)^{1/s} \le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\left\|D^{\alpha_j}A_j\right\|_{\mathrm{Osc}_{\exp L^{r_j}}}\right)M_{L(\log L)^{1/r}}\left(|f|_s\right)(\tilde{x}).$$

For $I_5^{(4)}$, similar to the proof of $I_5^{(1)}$, $I_5^{(2)}$ and I_2, we get

$$\begin{aligned}
&\left(\sum_{i=1}^{\infty}\left\|I_5^{(4)}\right\|^s\right)^{1/s} \\
&\quad\le C\sum_{|\alpha_1|=m_1}\int_{R^n\setminus\tilde{Q}}\left\|\frac{(x-y)^{\alpha_1}F_t(x,y)}{|x-y|^m}-\frac{(x_0-y)^{\alpha_1}F_t(x_0,y)}{|x_0-y|^m}\right\| \\
&\qquad\times\left|R_{m_2}(\tilde{A}_2;x,y)\right|\left|D^{\alpha_1}\tilde{A}_1(y)\right|\left|f(y)\right|_s\,dy \\
&\qquad+C\sum_{|\alpha_1|=m_1}\int_{R^n\setminus\tilde{Q}}\left|R_{m_2}(\tilde{A}_2;x,y)-R_{m_2}(\tilde{A}_2;x_0,y)\right| \\
&\qquad\times\frac{\|(x_0-y)^{\alpha_1}F_t(x_0,y)\|}{|x_0-y|^m}\left|D^{\alpha_1}\tilde{A}_1(y)\right|\left|f(y)\right|_s\,dy \\
&\quad\le C\sum_{|\alpha_2|=m_2}\left\|D^{\alpha_2}A_2\right\|_{\mathrm{Osc}_{\exp L^{r_2}}} \\
&\qquad\times\sum_{|\alpha_1|=m_1}\sum_{k=1}^{\infty}k\left(2^{-k}+2^{-\varepsilon k}\right)\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}\left|D^{\alpha_1}\tilde{A}_1(y)\right|\left|f(y)\right|_s\,dy \\
&\quad\le C\sum_{|\alpha_2|=m_2}\left\|D^{\alpha_2}A_2\right\|_{\mathrm{Osc}_{\exp L^{r_2}}}\sum_{|\alpha_1|=m_1}\sum_{k=1}^{\infty}k\left(2^{-k}+2^{-\varepsilon k}\right) \\
&\qquad\times\left\|D^{\alpha_1}A_1-\left(D^{\alpha_1}A_1\right)_{\tilde{Q}}\right\|_{\exp L^{r_1},2^k\tilde{Q}}\left\||f|_s\right\|_{L(\log L)^{1/r},2^k\tilde{Q}} \\
&\quad\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\left\|D^{\alpha_j}A_j\right\|_{\mathrm{Osc}_{\exp L^{r_j}}}\right)M_{L(\log L)^{1/r}}\left(|f|_s\right)(\tilde{x}).
\end{aligned}$$

Similarly,

$$\left(\sum_{i=1}^{\infty}\left\|I_5^{(5)}\right\|^s\right)^{1/s} \le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\left\|D^{\alpha_j}A_j\right\|_{\mathrm{Osc}_{\exp L^{r_j}}}\right)M_{L(\log L)^{1/r}}\left(|f|_s\right)(\tilde{x}).$$

For $I_5^{(6)}$, by using Lemma 3, we obtain

$$\begin{aligned}
&\left(\sum_{i=1}^{\infty}\left\|I_5^{(6)}\right\|^s\right)^{1/s} \\
&\quad\leq C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\int_{R^n\setminus\tilde{Q}}\left\|\frac{(x-y)^{\alpha_1+\alpha_2}F_t(x,y)}{|x-y|^m}-\frac{(x_0-y)^{\alpha_1+\alpha_2}F_t(x_0,y)}{|x_0-y|^m}\right\| \\
&\qquad\times\left|D^{\alpha_1}\tilde{A}_1(y)\right|\left|D^{\alpha_2}\tilde{A}_2(y)\right|\left|f(y)\right|_s\,dy \\
&\quad\leq C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\sum_{k=1}^{\infty}\left(2^{-k}+2^{-\varepsilon k}\right)\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}\left|D^{\alpha_1}\tilde{A}_1(y)\right|\left|D^{\alpha_2}\tilde{A}_2(y)\right|\left|f(y)\right|_s\,dy \\
&\quad\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\left\|D^{\alpha_j}A_j\right\|_{\mathrm{Osc}_{\exp L^{r_j}}}\right)M_{L(\log L)^{1/r}}\left(|f|_s\right)(\tilde{x}).
\end{aligned}$$

Thus

$$I_5\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\left\|D^{\alpha_j}A_j\right\|_{\mathrm{Osc}_{\exp L^{r_j}}}\right)M_{L(\log L)^{1/r}}\left(|f|_s\right)(\tilde{x}).$$

This completes the proof of Theorem 1. □

By Theorem 1 and the L^p-boundedness of $M_{L(\log L)^{1/r}}$, we may obtain the conclusions (1), (2) of Theorem 2.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors completed the paper together. They also read and approved the final manuscript.

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