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Complete convergence and complete moment convergence for negatively associated sequences of random variables

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Abstract

In this paper, we study the complete convergence and complete moment convergence for negatively associated sequences of random variables with $\mathbb{E}X = 0$, $\mathbb{E}\exp(\ln^{\alpha}|X|) < \infty$, $\alpha > 1$. As a result, we extend some complete convergence and complete moment convergence theorems for independent random variables to the case of negatively associated random variables without necessarily imposing any extra conditions. Our results generalize corresponding results obtained by Gut and Stadtmüller (Stat. Probab. Lett. 81:1486-1492, 2011) and Qiu and Chen (Stat. Probab. Lett. 91:76-82, 2014).

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1 Introduction and main results

Definition 1.1 Random variables $X_1, X_2, \ldots, X_n$, $n \geq 2$, are said to be negatively associated (NA) if for every pair of disjoint subsets A_1 and A_2 of $\{1, 2, \ldots, n\}$,

$$\operatorname{cov}\bigl(f_1(X_i; i \in A_1), f_2(X_j; j \in A_2)\bigr) \leq 0,$$

where f_1 and f_2 are increasing for every variable (or decreasing for every variable) functions such that this covariance exists. A sequence of random variables $\{X_i; i \geq 1\}$ is said to be NA if its every finite subfamily is NA.

By Joag-Dev and Proschan (1983 [3]), we have the following lemma.

Lemma 1.2 (Joag-Dev and Proschan, 1983 [3]) *Let $\{X_i; i \geq 1\}$ be a sequence of NA random variables.*

(i) *If $\{f_i; i \geq 1\}$ is a sequence of nondecreasing (or nonincreasing) functions, then $\{f_i(X_i); i \geq 1\}$ is also a sequence of NA random variables.*
(ii) *Increasing functions defined on disjoint subsets of a set of negatively associated random variables are negatively associated.*

This definition was introduced by Joag-Dev and Proschan (1983 [3]). Statistical test depends greatly on sampling. The random sampling without replacement from a finite population is NA, but is not independent. NA sampling has wide applications such as in multivariate statistical analysis and reliability theory. Because of the wide applications of NA sampling, the limit behaviors of NA random variables have received more and more attention recently. One can refer to: Joag-Dev and Proschan (1983 [3]) for fundamental properties, Newman (1984 [4]) for the central limit theorem, Matula (1992 [5]) for the three series theorem, Shao (2000 [6]) for the moment inequalities.

The concept of complete convergence of a sequence of random variables was introduced by Hsu and Robbins (1947 [7]). In view of the Borel-Cantelli lemma, complete convergence implies almost sure convergence. Chow (1988 [8]) first investigated the complete moment convergence, which is more exact than complete convergence. Thus, complete convergence and complete moment convergence are two of the most important problems in probability theory. Their recent results can be found in Wu (2012 [9], 2015 [10]), Xu and Tang (2014 [11]), Guo *et al.* (2014 [12]), Gut and Stadtmüller (2011 [1]), and Qiu and Chen (2014 [2]). In addition, Gut and Stadtmüller (2011 [1]) and Qiu and Chen (2014 [2]) obtained, respectively, complete convergence and complete moment convergence theorems for independent identically distributed sequences of random variable with $\mathbb{E}X = 0$, $\mathbb{E}\exp(\ln^\alpha |X|) < \infty$, $\alpha > 1$. In this paper, based on Gut and Stadtmüller (2011 [1]) and Qiu and Chen (2014 [2]), we extend the complete convergence and complete moment theorems for independent random variables to the negatively associated sequences of random variables without necessarily imposing any extra conditions, which extend the corresponding results of Gut and Stadtmüller (2011 [1]) and Qiu and Chen (2014 [2]).

In the following, the symbol c stands for a generic positive constant which may differ from one place to another. Let $a_n \ll b_n$ denote that there exists a constant $c > 0$ such that $a_n \leq cb_n$ for sufficiently large n, $\ln x$ means $\ln(\max(x, \mathrm{e}))$, and I denotes an indicator function.

Theorem 1.3 *Let $\alpha > 1$, $\{X, X_n; n \geq 1\}$ be a sequence of NA identically distributed random variables with partial sums $S_n = \sum_{i=1}^n X_i$, $n \geq 1$. Suppose that*

$$\mathbb{E}X = 0, \qquad \mathbb{E}\exp\bigl(\ln^\alpha |X|\bigr) < \infty, \tag{1.1}$$

then

$$\sum_{n=1}^\infty \exp\bigl(\ln^\alpha n\bigr)\frac{\ln^{\alpha-1} n}{n^2} P\Bigl(\max_{1\leq k\leq n} |S_k| > n\beta\Bigr) < \infty \quad \textit{for all } \beta > 1. \tag{1.2}$$

Conversely, if (1.2) *holds for some $\beta > 0$, then $\mathbb{E}\exp(\ln^\alpha |X/(2\beta)|) < \infty$; furthermore, if $\beta \leq 1/2$, then $\mathbb{E}\exp(\ln^\alpha |X|) < \infty$, if $\beta > 1/2$, then $\mathbb{E}\exp((1-\lambda)\ln^\alpha |X|) < \infty$ for any $\lambda > 0$.*

Theorem 1.4 *Assume that the conditions of Theorem* 1.3 *and* (1.1) *hold. Then*

$$\sum_{n=1}^\infty \exp\bigl(\ln^\alpha n\bigr)\frac{\ln^{\alpha-1} n}{n^{2+q}} \mathbb{E}\Bigl\{\max_{1\leq k\leq n} |S_k| - \beta n\Bigr\}_+^q < \infty \qquad \textit{for all } \beta > 1 \textit{ and all } q > 0. \tag{1.3}$$

Conversely, if (1.3) *holds for some $\beta > 0$, then $\mathbb{E}\exp(\ln^\alpha |X/(2\beta)|) < \infty$.*

Remark 1.5 By mimicking the analogous part in the proof of Theorem 2.1 in Qiu and Chen (2014 [2]), (1.2) and (1.3) imply, respectively,

$$\sum_{n=1}^{\infty}\exp\bigl(\ln^{\alpha} n\bigr)\frac{\ln^{\alpha-1} n}{n^2}P\left(\sup_{k\geq n}\left|\frac{S_k}{k}\right|>\beta\right)<\infty \quad \text{for all } \beta>1$$

and

$$\sum_{n=1}^{\infty}\exp\bigl(\ln^{\alpha} n\bigr)\frac{\ln^{\alpha-1} n}{n^2}\mathbb{E}\left\{\sup_{1\leq k\leq n}\left|\frac{S_k}{k}\right|-\beta\right\}_{+}^{q}<\infty \quad \text{for all } \beta>1 \text{ and all } q>0.$$

Remark 1.6 Corresponding results of Gut and Stadtmüller (2011 [1]) and Qiu and Chen (2014 [2]) are the special cases of our Theorems 1.3 and 1.4 when $\{X, X_n; n\geq 1\}$ is i.i.d.

2 Proofs

The following two lemmas will be useful in the proofs of our theorems, and the first is due to Shao (2000 [6]).

Lemma 2.1 (Shao, 2000 [6], Theorem 3) *Let $\{X_i; 1\leq i\leq n\}$ be a sequence of negatively associated random variables with zero means and finite second moments. Let $S_k=\sum_{i=1}^{k}X_i$ and $B_n=\sum_{i=1}^{n}\mathbb{E}X_i^2$. Then, for all $y>0$, $a>0$ and $0<\theta<1$,*

$$P\Bigl(\max_{1\leq k\leq n}S_k\geq y\Bigr)\leq P\Bigl(\max_{1\leq k\leq n}X_k>a\Bigr)+\frac{1}{1-\theta}\exp\left(-\frac{y^2\theta}{2(ay+B_n)}\left\{1+\frac{2}{3}\ln\left(1+\frac{ay}{B_n}\right)\right\}\right).$$

Lemma 2.2 *For any random variable X and $\alpha>0$,*

$$\mathbb{E}\exp\bigl(\ln^{\alpha}|X|\bigr)<\infty \quad \Leftrightarrow \quad \sum_{n=1}^{\infty}\exp\bigl(\ln^{\alpha} n\bigr)\frac{\ln^{\alpha-1} n}{n}P\bigl(|X|>n\bigr)<\infty.$$

Proof Let $a_n\approx b_n$ denote that there exist constants $c_1>0$ and $c_2>0$ such that $c_1a_n\leq b_n\leq c_2a_n$ for sufficiently large n. We have

$$\begin{aligned}
&\sum_{n=1}^{\infty}\exp\bigl(\ln^{\alpha} n\bigr)\frac{\ln^{\alpha-1} n}{n}P\bigl(|X|>n\bigr)\\
&\quad=\sum_{n=1}^{\infty}\exp\bigl(\ln^{\alpha} n\bigr)\frac{\ln^{\alpha-1} n}{n}\sum_{j=n}^{\infty}P\bigl(j<|X|\leq j+1\bigr)\\
&\quad=\sum_{j=1}^{\infty}P\bigl(j<|X|\leq j+1\bigr)\sum_{n=1}^{j}\exp\bigl(\ln^{\alpha} n\bigr)\frac{\ln^{\alpha-1} n}{n}\\
&\quad\approx\sum_{j=1}^{\infty}\exp\bigl(\ln^{\alpha} j\bigr)\mathbb{E}I\bigl(j<|X|\leq j+1\bigr)\\
&\quad\approx\sum_{j=1}^{\infty}\mathbb{E}\exp\bigl(\ln^{\alpha}|X|\bigr)I\bigl(j<|X|\leq j+1\bigr)\\
&\quad\approx\mathbb{E}\exp\bigl(\ln^{\alpha}|X|\bigr),
\end{aligned}$$

it follows that Lemma 2.2 holds. □

Proof of Theorem 1.3 Let $\beta > 1$ be arbitrary, set, for $n \geq 1$, $b_n = \beta n/(10 \ln^\alpha n)$, define, for $1 \leq k \leq n$,

$$X'_k = X_k I\{X_k \leq b_n\} + b_n I\{X_k > b_n\}, \qquad S'_n = \sum_{k=1}^n X'_k,$$
$$X''_k = (X_k - b_n) I\{b_n < X_k \leq n\}, \qquad X'''_k = (X_k - b_n) I\{X_k > n\}.$$

Obviously, $X_k = X'_k + X''_k + X'''_k$ and X'_k is increasing on X_k, thus, by Lemma 1.2(i), $\{X'_k; k \geq 1\}$ is also a sequence of NA random variables. Note that

$$\begin{aligned}
&\Bigl\{\max_{1\leq k\leq n} S_k > n\beta\Bigr\} \\
&\quad\subseteq \Bigl\{\max_{1\leq k\leq n} S_k > n\beta \text{ and } X_k \leq b_n \text{ for all } k \leq n\Bigr\} \\
&\qquad\cup \Bigl\{\max_{1\leq k\leq n} S_k > n\beta \text{ and } b_n < X_{k_0} \leq n \text{ for exactly one } k_0 \leq n \text{ and} \\
&\qquad X_j \leq b_n \text{ for all } j \neq k_0\Bigr\} \\
&\qquad\cup \bigl\{X''_k \neq 0 \text{ for at least two } k \leq n\bigr\} \\
&\qquad\cup \bigl\{X'''_k \neq 0 \text{ for at least one } k \leq n\bigr\} \\
&\quad\hat{=} A_n \cup B_n \cup C_n \cup D_n.
\end{aligned}$$

Therefore,

$$P\Bigl(\max_{1\leq k\leq n} S_k > n\beta\Bigr) \leq P(A_n) + P(B_n) + P(C_n) + P(D_n). \tag{2.1}$$

By condition (1.1), $\mathbb{E}X = 0$, and $\mathbb{E}\exp(\ln^\alpha |X|) < \infty$, $\alpha > 1$, we get $\mathbb{E}XI(X \leq b_n) = -\mathbb{E}XI(X > b_n)$ and $\mathbb{E}X^2 < \infty$. It is well known that $\mathbb{E}X^2 < \infty$ implies $\mathbb{E}X^2 I(|X| > b_n) \to 0$, $n \to \infty$, and we set $\delta \hat{=} 1 - \beta^{-1} > 0$, for sufficiently large n,

$$\begin{aligned}
\max_{1\leq k\leq n} \bigl|\mathbb{E}S'_k\bigr| &\leq \max_{1\leq k\leq n} \bigl|k\mathbb{E}XI(X \leq b_n)\bigr| + nb_n \mathbb{E}I(X > b_n) \\
&\leq n\mathbb{E}|X|I\bigl(|X| > b_n\bigr) + nb_n^{-1}\mathbb{E}X^2 I\bigl(|X| > b_n\bigr) \\
&\leq \frac{2n\mathbb{E}X^2 I(|X| > b_n)}{b_n} = \frac{20 \ln^\alpha n}{\beta} \mathbb{E}X^2 I\bigl(|X| > b_n\bigr) \\
&\leq \beta\delta \ln^\alpha n,
\end{aligned} \tag{2.2}$$

so that, taking $y = (n - \delta \ln^\alpha n)\beta$, $a = 2b_n$, $\theta = 4/5$ in Lemma 2.1, for sufficiently large n, we get

$$\begin{aligned}
P(A_n) &\leq P\Bigl(\max_{1\leq k\leq n} S'_k > n\beta\Bigr) \\
&\leq P\Bigl(\max_{1\leq k\leq n} \bigl(S'_k - \mathbb{E}S'_k\bigr) > \bigl(n - \delta \ln^\alpha n\bigr)\beta\Bigr) \\
&\ll \exp\biggl(-\frac{4(n - \delta \ln^\alpha n)^2 \beta^2}{10(\frac{\beta^2 n(n-\delta \ln^\alpha n)}{5 \ln^\alpha n} + n\mathbb{E}X^2)}\biggr)
\end{aligned}$$

$$= \exp\left(-\frac{2\beta^2(1-\frac{\delta\ln^\alpha n}{n})^2}{\beta^2(1-\frac{\delta\ln^\alpha n}{n})+\frac{5\mathbb{E}X^2\ln^\alpha n}{n}}\ln^\alpha n\right) \\ \le \exp\left(-\ln^\alpha n\right), \tag{2.3}$$

from $\frac{2\beta^2(1-\frac{\delta\ln^\alpha n}{n})^2}{\beta^2(1-\frac{\delta\ln^\alpha n}{n})+\frac{5\mathbb{E}X^2\ln^\alpha n}{n}} \to 2 > 1$ as $n \to \infty$. By the Markov inequality, (1.1), and $(\ln n + \ln(\beta/10) - \alpha\ln\ln n)^\alpha/\ln^\alpha n \to 1 < 1-\delta/2$ as $n \to \infty$, for sufficiently large n, $(\ln n + \ln(\beta/10) - \alpha\ln\ln n)^\alpha \le (1-\delta/2)\ln^\alpha n$, thus,

$$P\left(|X| > b_n\right) \le \frac{\mathbb{E}\exp(\ln^\alpha|X|)}{\exp(\ln^\alpha b_n)} \\ \ll \frac{1}{\exp(\ln n + \ln(\beta/10) - \alpha\ln\ln n)^\alpha} \\ \le \exp\left(-(1-\delta/2)\ln^\alpha n\right), \tag{2.4}$$

and, hence, by combining (2.3) and Lemma 1.2(ii), $\max_{1\le k\le n}\sum_{1\le i\le k, i\ne k_0} X'_i$ and X_{k_0} are NA random variable, we get

$$P(B_n) \le P\left(\exists 1 \le k_0 \le n \text{ such that } \max_{1\le k\le n}\sum_{1\le i\le k, i\ne k_0} X'_i > \beta n - n, X_{k_0} > b_n\right) \\ \le \sum_{k_0=1}^{n} P\left(\max_{1\le k\le n}\sum_{1\le i\le k, i\ne k_0} X'_i > \beta n - n = \beta\delta n\right) P(X_{k_0} > b_n). \tag{2.5}$$

Similar to the proof of (2.2), we have $\max_{1\le k\le n}|\mathbb{E}\sum_{1\le i\le k, i\ne k_0} X'_i| \le \beta\delta\ln^\alpha n$, so that, taking $y = \beta\delta(n - \ln^\alpha n)$, $a = 2b_n$, $\theta = 4/5$ in Lemma 2.1, using the fact that $\frac{2\beta^2\delta(1-\frac{\ln^\alpha n}{n})^2}{\beta^2\delta(1-\frac{\ln^\alpha n}{n})+\frac{5\mathbb{E}X^2(n-1)\ln^\alpha n}{n^2}} \to 2 > 1$ as $n \to \infty$, for sufficiently large n, we get

$$P\left(\max_{1\le k\le n}\sum_{1\le i\le k, i\ne k_0} X'_i > \beta\delta n\right) \\ \le P\left(\max_{1\le k\le n}\sum_{1\le i\le k, i\ne k_0}\left(X'_i - \mathbb{E}X'_i\right) > \beta\delta\left(n - \ln^\alpha n\right)\right) \\ \ll \exp\left(-\frac{4(n-\ln^\alpha n)^2\beta^2\delta^2}{10(\frac{\beta^2\delta n(n-\ln^\alpha n)}{5\ln^\alpha n} + (n-1)\mathbb{E}X^2)}\right) \\ = \exp\left(-\frac{2\beta^2\delta(1-\frac{\ln^\alpha n}{n})^2}{\beta^2\delta(1-\frac{\ln^\alpha n}{n})+\frac{5\mathbb{E}X^2(n-1)\ln^\alpha n}{n^2}}\delta\ln^\alpha n\right) \\ \le \exp\left(-\delta\ln^\alpha n\right).$$

Substituting the above inequality and (2.4) in (2.5), we obtain

$$P(B_n) \ll n\exp\left(-\delta\ln^\alpha n - (1-\delta/2)\ln^\alpha n\right) \\ = \exp\left(-\ln^\alpha n\right)\frac{n}{(\mathrm{e}^{\ln n})^{(\delta\ln^{\alpha-1} n)/2}} \\ \le \exp\left(-\ln^\alpha n\right). \tag{2.6}$$

By (2.4),

$$\begin{aligned} P(C_n) &= P\bigl(\exists 1 \le k_1 < k_2 \le n \text{ such that } X''_{k_1} \ne 0, X''_{k_2} \ne 0\bigr) \\ &\le \sum_{1\le k_1<k_2\le n} P(X_{k_1} > b_n, X_{k_2} > b_n) \le n^2 P^2\bigl(|X| > b_n\bigr) \\ &\ll n^2\bigl(\exp\bigl(-(1-\delta/2)\ln^\alpha n\bigr)\bigr)^2 \\ &= n^2\exp\bigl(-2(1-\delta/2)\ln^\alpha n\bigr) = n^2\exp\bigl(-1-(1-\delta)\ln^\alpha n\bigr) \\ &= \exp\bigl(-\ln^\alpha n\bigr)\frac{n^2}{(\mathrm{e}^{\ln n})^{(1-\delta)\ln^{\alpha-1} n}} \\ &\le \exp\bigl(-\ln^\alpha n\bigr). \end{aligned} \tag{2.7}$$

This, together with (2.1), (2.3), (2.5), and (2.6), shows

$$P\Bigl(\max_{1\le k\le n} S_k > \beta n\Bigr) \ll \exp\bigl(-\ln^\alpha n\bigr) + nP\bigl(|X| > n\bigr). \tag{2.8}$$

Because $-X_k$ is decreasing on X_k, by Lemma 1.2(i), $\{-X, -X_k; k \ge 1\}$ is also a sequence of NA random variables. Obviously, $\{-X, -X_k; k \ge 1\}$ also satisfies the condition (1.1). Therefore, replacing X_k by $-X_k$ in (2.8), we get

$$P\Bigl(\max_{1\le k\le n} (-S_k) > \beta n\Bigr) \ll \exp\bigl(-\ln^\alpha n\bigr) + nP\bigl(|X| > n\bigr).$$

Thus,

$$\begin{aligned} P\Bigl(\max_{1\le k\le n} |S_k| > \beta n\Bigr) &\le P\Bigl(\max_{1\le k\le n} S_k > \beta n\Bigr) + P\Bigl(\max_{1\le k\le n} (-S_k) > \beta n\Bigr) \\ &\ll \exp\bigl(-\ln^\alpha n\bigr) + nP\bigl(|X| > n\bigr). \end{aligned} \tag{2.9}$$

From (1.1) and Lemma 2.2,

$$\begin{aligned} &\sum_{n=1}^\infty \exp\bigl(\ln^\alpha n\bigr)\frac{\ln^{\alpha-1} n}{n^2} P\Bigl(\max_{1\le k\le n} |S_k| > \beta n\Bigr) \\ &\quad \ll \sum_{n=1}^\infty \frac{\ln^{\alpha-1} n}{n^2} + \sum_{n=1}^\infty \exp\bigl(\ln^\alpha n\bigr)\frac{\ln^{\alpha-1} n}{n} P\bigl(|X| > n\bigr) \\ &\quad < \infty. \end{aligned}$$

That is, (1.2) holds.

Conversely, if (1.2) holds, then combining with $\max_{1\le k\le n}|X_k| \le 2\max_{1\le k\le n}|S_k|$, it follows that

$$\sum_{n=1}^\infty \exp\bigl(\ln^\alpha n\bigr)\frac{\ln^{\alpha-1} n}{n^2} P\Bigl(\max_{1\le k\le n} |X_k| > 2\beta n\Bigr) < \infty, \tag{2.10}$$

it implies that $P(\max_{1\le k\le n}|X_k| > 2\beta n) \to 0$, $n \to \infty$, hence, for sufficiently large n,

$$P\Bigl(\max_{1\le k\le n} |X_k| > 2\beta n\Bigr) < \frac{1}{2}. \tag{2.11}$$

Obviously, NA implies pairwise negative quadrant dependent (PNQD) from their definitions. Thus, by Lemma 1.4 of Wu (2012 [9]),

$$\Big(1-P\Big(\max_{1\le k\le n}|X_k|>2\beta n\Big)\Big)^2\sum_{k=1}^{n}P\big(|X_k|>2\beta n\big)\le cP\Big(\max_{1\le k\le n}|X_k|>2\beta n\Big),$$

from which, combining with (2.11), we have

$$nP\big(|X|>2\beta n\big)\le cP\Big(\max_{1\le k\le n}|X_k|>2\beta n\Big).$$

Consequently, by (2.10),

$$\sum_{n=1}^{\infty}\exp\big(\ln^\alpha n\big)\frac{\ln^{\alpha-1}n}{n}P\Big(\frac{|X|}{2\beta}>n\Big)<\infty,$$

and, hence, we have $\mathbb{E}\exp(\ln^\alpha|X/(2\beta)|)<\infty$ from Lemma 2.1. Therefore, if $0<\beta\le 1/2$, then $\mathbb{E}\exp(\ln^\alpha|X|)\le\mathbb{E}\exp(\ln^\alpha|X/(2\beta)|)<\infty$, if $\beta>1/2$, then for any $\lambda>0$,

$$\frac{(1-\lambda)\ln^\alpha x}{\ln^\alpha(x/(2\beta))}\to 1-\lambda<1,\quad \text{as } x\to+\infty.$$

This implies that there exists a constant M such that for all $x\ge M$, we have $(1-\lambda)\ln^\alpha x\le\ln^\alpha(x/(2\beta))$. Hence,

$$\begin{aligned}\mathbb{E}\exp\big((1-\lambda)\ln^\alpha|X|\big) &= \mathbb{E}\exp\big((1-\lambda)\ln^\alpha|X|\big)I\big(|X|\le M\big)\\ &\quad+\mathbb{E}\exp\big((1-\lambda)\ln^\alpha|X|\big)I\big(|X|>M\big)\\ &\le c+\mathbb{E}\exp\big(\ln^\alpha\big|X/(2\beta)\big|\big)I\big(|X|>M\big)\\ &\ll \mathbb{E}\exp\big(\ln^\alpha\big|X/(2\beta)\big|\big)\\ &< \infty.\end{aligned}$$

This completes the proof of Theorem 1.3. □

Proof of Theorem 1.4 Note that

$$\begin{aligned}&\sum_{n=1}^{\infty}\exp\big(\ln^\alpha n\big)\frac{\ln^{\alpha-1}n}{n^{2+q}}\mathbb{E}\Big\{\max_{1\le k\le n}|S_k|-\beta n\Big\}_+^q\\ &\quad=\beta^q\sum_{n=1}^{\infty}\exp\big(\ln^\alpha n\big)\frac{\ln^{\alpha-1}n}{n^{2+q}}\int_0^n qx^{q-1}P\Big(\max_{1\le k\le n}|S_k|-\beta n>\beta x\Big)\,\mathrm{d}x\\ &\qquad+\beta^q\sum_{n=1}^{\infty}\exp\big(\ln^\alpha n\big)\frac{\ln^{\alpha-1}n}{n^{2+q}}\int_n^\infty qx^{q-1}P\Big(\max_{1\le k\le n}|S_k|-\beta n>\beta x\Big)\,\mathrm{d}x\\ &\quad\ll\sum_{n=1}^{\infty}\exp\big(\ln^\alpha n\big)\frac{\ln^{\alpha-1}n}{n^2}P\Big(\max_{1\le k\le n}|S_k|>\beta n\Big)\\ &\qquad+\sum_{n=1}^{\infty}\exp\big(\ln^\alpha n\big)\frac{\ln^{\alpha-1}n}{n^{2+q}}\int_n^\infty x^{q-1}P\Big(\max_{1\le k\le n}|S_k|>\beta x\Big)\,\mathrm{d}x.\end{aligned}$$

Hence, by (1.2), in order to establish (1.3), it suffices to prove that

$$\sum_{n=1}^{\infty} \exp\bigl(\ln^\alpha n\bigr) \frac{\ln^{\alpha-1} n}{n^{2+q}} \int_n^\infty x^{q-1} P\Bigl(\max_{1\le k\le n} |S_k| > \beta x\Bigr)\,\mathrm{d}x < \infty. \tag{2.12}$$

Let $\beta > 1$ be an arbitrary, set, for $x \ge n$, $b_x = \beta x/(10\ln^\alpha x)$, define, for $1 \le k \le n$,

$$\begin{aligned} &Y_k' = X_k I\{X_k \le b_x\} + b_x I\{X_k > b_x\}, \qquad U_n' = \sum_{k=1}^n Y_k', \\ &Y_k'' = (X_k - b_x) I\{b_x < X_k \le x\}, \qquad Y_k''' = (X_k - b_x) I\{X_k > x\}. \end{aligned}$$

By similar methods to the proof of (2.1), we have

$$P\Bigl(\max_{1\le k\le n} S_k > x\beta\Bigr) \le P(A_x) + P(B_x) + P(C_x) + P(D_x), \tag{2.13}$$

which leads to

$$\begin{aligned} &A_x = \Bigl\{\max_{1\le k\le n} U_k' > x\beta\Bigr\}, \\ &B_x = \Bigl\{\max_{1\le k\le n} S_k > x\beta \text{ and } b_x < X_{k_0} \le x \text{ for exactly one } k_0 \le n \text{ and } X_j \le b_x \text{ for all } j \ne k_0\Bigr\}, \\ &C_x = \bigl\{Y_k'' \ne 0 \text{ for at least two } k \le n\bigr\}, \qquad D_x = \bigl\{Y_k''' \ne 0 \text{ for at least one } k \le n\bigr\}. \end{aligned}$$

Using similar methods to those used in the proof of (2.3)-(2.7), for $\delta \hat{=} 1 - \beta^{-1} > 0$ and $x \ge n$, we have $\max_{1\le k\le n} |\mathbb{E}U_k'| \le \beta\delta \ln^\alpha x$, and

$$\begin{aligned} &P(A_x) \ll \exp\bigl(-\ln^\alpha x\bigr), \\ &P\bigl(|X| > b_x\bigr) \ll \exp\bigl(-(1-\delta/2)\ln^\alpha x\bigr), \\ &P(B_x) \ll n\exp\bigl(-\delta\ln^\alpha x - (1-\delta/2)\ln^\alpha x\bigr) \\ &\qquad = \exp\bigl(-\ln^\alpha x\bigr)\frac{n}{x^{(\delta\ln^{\alpha-1} n)/2}} \le \exp\bigl(-\ln^\alpha x\bigr), \\ &P(C_x) \le n^2P^2\bigl(|X| > b_x\bigr) \ll \exp\bigl(-\ln^\alpha x\bigr)n^2\exp\bigl(-(1-\delta)\ln^\alpha x\bigr) \\ &\qquad \le \exp\bigl(-\ln^\alpha x\bigr), \\ &P(D_x) \le nP(X > x) \le nP\bigl(|X| > x\bigr), \end{aligned}$$

which, combining with (2.13), shows

$$P\Bigl(\max_{1\le k\le n} S_k > x\beta\Bigr) \ll \exp\bigl(-\ln^\alpha x\bigr) + nP\bigl(|X| > x\bigr).$$

Replacing X_k by $-X_k$ in the above inequality, we have

$$P\Bigl(\max_{1\le k\le n} (-S_k) > x\beta\Bigr) \ll \exp\bigl(-\ln^\alpha x\bigr) + nP\bigl(|X| > x\bigr).$$

Therefore,

$$P\Big(\max_{1\le k\le n}|S_k| > x\beta\Big) \le P\Big(\max_{1\le k\le n} S_k > x\beta\Big) + P\Big(\max_{1\le k\le n}(-S_k) > x\beta\Big)$$
$$\ll \exp\big(-\ln^\alpha x\big) + nP\big(|X| > x\big).$$

Hence,

$$\begin{aligned}
&\int_n^\infty x^{q-1}P\Big(\max_{1\le k\le n}|S_k| > x\beta\Big)\,\mathrm{d}x \\
&\quad\ll \int_n^\infty x^{q-1}\exp\big(-\ln^\alpha x\big)\,\mathrm{d}x + \int_n^\infty x^{q-1}nP\big(|X|>x\big)\,\mathrm{d}x \\
&\quad\hat{=} I_1 + I_2.
\end{aligned} \tag{2.14}$$

By the fact that $(a+b)^\alpha \ge a^\alpha + b^\alpha$ for any $a, b > 0$ and $\alpha > 1$,

$$\begin{aligned}
&\sum_{n=1}^\infty \exp\big(\ln^\alpha n\big)\frac{\ln^{\alpha-1} n}{n^{2+q}} I_1 \\
&\quad= \sum_{n=1}^\infty \exp\big(\ln^\alpha n\big)\frac{\ln^{\alpha-1} n}{n^{2+q}} \int_1^\infty n^q t^{q-1}\exp\big(-(\ln n + \ln t)^\alpha\big)\,\mathrm{d}t \\
&\quad\le \sum_{n=1}^\infty \exp\big(\ln^\alpha n\big)\frac{\ln^{\alpha-1} n}{n^{2+q}} n^q \exp\big(-\ln^\alpha n\big)\int_1^\infty t^{q-1}\exp\big(-\ln^\alpha t\big)\,\mathrm{d}t \\
&\quad\ll \sum_{n=1}^\infty \exp\big(\ln^\alpha n\big)\frac{\ln^{\alpha-1} n}{n^{2+q}} n^q \exp\big(-\ln^\alpha n\big) \\
&\quad= \sum_{n=1}^\infty \frac{\ln^{\alpha-1} n}{n^2} < \infty.
\end{aligned} \tag{2.15}$$

By (1.1) and Lemma 2.2,

$$\begin{aligned}
&\sum_{n=1}^\infty \exp\big(\ln^\alpha n\big)\frac{\ln^{\alpha-1} n}{n^{2+q}} I_2 \\
&\quad= \sum_{n=1}^\infty \exp\big(\ln^\alpha n\big)\frac{\ln^{\alpha-1} n}{n^{1+q}} \int_n^\infty x^{q-1}P\big(|X|>x\big)\,\mathrm{d}x \\
&\quad= \sum_{n=1}^\infty \exp\big(\ln^\alpha n\big)\frac{\ln^{\alpha-1} n}{n^{1+q}} \sum_{j=n}^\infty \int_j^{j+1} x^{q-1}P\big(|X|>x\big)\,\mathrm{d}x \\
&\quad\ll \sum_{n=1}^\infty \exp\big(\ln^\alpha n\big)\frac{\ln^{\alpha-1} n}{n^{1+q}} \sum_{j=n}^\infty P\big(|X|>j\big)j^{q-1} \\
&\quad= \sum_{j=1}^\infty P\big(|X|>j\big)j^{q-1}\sum_{n=1}^j \exp\big(\ln^\alpha n\big)\frac{\ln^{\alpha-1} n}{n^{1+q}} \\
&\quad\ll \sum_{j=1}^\infty \exp\big(\ln^\alpha j\big)\frac{\ln^{\alpha-1} j}{j} P\big(|X|>j\big) < \infty,
\end{aligned}$$

from which, combining with (2.14) and (2.15), we see that (1.3) holds.

Conversely, (1.3) implies (1.2), that is, the conclusion was established. This completes the proof of Theorem 1.4. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
QW conceived of the study, drafted, completed, read, and approved the final manuscript. YJ conceived of the study, and drafted and approved the final manuscript.

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