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Mean square robust stability of stochastic switched discrete-time systems with convex polytopic uncertainties

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Abstract

This article is concerned with mean square robust stability of stochastic switched discrete time-delay systems with convex polytopic uncertainties. The system to be considered is subject to interval time-varying delays, which allows the delay to be a fast time-varying function and the lower bound is not restricted to zero. Based on the discrete Lyapunov functional, a switching rule for the mean square robust stability for the stochastic switched system with convex polytopic uncertainties is designed via linear matrix inequalities. Numerical examples are included to illustrate the effectiveness of the results.

Keywords: switching design, convex polytopic uncertainties, stochastic switched discrete system, mean square robust stability, Lyapunov function, linear matrix inequality.

1 Introduction

Since the time delay is frequently viewed as a source of instability and encountered in various engineering systems such as chemical processes, long transmission lines in pneumatic systems, networked control systems, etc., the study of delay systems has received much attention and various topics have been discussed over the past years.

A switched system is a hybrid dynamical system consisting of a finite number of subsystems and a logical rule that manages switching between these subsystems. Switched systems have drawn a great deal of attention in recent years, see [1-19] and references therein. The motivation for studying switched systems comes partly from the fact that switched systems and switched multi-controller systems have numerous applications in control of mechanical systems, process control, automotive industry, power systems, aircraft and traffic control, and many other fields. On the other hand, time-delay phenomena are very common in practical systems. A switched system with time-delay individual subsystems is called a switched time-delay system; in particular, when the subsystems are linear, it is then called a switched time-delay linear system. During the last decades, the stability analysis of switched linear continuous/discrete time-delay systems has attracted a lot of attention [4-8]. The main approach for stability analysis relies on the use of Lyapunov-Krasovskii functionals and linear matrix inequlity (LMI) approach for constructing a common Lyapunov function [8-10]. Although many important results have been obtained for switched linear continuous-time systems, there are few results concerning the stability of switched linear discrete systems with time-varying delays. It was shown in [5,7,11,12] that when all subsystems are asymptotically stable, the switching system is asymptotically stable under an arbitrary switching rule. The asymptotic stability for switching linear discrete time-delay systems has been studied in [10], but the result was limited to constant delays. In [11,12], a class of switching signals has been identified for the considered switched discrete-time delay systems to be stable under the average dwell time scheme.

This article studies mean square robust stability problem for stochastic switched linear discrete systems with convex polytopic uncertainties with interval time-varying delays. Specifically, our goal is to develop a constructive way to design switching rule to mean square robustly stable the system. By using improved Lyapunov-Krasovskii functionals combined with LMIs technique, we propose new criteria for the mean square robust stability of the system. Compared to the existing results, our result has its own advantages. First, the time delay is assumed to be a time-varying function belonging to a given interval, which means that the lower and upper bounds for the time-varying delay are available, the delay function is bounded but not restricted to zero. Second, the approach allows us to design the switching rule for mean square robust stability in terms of LMIs, which can be solvable by utilizing Matlab's LMI Control Toolbox available in the literature to date.

The article is organized as follows: Section 2 presents definitions and some well-known technical propositions needed for the proof of the main results. Switching rule for the mean square robust stability is presented in Section 3. Numerical example is provided to illustrate the theoretical results in Section 4, and the conclusions are drawn in Section 5.

2 Preliminaries

The following notations will be used throughout this article. R^+ denotes the set of all real non-negative numbers; R^n denotes the n-dimensional space with the scalar product of two vectors $\langle x, y\rangle$ or $x^T y$; $R^{n\times r}$ denotes the space of all matrices of $(n \times r)$ - dimension. A^T denotes the transpose of A; a matrix A is symmetric if $A = A^T$, I is the identity matrix of appropriate dimention.

Matrix A is semi-positive definite ($A \geq 0$) if $\langle Ax, x\rangle \geq 0$, for all $x \in R^n$; A is positive definite ($A > 0$) if $\langle Ax, x\rangle > 0$ for all $x \neq 0$; $A \geq B$ means $A - B \geq 0$. $\lambda(A)$ denotes the set of all eigenvalues of A; $\lambda_{\min}(A) = \min\{Re\lambda : \lambda \in \lambda(\mathrm{A})\}$.

Consider a stochastic switched linear discrete systems with convex polytopic uncertainties with interval time-varying delay of the form

$$\begin{aligned} x(k+1) &= A_\gamma(\zeta)x(k) + B_\gamma(\zeta)x(k-d(k)) + \sigma_\gamma(x(k),\ x(k-d(k)),\ k)\omega(k), \quad k = 0, 1, 2, \ldots \\ x(k) &= v_k, \quad k = -d_2,\ -d_2+1,\ \ldots,\ 0, \end{aligned} \tag{2.1}$$

where $x(k) \in R^n$ is the state, $\gamma(.) : R^n \to \mathcal{N} := \{1, 2, \ldots, N\}$ is the switching rule, which is a function depending on the state at each time and will be designed. A switching function is a rule which determines a switching sequence for a given switching system. Moreover, $\gamma(x(k)) = i$ implies that the system realization is chosen as the i^{th} system, $i = 1, 2,..., N$. It is seen that the system (2.1) can be viewed as an autonomous switched system in which the effective subsystem changes when the state $x(k)$ hits predefined boundaries. A_i, B_i, $i = 1, 2,..., N$ are given constant matrices. The system matrices are subjected to uncertainties and belong to the polytope Ω given by

$$\Omega = \{[A_i,\ B_i](\zeta)\ := \sum_{j=1}^{N} \zeta_j [A_{ij},\ B_{ij}],\quad \sum_{j=1}^{N} \zeta_j = 1,\ \zeta_j \geq 0\}, \tag{2.2}$$

where A_{ij}, B_{ij}, i, j = 1, 2,..., N, are given constant matrices with appropriate dimensions. $\omega(k)$ is a scalar Wiener process (Brownian Motion) on $(\Omega, \mathcal{F}, \mathcal{P})$ with

$$E\{\omega(k)\} = 0,\quad E\{\omega^2(k)\} = 1,\quad E\{\omega(i)\omega(j)\} = 0(i \neq j), \tag{2.3}$$

and σ_i: $R^n \times R^n \times R \to R^n$, i = 1, 2,..., N is the continuous function, and is assumed to satisfy that

$$\begin{aligned}&\sigma_i^T(x(k),\ x(k-d(k)),\ k)\sigma_i(x(k),\ x(k-d(k)),\ k) \leq \rho_{i1}x^T(k)x(k) + \rho_{i2}x^T(k-d(k))x(k-d(k),\\ &x(k), x(k-d(k) \in R^n,\end{aligned} \tag{2.4}$$

where $\rho_{i1} > 0$ and $\rho_{i2} > 0$, i = 1, 2,..., N are khown constant scalars. For simplicity, we denote $\sigma_i(x(k), x(k- d(k)), k)$ by σ_i, respectively.

The time-varying function $d(k)$ satisfies the following condition:

$$0 < d_1 \leq d(k) \leq d_2,\quad \forall k = 0, 1, 2,\ \ldots.$$

Remark 2.1. It is worth noting that the time delay is a time-varying function belonging to a given interval, in which the lower bound of delay is not restricted to zero.

Definition 2.1. The stochastic switched linear discrete systems with convex polytopic uncertainties (2.1) is robustly stable in the mean square if there exist a positive definite scalar function $V(k, x(k)$: $R^+ \times R^n \to R$ and a switching function $\gamma(\cdot)$ such that

$$E\{\Delta V(k,\ x(k))\} = E\{V(k+1,\ x(k+1)) - V(k,\ x(k))\} < 0,$$

along any trajectory of solution of the system (2.1) for all uncertainties which satisfy (2.2).

Definition 2.2. The system of matrices $\{J_i\}$, i = 1, 2,..., N, is said to be strictly complete if for every $x \in R^n \backslash \{0\}$ there is $i \in \{1, 2,..., N\}$ such that $x^T J_i x < 0$.

It is easy to see that the system $\{J_i\}$ is strictly complete if and only if

$$\bigcup_{i=1}^{N} \alpha_i = R^n \backslash \{0\},$$

where

$$\alpha_i = \{x \in R^n :\quad x^T J_i x < 0\},\ i = 1, 2,\ \ldots,\ N.$$

Proposition 2.1. [11]*The system* $\{J_i\}$, i = 1, 2,..., N, *is strictly complete if there exist* $\delta_i \geq 0,\ i = 1, 2, \ldots, N, \sum_{i=1}^{N} \delta_i > 0$ *such that*

$$\sum_{i=1}^{N} \delta_i J_i < 0.$$

If N = 2 *then the above condition is also necessary for the strict completeness.*

Proposition 2.2. *For real numbers* $\zeta_j \geq 0$, j = 1,2,...,N, $\sum_{j=1}^{N} \zeta_j = 1$ *the following inequality hold*

$$(N-1)\sum_{j=1}^{N}\zeta_j^2 - 2\sum_{j=1}^{N-1}\sum_{l=j+1}^{N}\zeta_j\zeta_l \geq 0.$$

Proof. The proof is followed from the completing the square:

$$(N-1)\sum_{j=1}^{N}\zeta_j^2 - 2\sum_{j=1}^{N-1}\sum_{l=j+1}^{N}\zeta_j\zeta_l = \sum_{j=1}^{N-1}\sum_{l=j+1}^{N}(\zeta_j-\zeta_l)^2 \geq 0.$$

3 Main results

Let us set

$$
\begin{gathered}
\| x_k \| = \sup_{s\in[-d_2,0]} \| x(k+s) \|, \\
W_{ijj} = \begin{pmatrix} (d_2-d_1)Q_j - P_j + 2\rho_{i1}I & S_j - S_jA_{ij} & -S_jB_{ij} \\ S_j^T - A_{ij}^TS_j^T & P_j + S_j + S_j^T & -S_jB_{ij} \\ -B_{ij}^TS_j^T & -B_{ij}^TS_j^T & -Q_j + 2\rho_{i2}I \end{pmatrix}, \\
W_{ijl} = \begin{pmatrix} (d_2-d_1)Q_j - P_j + 2\rho_{i1}I & S_j - S_jA_{il} & -S_jB_{il} \\ S_j^T - A_{il}^TS_j^T & P_j + S_j + S_j^T & -S_jB_{il} \\ -B_{il}^TS_j^T & -B_{il}^TS_j^T & -Q_j + 2\rho_{i2}I \end{pmatrix}, \\
W_{ilj} = \begin{pmatrix} (d_2-d_1)Q_l - P_l + 2\rho_{i1}I & S_l - S_lA_{ij} & -S_lB_{ij} \\ S_l^T - A_{ij}^TS_l^T & P_l + S_l + S_l^T & -S_lB_{ij} \\ -B_{ij}^TS_l^T & -B_{ij}^TS_l^T & -Q_l + 2\rho_{i2}I \end{pmatrix}, \\
P(\zeta) = \sum_{j=1}^{N}\zeta_jP_j, \quad Q(\zeta) = \sum_{j=1}^{N}\zeta_jQ_j, \quad S(\zeta) = \sum_{j=1}^{N}\zeta_jS_j, \\
\lambda_1 = \lambda_{\min}(P(\zeta)), \quad \lambda_2 = \lambda_{\max}(P(\zeta)), \quad \mathcal{R} = \begin{pmatrix} R & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
J_{ijj} := Q_j - S_jA_{ij} - A_{ij}^TS_j^T, \\
J_{ijl} = Q_j - S_jA_{il} - A_{il}^TS_j^T, \\
J_{ilj} := Q_l - S_lA_{ij} - A_{ij}^TS_l^T, \\
\alpha_{ijj} = \{x \in R^n : \quad x^TJ_{ijj}x < 0, \}, \quad i = 1,2,\dots,N, \quad j = 1,2,\dots,N, \\
\alpha_{ijl} = \{x \in R^n : \quad x^TJ_{ijl}x < 0, \}, \quad i = 1,2,\dots,N, \quad j = 1,2,\dots,N-1; \quad l = j+1,\dots,N, \\
\alpha_{ijl} = \{x \in R^n : \quad x^TJ_{ilj}x < 0, \ \}, \quad i = 1,2,\dots,N, \quad j = 1,2,\dots,N-1; \quad l = j+1,\dots,N, \\
\bar{\alpha}_{1jj} = \alpha_{1jj}, \quad \bar{\alpha}_{ijj} = \alpha_{ijj}\setminus\bigcup_{i=1}^{i-1}\bar{\alpha}_{ijj}, \quad i = 2,3,\dots,N, \quad j = 1,2,\dots,N \\
\bar{\alpha}_{1jl} = \alpha_{1jl}, \quad \bar{\alpha}_{ijl} = \alpha_{ijl}\setminus\bigcup_{i=1}^{i-1}\bar{\alpha}_{ijl}, \quad i = 2,3,\dots,N, \quad j = 1,2,\dots,N-1; \quad l = j+1,\dots,N, \\
\bar{\alpha}_{1lj} = \alpha_{1lj}, \quad \bar{\alpha}_{ilj} = \alpha_{ilj}\setminus\bigcup_{i=1}^{i-1}\bar{\alpha}_{ilj}, \quad i = 2,3,\dots,N, \quad j = 1,2,\dots,N-1; \quad l = j+1,\dots,N.
\end{gathered}
\tag{3.1}
$$

The main result of this article is summarized in the following theorem.

Theorem 3.1. *The stochastic switched system with convex polytopic uncertainties (2.1) is robustly stable in the mean square if there exist symmetric matrices $P_j > 0$, $Q_j > 0$, $R \geq 0$, $j = 1, 2..., N$ and matrix S_j, $j = 1, 2..., N$ satisfying the following conditions*

(*i*) $\exists\delta_i \geq 0,\ \sum_{i=1}^{N}\delta_i > 0 : \sum_{i=1}^{N}\delta_iJ_{ijj} < 0$, *and* $J_{ijj}+\mathcal{R} < 0$, $i = 1,2,...,N$, $j = 1,2,...,N$.

(*ii*) $\exists\delta_i \geq 0,\ \sum_{i=1}^{N}\delta_i > 0 : \sum_{i=1}^{N}[\delta_iJ_{ijl} + \delta_iJ_{ilj}] < 0$, *and* $J_{ijl} + J_{ilj} - \frac{2}{N-1}\mathcal{R} < 0$, $i = 1,2,...,N$, $j = 1,2,..., N - 1$; $l = j + 1,...,N$.

(*iii*) $W_{ijj}+\mathcal{R} < 0$, $i = 1, 2,...,N$, $j = 1, 2,...,N$.

(iv) $W_{ijl} + W_{ilj} - \frac{2}{N-1}\mathcal{R} < 0$, i = 1, 2,...,N, j = 1, 2,...,N - 1; l = j+1,...,N.

The switching rule is chosen as $\gamma(x(k)) = i$, *whenever* $x(k) \in \bar{\alpha}_{ijl}$.

Proof. Consider the following Lyapunov-Krasovskii functional for any ith system (2.1)

$$V(k) = V_1(k) + V_2(k) + V_3(k),$$

where

$$V_1(k) = x^T(k)P(\zeta)x(k),\ V_2(k) = \sum_{i=k-d(k)}^{k-1} x^T(i)Q(\zeta)x(i),$$
$$V_3(k) = \sum_{j=-d_2+2}^{-d_1+1}\sum_{l=k+j}^{k-1} x^T(l)Q(\zeta)x(l).$$

We can verify that

$$\lambda_1\|x(k)\|^2 \le V(k) \le \lambda_2\|x_k\|^2. \tag{3.2}$$

Let us set $\xi(k)$ = [x^T (k) x^T (k + 1) x^T (k - $d(k)$) ω^T (k)], and

$$H(\zeta) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & P(\zeta) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ G(\zeta) = \begin{pmatrix} P(\zeta) & 0 & 0 & 0 \\ I & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}.$$

Then, the difference of $V_1(k)$ along the solution of the system (2.1) and taking the mathematical expectation, we obtained

$$\begin{aligned} E\{\Delta V_1(k)\} &= E\{x^T(k+1)P(\zeta)x(k+1) - x^T(k)P(\zeta)x(k)\} \\ &= E\left\{\xi^T(k)H(\zeta)\xi(k) - 2\xi^T(k)G^T(\zeta)\begin{pmatrix} 0.5x(k) \\ 0 \\ 0 \\ 0 \end{pmatrix}\right\}. \end{aligned} \tag{3.3}$$

because of

$$E\{\xi^T(k)H(\zeta)\xi(k)\} = E\{x(k+1)P(\zeta)x(k+1)\},$$
$$E\left\{2\xi^T(k)G^T(\zeta)\begin{pmatrix} 0.5x(k) \\ 0 \\ 0 \\ 0 \end{pmatrix}\right\} = E\{x^T(k)P(\zeta)x(k)\}.$$

Using the expression of system (2.1)

$$0 = -S(\zeta)x(k+1) + S(\zeta)A_i(\zeta)x(k) + S(\zeta)B_i(\zeta)x(k-d(k)) + S(\zeta)\sigma_i\omega(k),$$
$$0 = -\sigma_i^T x(k+1) + \sigma_i^T A_i(\zeta)x(k) + \sigma_i^T B_i(\zeta)x(k-d(k)) + \sigma_i^T\sigma_i\omega(k),$$

we have

$$E\left\{-2\xi^T(k)G^T(\zeta)\times\begin{pmatrix}0.5x(k)\\ -S(\zeta)x(k+1)+S(\zeta)A_i(\zeta)x(k)+S(\zeta)B_i(\zeta)x(k-d(k))+S(\zeta)\sigma_i\omega(k)\\ 0\\ -\sigma_i^Tx(k+1)+\sigma_i^TA_i(\zeta)x(k)+\sigma_i^TB_i(\zeta)x(k-d(k))+\sigma_i^T\sigma_i\omega(k)\end{pmatrix}\xi(k)\right\}$$
$$=E\left\{-\xi^T(k)G^T(\zeta)\begin{pmatrix}0.5I & 0 & 0 & 0\\ S(\zeta)A_i(\zeta) & -S(\zeta) & S(\zeta)B_i(\zeta) & S(\zeta)\sigma_i\\ 0 & 0 & 0 & 0\\ \sigma_i^TA_i(\zeta) & -\sigma_i^T & \sigma_i^TB_i(\zeta) & \sigma_i^T\sigma_i\end{pmatrix}\xi^T(k)\right\}$$
$$-E\left\{\xi^T(k)\begin{pmatrix}0.5I & 0 & 0 & 0\\ S(\zeta)A_i(\zeta) & -S(\zeta) & S(\zeta)B_i(\zeta) & S(\zeta)\sigma_i\\ 0 & 0 & 0 & 0\\ \sigma_i^TA_i(\zeta) & -\sigma_i^T & \sigma_i^TB_i(\zeta) & \sigma_i^T\sigma_i\end{pmatrix}^T G(\zeta)\xi(k)\right\}.$$

Therefore, from (3.3) it follows that

$$\begin{aligned}E\{\Delta V_1(k)\} &= E\{x^T(k)[-P(\zeta)-S(\zeta)A_i(\zeta)-A_i^T(\zeta)S^T(\zeta)]x(k)\}\\ &+E\{2x^T(k)[S(\zeta)-S(\zeta)A_i(\zeta)]x(k+1)\}\\ &+E\{2x^T(k)[-S(\zeta)B_i(\zeta)]x(k-d(k))\}\\ &+E\{2x^T(k)[-S(\zeta)\sigma_i-\sigma_i^TA_i(\zeta)]\omega(k)\}\\ &+E\{x(k+1)[P(\zeta)+S(\zeta)+S^T(\zeta)]x(k+1)\}\\ &+E\{2x(k+1)[-S(\zeta)B_i(\zeta)]x(k-d(k))\}\\ &+E\{2x(k+1)[\sigma_i^T-S(\zeta)\sigma_i]\omega(k)\}\\ &+E\{2x^T(k-d(k))[-\sigma_i^TB_i(\zeta)]\omega(k)\}\\ &+E\{\omega^T(k)[-2\sigma_i^T\sigma_i]\omega(k)\}.\end{aligned}$$

By asumption (2.3), we have

$$\begin{aligned}E\{\Delta V_1(k)\} &= E\{x^T(k)[-P(\zeta)-S(\zeta)A_i(\zeta)-A_i^T(\zeta)S^T(\zeta)]x(k)\}\\ &+E\{2x^T(k)[S(\zeta)-S(\zeta)A_i(\zeta)]x(k+1)\}\\ &+E\{2x^T(k)[-S(\zeta)B_i(\zeta)]x(k-d(k))\}\\ &+E\{x(k+1)[P(\zeta)+S(\zeta)+S^T(\zeta)]x(k+1)\}\\ &+E\{2x(k+1)[-S(\zeta)B_i(\zeta)]x(k-d(k))\}\\ &+E\{2[-\sigma_i^T\sigma_i]\}.\end{aligned}$$

Applying asumption (2.4), the following estimations hold

$$E\{-\sigma_i^T(x(k),\ x(k-d(k)),\ k)\sigma_i(x(k),\ x(k-d(k)),\ k)\}\le E\{\rho_{i1}x^T(k)x(k)+\rho_{i2}x^T(k-d(k))x(k-d(k)\}.$$

Therefore, we have

$$\begin{aligned}E\{\Delta V_1(k)\} &= E\{x^T(k)[-P(\zeta)-S(\zeta)A_i(\zeta)-A_i^T(\zeta)S^T(\zeta)+2\rho_{i1}I]x(k)\}\\ &+E\{2x^T(k)[S(\zeta)-S(\zeta)A_i(\zeta)]x(k+1)\}\\ &+E\{2x^T(k)[-S(\zeta)B_i(\zeta)]x(k-d(k))\}\\ &+E\{x(k+1)[P(\zeta)+S(\zeta)+S^T(\zeta)]x(k+1)\}\\ &+E\{2x(k+1)[-S(\zeta)B_i(\zeta)]x(k-d(k))\}\\ &+E\{x^T(k-d(k))[2\rho_{i2}I]x(k-d(k))\}.\end{aligned}\tag{3.4}$$

The difference of $V_2(k)$ is given by

$$\begin{aligned} E\{\Delta V_2(k)\} &= E\left\{\sum_{i=k+1-d(k+1)}^{k} x^T(i)Q(\zeta)x(i) - \sum_{i=k-d(k)}^{k-1} x^T(i)Q(\zeta)x(i)\right\} \\ &= E\left\{\sum_{i=k+1-d(k+1)}^{k-d_1} x^T(i)Q(\zeta)x(i) + x^T(k)Q(\zeta)x(k) - x^T(k-d(k))Q(\zeta)x(k-d(k))\right\} \\ &+ E\left\{\sum_{i=k+1-d_1}^{k-1} x^T(i)Q(\zeta)x(i) - \sum_{i=k+1-d(k)}^{k-1} x^T(i)Q(\zeta)x(i)\right\}. \end{aligned} \tag{3.5}$$

Since $d(k) \geq d_1$ we have

$$E\left\{\sum_{i=k+1-d_1}^{k-1} x^T(i)Q(\zeta)x(i) - \sum_{i=k+1-d(k)}^{k-1} x^T(i)Q(\zeta)x(i)\right\} \leq 0,$$

and hence from (3.5) we have

$$E\{\Delta V_2(k)\} \leq E\{\sum_{i=k+1-d(k+1)}^{k-d_1} x^T(i)Q(\zeta)x(i) + x^T(k)Q(\zeta)x(k) - x^T(k-d(k))Q(\zeta)x(k-d(k))\}. \tag{3.6}$$

The difference of $V_3(k)$ is given by

$$\begin{aligned} E\{\Delta V_3(k)\} &= E\{\sum_{j=-d_2+2}^{-d_1+1} \sum_{l=k+j+1}^{k} x^T(l)Q(\zeta)x(l) - \sum_{j=-d_2+2}^{-d_1+1} \sum_{l=k+j}^{k-1} x^T(l)Q(\zeta)x(l)\} \\ &= E\left\{\sum_{j=-d_2+2}^{-d_1+1} [\sum_{l=k+j}^{k-1} x^T(l)Q(\zeta)x(l) + x^T(k)Q(\zeta)(\xi)x(k)\right\} \\ &- E\{\sum_{l=k+j}^{k-1} x^T(l)Q(\zeta)x(l) - x^T(k+j-1)Q(\zeta)x(k+j-1)]\} \\ &= E\left\{\sum_{j=-d_2+2}^{-d_1+1} [x^T(k)Q(\zeta)x(k) - x^T(k+j-1)Q(\zeta)x(k+j-1)]\right\} \\ &= E\left\{(d_2 - d_1)x^T(k)Q(\zeta)x(k) - \sum_{j=k+1-d_2}^{k-d_1} x^T(j)Q(\zeta)x(j)\right\} \end{aligned} \tag{3.7}$$

Since $d(k) \leq d_2$, and

$$E\left\{\sum_{i=k=1-d(k+1)}^{k-d_1} x^T(i)Q(\zeta)x(i) - \sum_{i=k+1-d_2}^{k-d_1} x^T(i)Q(\zeta)x(i)\right\} \leq 0,$$

we obtain from (3.6) and (3.7) that

$$E\{\Delta V_2(k) + \Delta V_3(k)\} \leq E\{(d_2 - d_1 + 1)x^T(k)Q(\zeta)x(k) - x^T(k-d(k))Q(\zeta)x(k-d(k))\}. \tag{3.8}$$

Therefore, combining the inequalities (3.4), (3.8) gives

$$E\{\Delta V(k)\} \leq E\{x^T(k)J_i(\zeta)x(k) + \psi^T(k)W_i(\zeta)\psi(k)\}, \tag{3.9}$$

where

$$\begin{aligned}
\psi(k) &= [x^T(k)x^T(k+1)x^T(k-d(k))],\\
W_i(\zeta) &= \begin{pmatrix} (d_2-d_1)Q(\zeta)-P(\zeta)+2\rho_{i1}I & S(\zeta)-S(\zeta)A_i(\zeta) & -S(\zeta)B_i(\zeta)\\ S^T(\zeta)-A_i^T(\zeta)S^T(\zeta) & P(\zeta)+S(\zeta)+S^T(\zeta) & -S(\zeta)B_i(\zeta)\\ -B_i^T(\zeta)S^T(\zeta) & -B_i^T(\zeta)S^T(\zeta) & -Q(\zeta)+2\rho_{i2}I \end{pmatrix},\\
J_i(\zeta) &= Q(\zeta)-S(\zeta)A_i(\zeta)-A_i^T(\zeta)S^T(\zeta).
\end{aligned}$$

From the convex combination of the expression of $P(\zeta)$, $Q(\zeta)$, $S_1(\zeta)$, $A(\zeta)$, $B(\zeta)$, we have

$$\begin{aligned}
W_i(\zeta) &= \sum_{j=1}^{N}\zeta_j^2\begin{pmatrix} (d_2-d_1)Q_j-P_j+2\rho_{i1}I & S_j-S_{1j}A_{ij} & -S_jB_{ij}\\ S_j^T-A_{ij}^TS_j^T & P_j+S_j+S_j^T & -S_jB_{ij}\\ -B_{ij}^TS_j^T & -B_{ij}^TS_j^T & -Q_j+2\rho_{i2}I \end{pmatrix}\\
&+\sum_{j=1}^{N-1}\sum_{l=j+1}^{N}\zeta_j\zeta_l\begin{pmatrix} (d_2-d_1)Q_{jl}-P_{jl}+2\rho_{i1}I & S_{jl}-(SA_i)_{jl} & -(SB_i)_{jl}\\ S_{jl}^T-(SA_i)_{jl}^T & P_{jl}+S_{jl}+S_{jl}^T & -(SB_i)_{jl}\\ -(SB_i)_{jl}^T & -(SB_i)_{jl}^T & -Q_{jl}+2\rho_{i2}I \end{pmatrix}\\
&= \sum_{j=1}^{N}\zeta_j^2W_{ijj}+\sum_{j=1}^{N-1}\sum_{l=j+1}^{N}\zeta_j\zeta_l[W_{ijl}+W_{ilj}],
\end{aligned}$$

$$\begin{aligned}
J_i(\zeta) &= \sum_{j=1}^{N}\zeta_j^2(Q_j-S_jA_{ij}-A_{ij}^TS_j^T)+\sum_{j=1}^{N-1}\sum_{l=j+1}^{N}\zeta_j\zeta_l(Q_{jl}-(SA_i)_{jl}-(SA_i)_{jl}^T\\
&= \sum_{j=1}^{N}\zeta_j^2J_{ijj}+\sum_{j=1}^{N-1}\sum_{l=j+1}^{N}\zeta_j\zeta_l[J_{ijl}+J_{ilj}],
\end{aligned}$$

where

$$\begin{aligned}
&(SA_i)_{jl} := S_jA_{il}+S_lA_{ij},\\
&(SB_i)_{jl} := S_jB_{il}+S_lB_{ij},\\
&Q_{jl} = Q_j+Q_l,,\ P_{jl}=P_j+P_l,\ S_{jl}=S_j+S_l.
\end{aligned}$$

Then the conditions (i)-(iv) give

$$\begin{aligned}
W_i(\zeta) &< -\sum_{j=1}^{N}\zeta_j^2\mathcal{R}+\frac{2}{N-1}\sum_{j=1}^{N-1}\sum_{l=j+1}^{N}\zeta_j\zeta_l\mathcal{R}\le 0,\\
J_i(\zeta) &< -\sum_{j=1}^{N}\zeta_j^2\mathcal{R}+\frac{2}{N-1}\sum_{j=1}^{N-1}\sum_{l=j+1}^{N}\zeta_j\zeta_l\mathcal{R}\le 0,
\end{aligned}$$

because of Proposition 2.2:

$$(N-1)\sum_{j=1}^{N}\zeta_j^2-2\sum_{j=1}^{N-1}\sum_{l=j+1}^{N}\zeta_j\zeta_l=\sum_{j=1}^{N-1}\sum_{l=j+1}^{N}(\zeta_j-\zeta_l)^2\ge 0.$$

Therefore, we finally obtain from (3.9) and the condition (iii), (iv) that

$$E\{\Delta V(k)\} < E\{x^T(k)J_i(\zeta)x(k)\},\ \forall i=1,2,\ \ldots,\ N,\ k=0,1,2,\ \ldots.$$

We now apply the condition (i), (ii) and Proposition 2.1, the system $J_i(\zeta)$ is strictly complete, and the sets α_{ijl} and $\bar{\alpha}_{ijl}$ by (3.1) are well defined such that

$$\bigcup_{i=1}^{N} \alpha_{ijl} = R^n \backslash \{0\},$$

$$\bigcup_{i=1}^{N} \bar{\alpha}_{ijl} = R^n \backslash \{0\}, \quad \bar{\alpha}_{ijl} \cap \bar{\alpha}_{tjl} = \emptyset, i \neq t.$$

Therefore, for any $x(k) \in R^n$, $k = 0, 1, 2,....$, there exists $i \in \{1, 2,..., N\}$ such that $x(k) \in \bar{\alpha}_{ijl}$. By choosing switching rule as $\gamma(x(k)) = i$ whenever $x(k) \in \bar{\alpha}_{ijl}$, from the condition (i) and (ii) we have

$$E\{\Delta V(k)\} \leq E\{x^T(k)J_i(\zeta)x(k)\} < 0, \; k = 0, 1, 2, \; \ldots,$$

which, combining the condition (3.2), and Definition 2.1, the system (2.1) is exponentially stable in the mean square. The proof is complete.

Remark 3.1. Note that the results purposed in [4-6,9,12] for switching systems to be asymptotically stable under an arbitrary switching rule. The asymptotic stability for switching linear discrete time-delay systems studied in [9] was limited to constant delays. In [10], a class of switching signals has been identified for the considered switched discrete-time delay systems to be stable under the averaged well time scheme.

Remark 3.2. It is worth noting that the time delay is a time-varying function belonging to a given interval, in which the lower bound of delay is not restricted to zero but in [18,19] were limited to constant delays and the lower bound of delay was restricted to zero.

4 Numerical examples

Example 4.1. Consider the stochastic switched discrete-time system with convex polytopic uncertainties (2.1) for $N = 2$, where the delay function $d(k)$ is given by

$$d(k) = 1 + 28sin^2\frac{k\pi}{2}, \; k = 0, 1, 2, \; \ldots.$$

and

$$(A_{11}, \; B_{11}) = \left(\begin{bmatrix} -1 & 0.1 \\ 0.2 & -2 \end{bmatrix}, \begin{bmatrix} -1 & 0.2 \\ 0.1 & -3 \end{bmatrix}\right), \; (A_{12}, \; B_{12}) = \left(\begin{bmatrix} -2 & 0.3 \\ 0.5 & -3 \end{bmatrix}, \begin{bmatrix} -3 & 0.01 \\ 0.24 & -1.8 \end{bmatrix}\right),$$

$$(A_{21}, \; B_{21}) = \left(\begin{bmatrix} -0.1 & 0.01 \\ 0.02 & -0.2 \end{bmatrix}, \begin{bmatrix} -0.1 & 0.02 \\ 0.01 & -0.3 \end{bmatrix}\right), \; (A_{22}, \; B_{22}) = \left(\begin{bmatrix} -0.2 & 0.03 \\ 0.05 & -0.3 \end{bmatrix}, \begin{bmatrix} -0.3 & 0.01 \\ 0.024 & -0.18 \end{bmatrix}\right).$$

By LMI toolbox of Matlab, we find that the conditions (i) - (iv) of Theorem 3.1 are satisfied with $\delta_1 = 0.01$, $\delta_2 = 0{:}01$, $\zeta_1 = 0{:}5$, $\zeta_2 = 0.5$, $\rho_{11} = 0.1$, $\rho_{12} = 0.1$, $\rho_{21} = 0.1$, $\rho_{22} = 0.1$, $d_1 = 1$, $d_2 = 29$, and

$$P_1 = \begin{bmatrix} 28.0877 & 0.0742 \\ 0.0742 & 22.3782 \end{bmatrix}, P_2 = \begin{bmatrix} 23.3129 & 0.0057 \\ 0.0057 & 29.0647 \end{bmatrix}, Q_1 = \begin{bmatrix} 0.3324 & 0.0128 \\ 0.0128 & 0.0868 \end{bmatrix}, Q_2 = \begin{bmatrix} 0.0160 & 0.0097 \\ 0.0097 & 0.1173 \end{bmatrix},$$

$$S_1 = \begin{bmatrix} -5.0578 & -0.1508 \\ -0.6586 & -3.1947 \end{bmatrix},\ S_2 = \begin{bmatrix} -3.9473 & -0.2175 \\ -0.6347 & -2.9459 \end{bmatrix}, \mathcal{R} = \begin{bmatrix} 0.3270 & -0.0153 \\ -0.0153 & 0.4768 \end{bmatrix}.$$

In this case, we have

$$(J_{111}(R,\ Q),\ J_{211}(R,\ Q)) = \left(\begin{bmatrix} -9.7229 & 0.1973 \\ 0.1973 & -12.5603 \end{bmatrix}, \begin{bmatrix} -0.6731 & 0.0313 \\ 0.0313 & -1.1780 \end{bmatrix} \right),$$

$$(J_{122}(R,\ Q),\ J_{222}(R,\ Q)) = \left(\begin{bmatrix} -15.5556 & 0.7448 \\ 0.7448 & -17.1775 \end{bmatrix}, \begin{bmatrix} -1.5412 & 0.0833 \\ 0.0833 & -1.6122 \end{bmatrix} \right),$$

$$(J_{112}(R,\ Q),\ J_{212}(R,\ Q)) = \left(\begin{bmatrix} -19.7481 & 1.3578 \\ 1.3578 & -18.6862 \end{bmatrix}, \begin{bmatrix} -1.6756 & 0.1473 \\ 0.1473 & -1.7905 \end{bmatrix} \right),$$

$$(J_{121}(R,\ Q),\ J_{221}(R,\ Q)) = \left(\begin{bmatrix} -7.7916 & -0.0761 \\ -0.0761 & -11.5395 \end{bmatrix}, \begin{bmatrix} -0.7648 & 0.0012 \\ 0.0012 & -1.0484 \end{bmatrix} \right).$$

Moreover, the sum

$$\begin{aligned} &\delta_1 J_{111}(R,\ Q) + \delta_1 J_{211}(R,\ Q) + \delta_1 J_{122}(R,\ Q) + \delta_1 J_{222}(R,\ Q) + \delta_1 J_{112}(R,\ Q) + \delta_1 J_{121}(R,\ Q) + \delta_1 J_{212}(R,\ Q) \\ &+\delta_1 J_{221}(R,\ Q) + \delta_2 J_{111}(R,\ Q) + \delta_2 J_{211}(R,\ Q) + \delta_2 J_{122}(R,\ Q) + \delta_2 J_{222}(R,\ Q) + \delta_2 J_{112}(R,\ Q) + \delta_2 J_{121}(R,\ Q) \\ &+\delta_2 J_{212}(R,\ Q) + \delta_2 J_{221}(R,\ Q) = \begin{bmatrix} -1.1495 & 0.0497 \\ 0.0497 & -1.3119 \end{bmatrix}, \end{aligned}$$

is negative definite; i.e. the first entry in the first row and the first column -1:1495 <0 is negative and the determinant of the matrix is positive. The sets α_{ijl}, $i, j, l = 1, 2$, are given as

$$\begin{aligned} \alpha_{111} &= \{(x_1,\ x_2)\ :\ -9.7229x_1^2 + 0.3946x_1x_2 - 12.5603x_2^2 < 0\}, \\ \alpha_{211} &= \{(x_1,\ x_2) : 0.6731x_1^2 - 0.0626x_1x_2 + 1.1780x_2^2 > 0\}, \\ \alpha_{122} &= \{(x_1,\ x_2)\ :\ -15.5556x_1^2 + 1.4896x_1x_2 - 17.1775x_2^2 < 0\}, \\ \alpha_{222} &= \{(x_1,\ x_2) : 1.5412x_1^2 - 0.1666x_1x_2 + 1.6122x_2^2 > 0\}, \\ \alpha_{112} &= \{(x_1,\ x_2)\ :\ -19.7481x_1^2 + 2.7156x_1x_2 - 18.6862x_2^2 < 0\}, \\ \alpha_{212} &= \{(x_1,\ x_2) : 1.6756x_1^2 - 0.2946x_1x_2 + 1.7905x_2^2 > 0\}, \\ \alpha_{121} &= \{(x_1,\ x_2)\ :\ -7.7916x_1^2 - 0.1522x_1x_2 - 11.5395x_2^2 < 0\}, \\ \alpha_{221} &= \{(x_1,\ x_2) : 0.7648x_1^2 - 0.0024x_1x_2 + 1.0484x_2^2 > 0\}. \end{aligned}$$

Obviously, the union of these sets is equal to $R_2 \setminus \{0\}$. The switching regions are defined as

$$\begin{aligned} \bar{\alpha}_{111} &= \{(x_1,\ x_2)\ :\ -9.7229x_1^2 + 0.3946x_1x_2 - 12.5603x_2^2 < 0\}, \\ \bar{\alpha}_{211} &= \alpha_{211} \backslash \bar{\alpha}_{111}, \\ \bar{\alpha}_{122} &= \{(x_1,\ x_2)\ :\ -15.5556x_1^2 + 1.4896x_1x_2 - 17.1775x_2^2 < 0\}, \\ \bar{\alpha}_{222} &= \alpha_{222} \backslash \bar{\alpha}_{122}, \\ \bar{\alpha}_{112} &= \{(x_1,\ x_2)\ :\ -19.7481x_1^2 + 2.7156x_1x_2 - 18.6862x_2^2 < 0\}, \\ \bar{\alpha}_{212} &= \alpha_{212} \backslash \bar{\alpha}_{112}, \\ \bar{\alpha}_{121} &= \{(x_1,\ x_2)\ :\ -7.7916x_1^2 - 0.1522x_1x_2 - 11.5395x_2^2 < 0\}, \\ \bar{\alpha}_{221} &= \alpha_{221} \backslash \bar{\alpha}_{121}. \end{aligned}$$

By Theorem 3.1 the system is robustly stable in the mean square and the switching rule is chosen as $\gamma(x(k)) = i$ whenever $x(k) \in \bar{\alpha}_{ijl}$.

5 Conclusion

This article has proposed a switching design for the mean square robust stability of stochastic switched linear discrete-time systems with convex polytopic uncertainties with interval time-varying delays. Based on the discrete Lyapunov functional, a switching rule for the mean square robust stability for the stochastic system with convex polytopic uncertainties is designed via linear matrix inequalities.

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Authors' contributions
The authors contributed equally and significantly in writing this article. The authors read and approved the final manuscript.

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The authors declare that they have no competing interests.

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References

1. Liberzon, D, Morse, AS: Basic problems in stability and design of switched systems. IEEE Control Syst Mag. **19**, 57–70 (1999)
2. Savkin, AV, Evans, RJ: Hybrid Dynamical Systems: Controller and Sensor Switching Problems. Springer, New York (2001)
3. Sun, Z, Ge, SS: Switched Linear Systems: Control and Design. Springer, London (2005)
4. Gao, F, Zhong, S, Gao, X: Delay-dependent stability of a type of linear switching systems with discrete and distributed time delays. Appl Math Comput. **196**, 24–39 (2008). doi:10.1016/j.amc.2007.05.053
5. Lien, CH, Yu, KW, Chung, YJ, Lin, YF, Chung, LY, Chen, JD: Exponential stability analysis for uncertain switched neutral systems with interval-time-varying state delay. Nonlinear Anal Hybrid Syst. **3**, 334–342 (2009). doi:10.1016/j.nahs.2009.02.010
6. Xie, G, Wang, L: Quadratic stability and stabilization of discrete-time switched systems with state delay. Proceedings of the IEEE Conference on Decision and Control, Atlantics, December 2004. 3235–3240 (2004)
7. Boyd, S, Ghaoui, LE, Feron, E, Balakrishnan, V: Linear Matrix Inequalities in System and Control Theory. SIAM, Philadelphia (1994)
8. Ratchagit, K, Phat, VN: Robust stability and stabilization of linear polytopic delay-difference equations with interval time-varying delays. Neural Parallel Sci Comput. **19**, 361–372 (2011)
9. Zhai, GS, Hu, B, Yasuda, K, Michel, A: Qualitative analysis of discrete-time switched systems. Proceedings of the American Control Conference. 1880–1885 (2002)
10. Zhang, WA, Yu, L: Stability analysis for discrete-time switched time-delay systems. Automatica. **45**, 2265–2271 (2009). doi:10.1016/j.automatica.2009.05.027
11. Uhlig, F: A recurring theorem about pairs of quadratic forms and extensions. Linear Algeb Appl. **25**, 219–237 (1979)
12. Phat, VN, Ratchagit, K: Stability and stabilization of switched linear discrete-time systems with interval time-varying delay. Nonlinear Anal Hybrid Syst. **5**, 605–612 (2011). doi:10.1016/j.nahs.2011.05.006
13. Tian, L, Liang, J, Cao, J: Robust observer for discrete-time Markovian jumping neural networks with mixed mode-dependent delays. Nonlinear Dyn. **67**, 47–61 (2012). doi:10.1007/s11071-011-9956-y
14. Liang, J, Lam, J, Wang, Z: State estimation for Markov-type genetic regulatory networks with delays and uncertain mode transition rates. Phys Lett A. **373**, 4328–4337 (2009). doi:10.1016/j.physleta.2009.09.055
15. Dong, H, Wang, Z, Ho, DWC, Gao, H: Robust H_∞ filtering for Markovian jump systems with randomly occurring nonlinearities and sensor saturation: the finite-horizon case. IEEE Trans Sig Process. **59**, 3048–3057 (2011)
16. Wang, Z, Wei, G, Feng, G: Reliable H_∞ control for discrete-time piecewise linear systems with infinite distributed delays. Automatica. **45**, 2991–2994 (2009). doi:10.1016/j.automatica.2009.09.012
17. Shen, B, Wang, Z, Liu, X: Bounded H_∞ synchronization and state estimation for discrete time-varying stochastic complex networks over a finite horizon. IEEE Trans Neural Netw. **22**, 145–157 (2011)
18. Wang, Y, Wang, Z, Liang, J: A delay fractioning approach to global synchronization of delayed complex networks with stochastic disturbances. Phys Lett A. **372**, 6066–6073 (2008). doi:10.1016/j.physleta.2008.08.008
19. Wang, Z, Wang, Y, Liu, Y: Global synchronization for discrete-time stochastic complex networks with randomly occurred nonlinearities and mixed time delays. IEEE Trans Neural Netw. **21**, 11–25 (2010)

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