Dennis Burke, Vladimir V. Tkachuk

Diagonals and discrete subsets of squares

Comment.Math.Univ.Carolin. 54,1 (2013) 69 –82.

Abstract: In 2008 Juhász and Szentmiklóssy established that for every compact space X there exists a discrete $D \subset X \times X$ with $|D| = d(X)$. We generalize this result in two directions: the first one is to prove that the same holds for any Lindelöf Σ-space X and hence X^ω is d-separable. We give an example of a countably compact space X such that X^ω is not d-separable. On the other hand, we show that for any Lindelöf p-space X there exists a discrete subset $D \subset X \times X$ such that $\Delta = \{(x,x) : x \in X\} \subset \overline{D}$; in particular, the diagonal Δ is a retract of $\overline{D}$ and the projection of D on the first coordinate is dense in X. As a consequence, some properties that are not discretely reflexive in X become discretely reflexive in $X \times X$. In particular, if X is compact and $\overline{D}$ is Corson (Eberlein) compact for any discrete $D \subset X \times X$ then X itself is Corson (Eberlein). Besides, a Lindelöf p-space X is zero-dimensional if and only if $\overline{D}$ is zero-dimensional for any discrete $D \subset X \times X$. Under CH, we give an example of a crowded countable space X such that every discrete subset of $X \times X$ is closed. In particular, the diagonal of X cannot be contained in the closure of a discrete subspace of $X \times X$.

Keywords: diagonal, discrete subspaces, d-separable space, discrete reflexivity, Lindelöf p-space, Lindelöf Σ-space, finite powers, Corson compact spaces, Eberlein compact spaces, countably compact spaces

AMS Subject Classification: Primary 54H11, 54C10; Secondary 54D25, 54C25

References

[ATW] Alas O., Tkachuk V.V., Wilson R.G., *Closures of discrete sets often reflect global properties*, Topology Proc. **25** (2000), 27–44.

[Ar] Arhangel'skii A.V., *A class of spaces which contains all metric and all locally compact spaces* (in Russian), Mat. Sb. **67 (109)** (1965), no. 1, 55–88.

[BT] Burke D., Tkachuk V.V., *Discrete reflexivity and complements of the diagonal*, Acta Math. Hungarica, to appear.

[DTTW] Dow A., Tkachenko M.G., Tkachuk V.V., Wilson R.G., *Topologies generated by discrete subspaces*, Glasnik Mat. **37(57)** (2002), 189–212.

[vD] van Douwen E.K., *Applications of maximal topologies*, Topology Appl. **51** (1993), 125–139.

[En] Engelking R., *General Topology*, PWN, Warszawa, 1977.

[Gr] Gruenhage G., *Generalized Metric Spaces*, Handbook of Set-Theoretic Topology, Ed. by K. Kunen and J.E. Vaughan, Elsevier Science Publisher, New York, 1984, pp. 423–501.

[JSz] Juhász I., Szentmiklossy Z., *On d-separability of powers and $C_p(X)$*, Topology Appl. **155** (2008), 277–281.

[Tk1] Tkachuk V.V., *Spaces that are projective with respect to classes of mappings*, Trans. Moscow Math. Soc. **50** (1988), 139–156.

[Tk2] Tkachuk V.V., *A C_p-theory Problem Book. Topological and Function Spaces*, Springer, New York, 2011.