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Global Finite-time Stabilization for Nonholonomic Mobile Robots Based on Visual Servoing

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Abstract In this paper, the global finite-time stabilization problem is considered for nonholonomic mobile robots based on visual servoing with uncalibrated visual parameters, control direction and unmatched external disturbances. Firstly, the simple dynamic chained-form systems is obtained by using a state and input transformation of the kinematic robot systems. Secondly, a new discontinuous switching controller is presented in the presence of uncertainties and disturbances, it is rigorously proved that the corresponding closed-loop system can be stabilized to the origin equilibrium point in a finite time. Finally, the simulation results show the effectiveness of the proposed control design approach.

Keywords Nonholonomic Mobile Robots, Chained-form System, Visual Servoing, Finite-time Stabilization, Switching Control

1. Introduction

Addressing the stabilization problem of nonholonomic systems is a challenging task which has attracted a continuously increasing attention in the control community. As pointed out in [1], such a class of nonlinear systems can not be stabilized to a point with pure smooth (or even continuous) state feedback control. To overcome this difficult, up to now, there have been a lot of control methods to stabilize such systems, which includes discontinuous feedback control laws [2]-[5], continuous time-varying feedback control laws [6]-[8] and hybrid feedback control laws [9]-[11]. As a typical model of the nonholonomic system, the nonholonomic characteristic of wheeled mobile robots arises from the wheel which is rolling without slipping. Many research results of controlling nonholonomic mobile robots have been given in recent decades, such as formation control or cooperative control of multi-robot systems [12]-[15], motion planning [16], trajectories tracking [17]-[19] and point stabilization [20]-[24], etc.

Recently, based on visual servoing model, a new robust control issue is considered in [25]-[31] for nonholonomic mobile robots with uncalibrated camera parameters.Under a single camera fixed on the ceiling, the trajectory tracking and point stabilization (practical stabilization) problems are discussed for the kinematic model with uncertain visual parameters in [25], [27] and [28]. Detailedly, by using Barbalat theorem and Lyapunov techniques, a dynamic feedback robust controller is proposed in [25] that can enable the mobile robot configuration tracking despite the lack of depth information and the lack of precise visual parameters. In [27] and [28], two different switching control design strategies are proposed to address the stabilization problem of the mobile robots, and compared with other results on the same subject (visual servoing feedback control of nonholonomic mobile robots), it is more realistic to suppose that all the parameters of the camera system are unknown in this two papers. In addition, in [26], a new time varying feedback controller is proposed for the exponential stabilization of the nonholonomic chained system with unknown parameters by using state-scaling and switching technique, in [29], the authors have presented a robust adaptive tracking controller for the dynamic mobile robots system.

Additionally, it's worth mentioning that based on visual servoing, the finite-time tracking control for nonholonomic mobile robots and the finite-time tracking control for multiple nonholonomic mobile robots have been discussed in [30] and [31], respectively. However, these results have not involved the finite-time stabilization problem for nonholonomic mobile robots with uncertain camera parameters, as is known to all, two classes of problems-stabilization and tracking control for nonholonomic systems are not the same at all.

Finite-time stabilization problems have been studied mostly in the contexts of optimality, controllability, and deadbeat control for several decades [32]-[38], in which, compared to the regular asymptotic stabilization, it was demonstrated that finite-time stable systems might enjoy not only faster convergence but also better robustness and disturbance rejection properties.

This article considers the global finite-time stabilization problem for a class of nonholonomic mobile robots based on visual servoing with uncalibrated visual parameters and external disturbances. The main contributions can be summarized as the following two respects:

1) By using a state and input transformation, the dynamic extended chained-form systems is introduced, then according to its special chained structure, two uncertain subsystems is used to designed the discontinuous switching controller.

2) To propose the step-by-step switching control law, the systematic strategy of combining the finite-time stability theory and a three-step discontinuous design method is adopted to deal with the uncertainties and disturbances. Moreover, the rigorous proof is presented to demonstrate that the corresponding closed-loop system can be stabilized to the origin equilibrium point in a finite time.

The structure of the article is as follows: Section 2 gives a formalization of the problem considered in this paper. A proper assumption and some lemmas are also presented in this section. Section 3 states our main results including switch controller design and stability analysis. Section 4 provides an illustrative numerical example and the corresponding simulation results of the proposed methodology. Finally, a conclusion is shown in Section 5.

2. Problem Statement

As shown in Figure 1, the two fixed rear wheels of the robot are controlled independently by motors, and a front castor wheel prevents the robot from tipping over as it moves on a plane. Assuming that the geometric center point and the mass center point of the robot are the same, and that the radii r are identical for all the wheels and the distance $2R$ between the fixed wheels is a known positive constant. Its kinematic model can be described by the following differential equations [39]:

$$\begin{cases} \dot{x} = v\cos\theta, \\ \dot{y} = v\sin\theta, \\ \dot{\theta} = \omega, \end{cases} \tag{1}$$

where (x, y) is the position of the mass center of the robot moving in the plane, v is the forward velocity, ω is the steering velocity and θ denotes its heading angle from the horizontal axis.

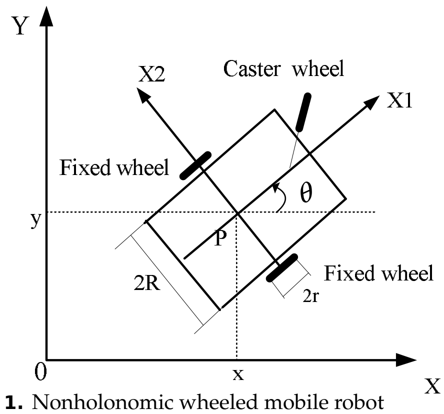

Figure 1. Nonholonomic wheeled mobile robot

We consider that the movement of the mobile robot above can be measured by using a pinhole camera fixed to the ceiling (as shown in Figure 2). Assuming that the camera plane, the image plane and the robot plane are parallel. There are four coordinate frames, namely the inertial frame $X - Y - Z$, the camera frame $x - y - z$, the image frame $u - o_1 - v$, and the attached robot frame $X_1 - P - X_2$. Point C is the crossing point between the optical axis of the camera and $X - Y$ plane. Its coordinate relative to $X - Y$ plane is (c_x, c_y). The coordinate of the original point of the camera frame with respect to the image frame is defined by (O_{c_1}, O_{c_2}), and (x, y) is the coordinate of the mass center P of the robot with respective to $X - Y$ plane, its image position is noted as (x_m, y_m).

The pinhole camera model can be expressed as [25]-[31], [37]:

$$\begin{pmatrix} x_m \\ y_m \end{pmatrix} = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} R \begin{pmatrix} x - c_x \\ y - c_y \end{pmatrix} + \begin{pmatrix} O_{c_1} \\ O_{c_2} \end{pmatrix} \tag{2}$$

where α_1 and α_2 are positive constants, which are dependent on the depth information, focal length, scalar factors along u axis, and v axis, respectively; and

$$R = \begin{pmatrix} \cos\theta_0 & \sin\theta_0 \\ -\sin\theta_0 & \cos\theta_0 \end{pmatrix}, \tag{3}$$

where θ_0 denotes the angle between X axis and y axis with a positive anticlockwise orientation.

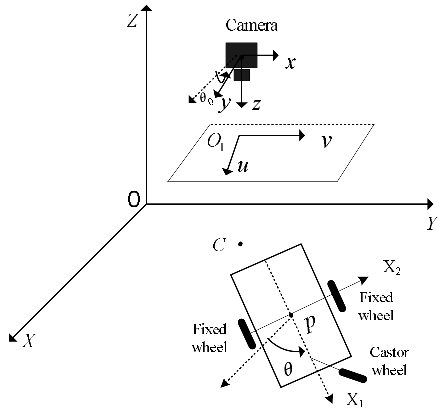

Figure 2. Nonholonomic wheeled mobile robot under a fixed camera

From (1), (2) and (3), by using a simple derivation, the image-based kinematical equation of the robot can be obtained:

$$\begin{cases} \dot{x}_m = \alpha_1 v \cos(\theta - \theta_0), \\ \dot{y}_m = \alpha_2 v \sin(\theta - \theta_0), \\ \dot{\theta} = \omega. \end{cases} \tag{4}$$

In the field of visual servoing control of robots, usually the camera parameters α_1, α_2 and the angle θ_0 can be gotten by calibration. But this process will take a lot of time, which implies that it is impossible to use this method in high requirement of real-time. Therefore, it is necessary to consider how to design a control law in the case of dealing with these uncalibrated parameters.

As in [25], we make the following assumption:

Assumption 1: $\theta_0 = 0$, α_1, α_2 are unknown and bounded, the bounds of which are known positive constants:

$$0 < \alpha_i^{\min} \le \alpha_i \le \alpha_i^{\max}, i = 1, 2.$$

Remark 1: Compared with it in [25], our assumptions are more relaxed since it is not necessary to suppose $\alpha_1 = \alpha_2$ in this paper.

Under this case, system (4) can be rewritten as

$$\begin{cases} \dot{x}_m = \alpha_1 v \cos\theta, \\ \dot{y}_m = \alpha_2 v \sin\theta, \\ \dot{\theta} = \omega, \end{cases}$$

taking a state and input transformation [41]:

$$x_0 = x_m, \;\; x_1 = y_m, \;\; x_2 = \tan\theta,$$

$$u_0 = v\cos\theta, \;\; u_1 = (\sec\theta)^2\omega,$$

we obtain

$$\begin{cases} \dot{x}_0 = \alpha_1 u_0, \\ \dot{x}_1 = \alpha_2 x_2 u_0, \\ \dot{x}_2 = u_1. \end{cases} \tag{5}$$

It is noted that system (5) is so-called canonical chained-form with three-order and two control inputs u_0, u_1. The finite-time stabilization problem of (5) can be completely addressed by applying the controller given in [35], moreover, the authors have dealt with the nonholonomic chained systems with uncertain parameters and a matched disturbance.

In this paper, we will consider the finite-time stabilization problem of the extended chained-form systems (6) with unknown parameters α_1, α_2, uncertain control direction $\gamma_i(t), (i = 1, 2)$ and unmatched un-modeled dynamics (or external disturbance) $\varphi_i(t), (i = 1, 2)$.

$$\begin{cases} \dot{x}_0 = \alpha_1 u_0, \\ \dot{x}_1 = \alpha_2 x_2 u_0, \\ \dot{x}_2 = u_1, \\ \dot{u}_0 = \gamma_1(t)\tau_1 + \varphi_1(t), \\ \dot{u}_1 = \gamma_2(t)\tau_2 + \varphi_2(t), \end{cases} \tag{6}$$

where τ_1 and τ_2 are the new control inputs, the bounded measurable functions $\gamma_i(t), \varphi_i(t), (i = 1, 2)$ are supposed to satisfy that

$$\gamma_i^m \le \gamma_i(t) \le \gamma_i^M, \;\; |\varphi_i(t)| \le \varphi_i^M, \;\; (i = 1, 2)$$

here $\gamma_i^m, \gamma_i^M, \varphi_i^m, \varphi_i^M, (i = 1, 2)$ are positive constants.

Remark 2: Usually, the new control inputs τ_1 and τ_2 in the extended nonholonomic system (6) can be seen as the form of force or torque inputs, which is more practical than the form of velocity or acceleration controller u_0 and u_1 of (5) in the engineering application, because the new controller can be easier to implement for electrical engineer.

The following lemmas and conclusions are needed for our controller design later.

Lemma 1 ([42]): Considering the following system

$$\dot{\bar{x}} = f(\bar{x}), \; f(0) = 0, \; \bar{x} \in R^n, \tag{7}$$

suppose there exists a continuous function $\bar{V}(\bar{x}) : U \to R$ such that the following conditions hold

(i) $\bar{V}(\bar{x})$ is positive definite.

(ii)There exist real numbers $\bar{c} > 0$ and $\bar{\alpha} \in (0, 1)$ and an open neighbourhood $U_0 \in U$ of the origin such that $\dot{\bar{V}}(\bar{x}) + \bar{c}\bar{V}^{\bar{\alpha}}(\bar{x}) \le 0, \; \bar{x} \in U_0 \backslash \{0\}$.

Then the origin is a finite-time stable equilibrium of system (7). If $U = U_0 = R^n$, then the origin is a globally finite-time stable equilibrium of system (7).

Lemma 2 ([43]): If $0 < p = p_1/p_2 \leq 1$, where $p_1 > 0, p_2 > 0$ are positive odd integers, then

$$|x^p - y^p| \leq 2^{1-p}|x-y|^p. \quad (8)$$

Lemma 3 ([44]): For $x \in R, y \in R$, let c, d be positive real numbers, then

$$|x|^c|y|^d \leq \frac{c}{c+d}|x|^{c+d} + \frac{d}{c+d}|y|^{c+d}. \quad (9)$$

Lemma 4 ([43]): For $x_i \in R, i = 1, 2, ..., n$, $0 < p \leq 1$ is a real number, then the following inequality holds:

$$(|x_1| + |x_2| + ... + |x_n|)^p \leq |x_1|^p + ... + |x_n|^p$$

$$\leq n^{1-p}(|x_1| + ... + |x_n|)^p. \quad (10)$$

Theorem 1: Consider an uncertain nonlinear system:

$$\begin{aligned} \dot{z}_i &= \beta_i z_{i+1},\ i = 1, ..., r-1, \\ \dot{z}_r &= \gamma(t)u + \varphi(t), \end{aligned} \quad (11)$$

where $z \in R^r$ is the state vector and $u \in R$ is the control input. Unknown parameters $\beta_i > 0, (i = 1, 2, ..., r-1)$, the functions $\varphi(\cdot)$ and $\gamma(\cdot)$ are arbitrary measurable functions that represent bounded uncertainty:

$$|\varphi(t)| \leq \bar{\varphi},\ \ \gamma_m \leq \gamma(t) \leq \gamma_M,$$

where $\bar{\varphi}, \gamma_m, \gamma_M$ are positive constants. For the following system (12),

$$\begin{aligned} \dot{z}_i &= \beta_i z_{i+1},\ i = 1, ..., r-1, \\ \dot{z}_r &= \bar{u}_0, \end{aligned} \quad (12)$$

if there exist a state-feedback control law $\bar{u}_0(z)$, a positive definite C^1 function $\tilde{V}$ and real numbers $\tilde{c} > 0$ and $\tilde{\alpha} \in (0, 1)$ satisfying the conditions (for every $z \in R^r$):

(i) $\dot{\tilde{V}}(z) + \tilde{c}\tilde{V}^{\tilde{\alpha}}(z) \leq 0.$

(ii)$\frac{\partial \tilde{V}}{\partial z_r}(z)\bar{u}_0(z) \leq 0, \text{ and } \bar{u}_0(z) = 0 \Rightarrow \frac{\partial \tilde{V}}{\partial z_r} = 0.$

Let

$$u = (\bar{u}_0 + \bar{\varphi} sign(\bar{u}_0))/\gamma_m,$$

then system (11) is finite-time stable with respect to the origin.

Proof. :

See Appendix A. □

Next, the control task is to present a switching controller for system (6) such that all the states converge to the origin equilibrium point in a finite time.

3. Main Results

In this section, the main results will be presented. Firstly, we will state the basic idea to design a switching controller for system (6).

Motivated by the results of *Theorem 1*, we give a finite-time stable controller for the following system:

$$\begin{cases} \dot{x}_0 = \alpha_1 u_0, \\ \dot{x}_1 = \alpha_2 x_2 u_0, \\ \dot{x}_2 = u_1, \\ \dot{u}_0 = \tilde{\tau}_1, \\ \dot{u}_1 = \tilde{\tau}_2, \end{cases}$$

where $\tilde{\tau}_1$ and $\tilde{\tau}_2$ are the control inputs. In the first place, according to the special structure of the system above, two subsystems are considered, respectively. One is

$$\begin{cases} \dot{x}_1 = \alpha_2 x_2 u_0, \\ \dot{x}_2 = u_1, \\ \dot{u}_1 = \tilde{\tau}_2, \end{cases} \quad (13)$$

and the other is

$$\begin{cases} \dot{x}_0 = \alpha_1 u_0, \\ \dot{u}_0 = \tilde{\tau}_1. \end{cases} \quad (14)$$

Based on which, by using the design method of *Theorem 1*, we will propose a switching controller such that all the states of system (6) can be stabilized to zero in a finite time.

3.1 Switching controller design

According to the design idea above, a discontinuous switching controller is presented as follows.

Step 1: Let

$$\tau_1 = \left(\tilde{\tau}_1 + \varphi_1^M sign(\tilde{\tau}_1)\right)/\gamma_1^m,\ \ \tau_2 = 0,$$

where

$$\tilde{\tau}_1 = -\lambda_1 sign(u_0 - 1)|u_0 - 1|^{q_0},$$

where $\lambda_1 > 0$, $q_0 \in (0, 1)$ are design parameters. Then there exists a finite time $T_1 < +\infty$ such that $u_0(t) \equiv 1$ as $t \geq T_1$, and go to *Step 2*.

Step 2: Let

$$\tau_1 = \left(\tilde{\tau}_1 + \varphi_1^M sign(\tilde{\tau}_1)\right)/\gamma_1^m,$$

$$\tau_2 = \left(\tilde{\tau}_2 + \varphi_2^M sign(\tilde{\tau}_2)\right)/\gamma_2^m,$$

where

$$\tilde{\tau}_2 = -l_3\Big(u_1^{\frac{1}{2q_1-1}} + l_2^{\frac{1}{2q_1-1}}\big(x_2^{\frac{1}{q_1}} + l_1^{\frac{1}{q_1}}x_1\big)\Big)^{3q_1-2},$$

$q_1 = \frac{p}{q} \in (\frac{2}{3}, 1)$, p, q are positive odd integers, $l_i > 0, (i = 1,2,3)$ $k_j > 0, (j = 1,2)$ are design parameters satisfying the following conditions:

$$\begin{aligned}
1)\ & -l_1 + \frac{2^{1-q_1}}{1+q_1} + \frac{\alpha_2(2-q_1)2^{1-q_1}}{k_1(1+q_1)}l_1^{1+\frac{1}{q_1}}q_1 \\
& + \frac{(3-2q_1)(1-q_1)}{k_2 q_1(1+q_1)}l_1^{\frac{1}{q_1}-1}\Big(2^{2-2q_1}l_2^{\frac{2q_1}{2q_1-1}} + 2^{4-4q_1}l_2^{\frac{1}{2q_1-1}}\Big) \\
& + \frac{\alpha_2(3-2q_1)q_1}{k_2(1+q_1)}2^{2-2q_1}l_1^{1+\frac{1}{q_1}}l_2^{\frac{1}{2q_1-1}} < 0,
\end{aligned} \tag{15}$$

$$\begin{aligned}
2)\ & \frac{2^{1-q_1}q_1}{1+q_1} - \frac{l_2}{k_1} + \frac{\alpha_2(2-q_1)l_1^{\frac{1}{q_1}}}{k_1}\Big(\frac{2^{1-q_1}l_1}{1+q_1} + 2^{2-2q_1}\Big) \\
& + \frac{(3-2q_1)q_1}{k_2(1+q_1)}2^{3-3q_1}l_1^{\frac{1}{q_1}}l_2^{\frac{1}{2q_1-1}} + \frac{(2-q_1)2^{2-2q_1}}{k_1(1+q_1)} \\
& + \frac{(3-2q_1)l_2^{\frac{1}{2q_1-1}}}{k_2 q_1(1+q_1)}\Big(2^{2-2q_1}l_1^{\frac{1}{q_1}-1}l_2(2q_1-1) \\
& + 2^{2-q_1}q_1 l_2 + 2^{4-3q_1}(1-q_1)\Big) < 0,
\end{aligned} \tag{16}$$

$$\begin{aligned}
3)\ & \frac{(2q_1-1)2^{2-2q_1}}{k_1(1+q_1)} + \frac{(3-2q_1)l_2^{\frac{1}{2q_1-1}}}{k_2 q_1(1+q_1)}\Big(2^{2-2q_1}l_1^{\frac{1}{q_1}-1}l_2 \\
& + 2^{2-q_1}l_2 + 2^{5-3q_1}q_1 + 2^{5-4q_1}q_1 l_1^{\frac{1}{q_1}-1}\Big) + \frac{\alpha_2(3-2q_1)}{k_2(1+q_1)} \\
& \cdot l_1^{\frac{1}{q_1}}l_2^{\frac{1}{2q_1-1}}\big(2^{2-2q_1}l_1 + 2^{3-3q_1}\big) - \frac{l_3}{k_2} < 0.
\end{aligned} \tag{17}$$

Then there exists a finite time $T_2 < +\infty$ such that $x_1(t) = x_2(t) = u_1(t) \equiv 0$ as $t \geq T_2$, and go to *Step 3*.

Step 3: Let

$$\tau_1 = \big(\tilde{\tau}_1 + \varphi_1^M sign(\tilde{\tau}_1)\big)/\gamma_1^m,$$

$$\tau_2 = \big(\tilde{\tau}_2 + \varphi_2^M sign(\tilde{\tau}_2)\big)/\gamma_2^m,$$

where

$$\tilde{\tau}_1 = -\tilde{k}_3\big(u_0^{q_2} + \tilde{k}_1^{q_2}x_0\big)^{\frac{2}{q_2}-1}$$

$q_2 = \frac{p_1}{q_1} \in (0,1)$, p_1, q_1 are positive integers, $\tilde{k}_i > 0, (i = 1,2,3)$ are design parameters satisfying that

$$-\frac{3}{4}\tilde{k}_1 + \frac{\tilde{k}_1^{2+q_2}\alpha_1}{4\tilde{k}_2}2^{1-\frac{1}{q_2}} < 0, \tag{18}$$

$$\begin{aligned}
& \frac{2^{1-\frac{1}{q_2}}}{1+q_2}\Big(\frac{2^{3-\frac{1}{q_2}}q_2}{(1+q_2)\tilde{k}_1}\Big)^{q_2} + \frac{\tilde{k}_1^{q_2}\alpha_1}{\tilde{k}_2}2^{1-\frac{1}{q_2}} \\
& + \frac{\tilde{k}_1^{1+q_2}q_2\alpha_1 2^{1-\frac{1}{q_2}}}{(1+q_2)\tilde{k}_2}\Big(\frac{4}{(1+q_2)\tilde{k}_1}\Big)^{\frac{1}{q_2}} - \frac{\tilde{k}_3}{\tilde{k}_2} < 0.
\end{aligned} \tag{19}$$

Then there exists a finite time $T_3 < +\infty$ such that $x_0(t) = u_0(t) \equiv 0$ as $t \geq T_3$, and let $\tau_1 = \tau_2 = 0$, stop.

3.2 Stability analysis

Theorem 2: Under *Assumption 1*, the switching controller composed by *Step 1* $\sim$ *Step 3* in Section A ensures that system (6) can be stabilized to zero in a finite time.

Proof. Firstly, in *Step 1*, for a first-order system $\dot{u}_0 = \tilde{\tau}_1$, let $u_0 - 1 = \tilde{u}_0$, we have

$$\dot{\tilde{u}}_0 = \tilde{\tau}_1 = -\lambda_1 sign(\tilde{u}_0)|\tilde{u}_0|^{q_0}.$$

Choosing a Lyapunov function $\tilde{V}_0 = \frac{1}{2}\tilde{u}_0^2$, then its time derivative along the $\{\tilde{u}_0\}-$system is

$$\dot{\tilde{V}}_0 = \tilde{u}_0\dot{\tilde{u}}_0 = \tilde{u}_0\tilde{\tau}_1 = -\lambda_1|\tilde{u}_0|^{1+q_0} \leq 0,$$

which can be rewritten by

$$\dot{\tilde{V}}_0 + \lambda_1\sqrt{2}(\tilde{V}_0)^{\frac{1}{2}} \leq 0. \tag{20}$$

Hence, according to *Lemma 1*, it's clear that under the controller $\tilde{\tau}_1$, there exists a finite time $T_1 < +\infty$ such that $\tilde{u}_0(t) \equiv 0$ as $t \geq T_1$. As for the following uncertain system

$$\dot{\tilde{u}}_0 = \gamma_1(t)\tau_1 + \varphi_1(t), \tag{21}$$

because

$$\begin{aligned}
\frac{\partial\tilde{V}_0}{\partial\tilde{u}_0}\tilde{\tau}_1(\tilde{u}_0) &= \tilde{u}_0\big(-\lambda_1 sign(\tilde{u}_0)|\tilde{u}_0|^{q_0}\big) \\
&= -\lambda_1|\tilde{u}_0|^{1+q_0} \leq 0,
\end{aligned} \tag{22}$$

and

$$\tilde{\tau}_1(\tilde{u}_0) = 0 \Rightarrow \tilde{u}_0 = 0 \Rightarrow \frac{\partial\tilde{V}_0}{\partial\tilde{u}_0} = 0. \tag{23}$$

By (20), (22)-(23) and from *Theorem 1*, system (21) can be stabilized to zero in the finite time T_1 with the controller τ_1, i.e., $u_0(t) \equiv 1$ as $t \geq T_1$.

Next, in *Step 2*, substituting $u_0 \equiv 1$ into the subsystem (13), it has

$$\begin{cases} \dot{x}_1 = \alpha_2 x_2, \\ \dot{x}_2 = u_1, \\ \dot{u}_1 = \tilde{\tau}_2. \end{cases} \tag{24}$$

Take a positive definite, radially unbounded function about x_1

$$V_1(x_1) = \frac{x_1^2}{2\alpha_2}.$$

Its time derivative along (24) is

$$\dot{V}_1(x_1) = x_1 x_2 = x_1(x_2 - x_2^*) + x_1 x_2^*$$

$$\leq |x_1| \cdot |x_2 - x_2^*| + x_1 x_2^*,$$

where $x_2^* = -l_1 x_1^{q_1}$ is a virtual controller. Then we have

$$\dot{V}_1(x_1) \leq -l_1 x_1^{1+q_1} + |x_1||x_2 - x_2^*|. \tag{25}$$

As in [44], we take a C^1, positive definite and proper Lyapunov function about (x_1, x_2) as follows

$$V_2(x_1, x_2) = V_1(x_1) + \frac{1}{k_1} \int_{x_2^*}^{x_2} (s^{\frac{1}{q_1}} - x_2^{*\frac{1}{q_1}})^{2-q_1} ds.$$

We can obtain

$$\dot{V}_2(x_1, x_2) = \dot{V}_1(x_1) + \frac{1}{k_1} |\xi_1|^{2-q_1} u_1$$

$$+ \frac{1}{k_1}(2 - q_1) \frac{\partial x_2^{*\frac{1}{q_1}}}{\partial x_1} \dot{x}_1 \int_{x_2^*}^{x_2} (s^{\frac{1}{q_1}} - x_2^{*\frac{1}{q_1}})^{1-q_1} ds,$$

where $\xi_1 = x_2^{\frac{1}{q_1}} - x_2^{*\frac{1}{q_1}}$. So

$$\dot{V}_2(x_1, x_2) = \dot{V}_1(x_1) + \frac{1}{k_1} |\xi_1|^{2-q_1} u_1$$

$$- \frac{\alpha_2}{k_1}(2 - q_1) l_1^{\frac{1}{q_1}} x_2 \int_{x_2^*}^{x_2} (s^{\frac{1}{q_1}} - x_2^{*\frac{1}{q_1}})^{1-q_1} ds,$$

then

$$\dot{V}_2(x_1, x_2) \leq \dot{V}_1(x_1) + \frac{1}{k_1} |\xi_1|^{2-q_1} x_3^* + \frac{1}{k_1} |\xi_1|^{2-q_1}$$

$$\cdot |u_1 - x_3^*| + \frac{\alpha_2(2 - q_1) l_1^{\frac{1}{q_1}}}{k_1} |x_2||x_2 - x_2^*||\xi_1|^{1-q_1}, \tag{26}$$

where $x_3^* = -l_2 \xi_1^{2q_1 - 1}$ is seen as a virtual controller.

By using (8) in *Lemma 2*, we have

$$|x_2 - x_2^*| \leq 2^{1-q_1} |\xi_1|^{q_1}. \tag{27}$$

Note that $x_2 = x_2^* + (x_2 - x_2^*)$, from (27), it has

$$|x_2| \leq |x_2^*| + |x_2 - x_2^*| \leq l_1 |x_1|^{q_1} + 2^{1-q_1} |\xi_1|^{q_1}. \tag{28}$$

From (27)-(28) and (9) in *Lemma 3*, we have

$$|x_2||x_2 - x_2^*||\xi_1|^{1-q_1}$$

$$\leq \left(l_1 |x_1|^{q_1} + 2^{1-q_1} |\xi_1|^{q_1}\right) 2^{1-q_1} |\xi_1|^{q_1} |\xi_1|^{1-q_1}$$

$$\leq 2^{1-q_1} l_1 \frac{q_1 x_1^{1+q_1} + \xi_1^{1+q_1}}{1 + q_1} + 2^{2-2q_1} \xi_1^{1+q_1}.$$

Substituting (25), (27) and the formula above into (26), we have

$$\dot{V}_2(x_1, x_2) \leq -l_1 x_1^{1+q_1} + 2^{1-q_1} |x_1||\xi_1|^{q_1} + \frac{1}{k_1} |\xi_1|^{2-q_1} x_3^*$$

$$+ \frac{1}{k_1} |\xi_1|^{2-q_1} |u_1 - x_3^*| + \frac{\alpha_2(2 - q_1) l_1^{\frac{1}{q_1}}}{k_1}$$

$$\cdot \left(2^{1-q_1} l_1 \frac{q_1 x_1^{1+q_1} + \xi_1^{1+q_1}}{1 + q_1} + 2^{2-2q_1} \xi_1^{1+q_1}\right).$$

Because $x_3^* = -l_2 \xi_1^{2q_1 - 1}$, therefore

$$\dot{V}_2(x_1, x_2) \leq -l_1 x_1^{1+q_1} + 2^{1-q_1} |x_1||\xi_1|^{q_1} - \frac{l_2}{k_1} \xi_1^{1+q_1}$$

$$+ \frac{1}{k_1} |\xi_1|^{2-q_1} |u_1 - x_3^*| + \frac{\alpha_2(2 - q_1) l_1^{\frac{1}{q_1}}}{k_1}$$

$$\cdot \left(2^{1-q_1} l_1 \frac{q_1 x_1^{1+q_1} + \xi_1^{1+q_1}}{1 + q_1} + 2^{2-2q_1} \xi_1^{1+q_1}\right).$$

Applying *Lemma 3* again, we have

$$\dot{V}_2(x_1, x_2) \leq \Big(-l_1 + \frac{2^{1-q_1}}{1 + q_1} + \frac{\alpha_2(2 - q_1) 2^{1-q_1}}{k_1(1 + q_1)} l_1^{1+\frac{1}{q_1}} q_1 \Big)$$

$$\cdot x_1^{1+q_1} + \frac{1}{k_1} |\xi_1|^{2-q_1} |u_1 - x_3^*| + \Big(\frac{2^{1-q_1} q_1}{1 + q_1} - \frac{l_2}{k_1}$$

$$+\frac{\alpha_2(2-q_1)l_1^{\frac{1}{q_1}}}{k_1}\Big(\frac{2^{1-q_1}l_1}{1+q_1}+2^{2-2q_1}\Big)\Big)\xi_1^{1+q_1}. \quad (29)$$

Taking a Lyapunov function about (x_1, x_2, u_1) for system (24) as follows

$$V_3(x_1,x_2,u_1)=V_2(x_1,x_2)$$

$$+\frac{1}{k_2}\int_{x_3^*}^{u_1}\left(s^{\frac{1}{2q_1-1}}-x_3^{*\frac{1}{2q_1-1}}\right)^{3-2q_1}ds.$$

Let

$$\xi_2=u_1^{\frac{1}{2q_1-1}}-x_3^{*\frac{1}{2q_1-1}},$$

and the time derivative along (24) is

$$\dot V_3(x_1,x_2,u_1)=\dot V_2(x_1,x_2)$$

$$-\frac{3-2q_1}{k_2}\Big(\frac{\partial x_3^{*\frac{1}{2q_1-1}}}{\partial x_1}\alpha_2x_2+\frac{\partial x_3^{*\frac{1}{2q_1-1}}}{\partial x_2}u_1\Big)$$

$$\cdot\int_{x_3^*}^{u_1}\left(s^{\frac{1}{2q_1-1}}-x_3^{*\frac{1}{2q_1-1}}\right)^{2-2q_1}ds+\frac{\xi_2^{3-2q_1}}{k_2}\tilde\tau_2.$$

Note that

$$\frac{\partial x_3^{*\frac{1}{2q_1-1}}}{\partial x_1}=l_1^{\frac{1}{q_1}}(-l_2)^{\frac{1}{2q_1-1}},$$

$$\frac{\partial x_3^{*\frac{1}{2q_1-1}}}{\partial x_2}=\frac{1}{q_1}(-l_2)^{\frac{1}{2q_1-1}}x_2^{\frac{1}{q_1}-1},$$

thus

$$\dot V_3(x_1,x_2,u_1)\le\dot V_2(x_1,x_2)+\frac{3-2q_1}{k_2}l_2^{\frac{1}{2q_1-1}}$$

$$\Big(l_1^{\frac{1}{q_1}}\alpha_2|x_2|+\frac{1}{q_1}|x_2|^{\frac{1}{q_1}-1}|u_1|\Big)|\xi_2|^{2-2q_1}|u_1-x_3^*|$$

$$+\frac{\xi_2^{3-2q_1}}{k_2}\tilde\tau_2. \quad (30)$$

From (8) in *Lemma 2*, it has

$$|\xi_2|^{2-2q_1}|u_1-x_3^*|\le2^{2-2q_1}|\xi_2|. \quad (31)$$

From (28), (31) and (10) in *Lemma 4*, it has

$$|x_2||\xi_2|^{2-2q_1}|u_1-x_3^*|$$

$$\le(l_1|x_1|^{q_1}+2^{1-q_1}|\xi_1|^{q_1})2^{2-2q_1}|\xi_2|$$

$$\le2^{2-2q_1}l_1\frac{q_1x_1^{1+q_1}+\xi_2^{1+q_1}}{1+q_1}+2^{3-3q_1}\frac{q_1\xi_1^{1+q_1}+\xi_2^{1+q_1}}{1+q_1} \quad (32)$$

Therefore, from (32), we have

$$|u_1||\xi_2|^{2-2q_1}|u_1-x_3^*|\le|u_1-x_3^*+x_3^*|2^{2-2q_1}|\xi_2|$$

$$\le2^{2-2q_1}l_2\frac{(2q_1-1)\xi_1^{2q_1}+\xi_2^{2q_1}}{2q_1}+2^{4-4q_1}\xi_2^{2q_1} \quad (33)$$

By using (33) and Lemmas 2-4, we have

$$|x_2|^{\frac{1}{q_1}-1}|u_1||\xi_2|^{2-2q_1}|u_1-x_3^*|$$

$$\le\Big(|x_2^*|^{\frac{1}{q_1}-1}+|x_2^{\frac{1}{q_1}-1}-x_2^{*\frac{1}{q_1}-1}|\Big)$$

$$\cdot\Big(2^{2-2q_1}l_2\frac{(2q_1-1)\xi_1^{2q_1}+\xi_2^{2q_1}}{2q_1}+2^{4-4q_1}\xi_2^{2q_1}\Big)$$

$$\le\frac{2^{2-2q_1}l_1^{\frac{1}{q_1}-1}l_2}{1+q_1}\Big((1-q_1)x_1^{1+q_1}+(2q_1-1)\xi_1^{1+q_1}$$

$$+\xi_2^{1+q_1}\Big)+2^{2-q_1}l_2\frac{q_1\xi_1^{1+q_1}+\xi_2^{1+q_1}}{1+q_1}$$

$$+\frac{2^{4-4q_1}l_1^{\frac{1}{q_1}-1}}{1+q_1}\Big((1-q_1)x_1^{1+q_1}+2q_1\xi_2^{1+q_1}\Big)$$

$$+2^{4-3q_1}\frac{(1-q_1)\xi_1^{1+q_1}+2q_1\xi_2^{1+q_1}}{1+q_1} \quad (34)$$

Substituting (29), (32)-(34) into (30), we have

$$\dot V_3(x_1,x_2,u_1)$$

$$\le\Big(-l_1+\frac{2^{1-q_1}}{1+q_1}+\frac{\alpha_2(2-q_1)2^{1-q_1}}{k_1(1+q_1)}l_1^{1+\frac{1}{q_1}}q_1\Big)$$

$$\cdot x_1^{1+q_1}+\frac{1}{k_1}|\xi_1|^{2-q_1}2^{2-2q_1}|\xi_2|^{2q_1-1}$$

$$+\Big(\frac{2^{1-q_1}q_1}{1+q_1}-\frac{l_2}{k_1}+\frac{\alpha_2(2-q_1)l_1^{\frac{1}{q_1}}}{k_1}\Big(\frac{2^{1-q_1}l_1}{1+q_1}$$

$$+2^{2-2q_1}\Big)\Big)\xi_1^{1+q_1}+\frac{3-2q_1}{k_2}l_2^{\frac{1}{2q_1-1}}l_1^{\frac{1}{q_1}}\alpha_2$$

$$\cdot\Big(2^{2-2q_1}l_1\frac{q_1x_1^{1+q_1}+\xi_2^{1+q_1}}{1+q_1}+2^{3-2q_1}\frac{q_1\xi_1^{1+q_1}+\xi_2^{1+q_1}}{1+q_1}\Big)$$

$$+\frac{3-2q_1}{k_2q_1}l_2^{\frac{1}{2q_1-1}}\Big(2^{2-2q_1}l_1^{\frac{1}{q_1}-1}l_2$$

$$\frac{(1-q_1)x_1^{1+q_1}+(2q_1-1)\xi_1^{1+q_1}+\xi_2^{1+q_1}}{1+q_1}$$

$$+2^{2-q_1}l_2\frac{q_1\xi_1^{1+q_1}+\xi_2^{1+q_1}}{1+q_1}$$

$$+2^{4-4q_1}l_1^{\frac{1}{q_1}-1}\frac{(1-q_1)x_1^{1+q_1}+2q_1\xi_2^{1+q_1}}{1+q_1}$$

$$+2^{4-3q_1}\frac{(1-q_1)\xi_1^{1+q_1}+2q_1\xi_2^{1+q_1}}{1+q_1}\Big)+\frac{\xi_2^{3-2q_1}}{k_2}\tilde{\tau}_2, \tag{35}$$

Because

$$\frac{1}{k_1}|\xi_1|^{2-q_1}2^{2-2q_1}|\xi_2|^{2q_1-1}$$

$$\le\frac{1}{k_1}2^{2-2q_1}\frac{(2-q_1)\xi_1^{1+q_1}+(2q_1-1)\xi_2^{1+q_1}}{1+q_1}, \tag{36}$$

and

$$\tilde{\tau}_2=-l_3\Big(u_1^{\frac{1}{2q_1-1}}+l_2^{\frac{1}{2q_1-1}}(x_2^{\frac{1}{q_1}}+l_1^{\frac{1}{q_1}}x_1)\Big)^{3q_1-2}$$

$$=-l_3\xi_2^{3q_1-2}, \tag{37}$$

substituting (36) and (37) into (35) has

$$\dot{V}_3(x_1,x_2,u_1)$$

$$\le\Big(-l_1+\frac{2^{1-q_1}}{1+q_1}+\frac{\alpha_2(2-q_1)2^{1-q_1}}{k_1(1+q_1)}l_1^{1+\frac{1}{q_1}}q_1$$

$$+\frac{\alpha_2(3-2q_1)q_1}{k_2(1+q_1)}2^{2-2q_1}l_1^{1+\frac{1}{q_1}}l_2^{\frac{1}{2q_1-1}}+\frac{(3-2q_1)(1-q_1)}{k_2q_1(1+q_1)}$$

$$\cdot l_1^{\frac{1}{q_1}-1}\Big(2^{2-2q_1}l_2^{\frac{2q_1}{2q_1-1}}+2^{4-4q_1}l_2^{\frac{1}{2q_1-1}}\Big)\Big)x_1^{1+q_1}$$

$$+\Big(\frac{2^{1-q_1}q_1}{1+q_1}-\frac{l_2}{k_1}+\frac{\alpha_2(2-q_1)l_1^{\frac{1}{q_1}}}{k_1}(\frac{2^{1-q_1}l_1}{1+q_1}+2^{2-2q_1})$$

$$+\frac{(2-q_1)2^{2-2q_1}}{k_1(1+q_1)}+\frac{(3-2q_1)q_1}{k_2(1+q_1)}2^{3-3q_1}l_1^{\frac{1}{q_1}}l_2^{\frac{1}{2q_1-1}}$$

$$+\frac{(3-2q_1)l_2^{\frac{1}{2q_1-1}}}{k_2q_1(1+q_1)}\Big(2^{2-2q_1}l_1^{\frac{1}{q_1}-1}l_2(2q_1-1)+2^{2-q_1}q_1l_2$$

$$+2^{4-3q_1}(1-q_1)\Big)\Big)\xi_1^{1+q_1}+\Big(\frac{(2q_1-1)2^{2-2q_1}}{k_1(1+q_1)}$$

$$+\frac{\alpha_2(3-2q_1)}{k_2(1+q_1)}l_1^{\frac{1}{q_1}}l_2^{\frac{1}{2q_1-1}}(2^{2-2q_1}l_1+2^{3-3q_1})-\frac{l_3}{k_2}$$

$$+\frac{(3-2q_1)l_2^{\frac{1}{2q_1-1}}}{k_2q_1(1+q_1)}\Big(2^{2-2q_1}l_1^{\frac{1}{q_1}-1}l_2+2^{2-q_1}l_2$$

$$+2^{5-4q_1}q_1l_1^{\frac{1}{q_1}-1}+2^{5-3q_1}q_1\Big)\Big)\xi_2^{1+q_1}.$$

From (15)-(17), we can rewrite the formula above as follows

$$\dot{V}_3(x_1,x_2,u_1)\le\beta_1x_1^{1+q_1}+\beta_2\xi_1^{1+q_1}+\beta_3\xi_2^{1+q_1}, \tag{38}$$

where

$$\beta_1=-l_1+\frac{2^{1-q_1}}{1+q_1}+\frac{\alpha_2(2-q_1)2^{1-q_1}}{k_1(1+q_1)}l_1^{1+\frac{1}{q_1}}q_1$$

$$+\frac{\alpha_2(3-2q_1)q_1}{k_2(1+q_1)}2^{2-2q_1}l_1^{1+\frac{1}{q_1}}l_2^{\frac{1}{2q_1-1}}+\frac{(3-2q_1)(1-q_1)}{k_2q_1(1+q_1)}$$

$$\cdot l_1^{\frac{1}{q_1}-1}\Big(2^{2-2q_1}l_2^{\frac{2q_1}{2q_1-1}}+2^{4-4q_1}l_2^{\frac{1}{2q_1-1}}\Big)<0, \tag{39}$$

$$\beta_2=\frac{2^{1-q_1}q_1}{1+q_1}-\frac{l_2}{k_1}+\frac{\alpha_2(2-q_1)l_1^{\frac{1}{q_1}}}{k_1}(\frac{2^{1-q_1}l_1}{1+q_1}+2^{2-2q_1})$$

$$+\frac{(2-q_1)2^{2-2q_1}}{k_1(1+q_1)}+\frac{(3-2q_1)q_1}{k_2(1+q_1)}2^{3-3q_1}l_1^{\frac{1}{q_1}}l_2^{\frac{1}{2q_1-1}}$$

$$+\frac{(3-2q_1)l_2^{\frac{1}{2q_1-1}}}{k_2q_1(1+q_1)}\Big(2^{2-2q_1}l_1^{\frac{1}{q_1}-1}l_2(2q_1-1)$$

$$+2^{2-q_1}q_1l_2+2^{4-3q_1}(1-q_1)\Big)<0, \tag{40}$$

$$\beta_3=\frac{(2q_1-1)2^{2-2q_1}}{k_1(1+q_1)}+\frac{\alpha_2(3-2q_1)}{k_2(1+q_1)}l_1^{\frac{1}{q_1}}l_2^{\frac{1}{2q_1-1}}(2^{2-2q_1}l_1$$

$$+2^{3-3q_1}\Big) - \frac{l_3}{k_2} + \frac{(3-2q_1)l_2^{\frac{1}{2q_1-1}}}{k_2 q_1(1+q_1)}\Big(2^{2-2q_1} l_1^{\frac{1}{q_1}-1} l_2$$

$$+2^{2-q_1} l_2 + 2^{5-4q_1} q_1 l_1^{\frac{1}{q_1}-1} + 2^{5-3q_1} q_1\Big) < 0. \tag{41}$$

According to the definition of $V_3(x_1, x_2, u_1)$, by using (27) and (31), we have

$$V_3(x_1, x_2, u_1) \le \frac{x_1^2}{2\alpha_2^{\min}} + \frac{2^{1-q_1}}{k_1}\xi_1^2 + \frac{2^{2-2q_1}}{k_2}\xi_2^2.$$

Let

$$\rho_1 = \max\Big\{\frac{1}{2\alpha_2^{\min}}, \frac{2^{1-q_1}}{k_1}, \frac{2^{2-2q_1}}{k_2}\Big\},$$

it has

$$V_3(x_1, x_2, u_1) \le \rho_1(x_1^2 + \xi_1^2 + \xi_2^2). \tag{42}$$

From (38)-(41), take

$$\rho_2 = \max\Big\{\beta_1, \beta_2, \beta_3\Big\} < 0, \ \ 0 < c < -\frac{\rho_2}{\rho_1^{\alpha}},$$

$$\alpha = \frac{1+q_1}{2} \in (0, 1),$$

then we can obtain

$$\dot V_3(x_1, x_2, u_1) + cV_3^{\alpha}(x_1, x_2, u_1)$$

$$\le (\rho_2 + c\rho_1^{\alpha})\Big(x_1^{1+q_1} + \xi_1^{1+q1} + \xi_2^{1+q_1}\Big) \le 0. \tag{43}$$

Formulas (42)-(43) ensure that controller (37) can stabilize system (24) to zero in a finite time. Furthermore,

$$\frac{\partial V_3}{\partial u_1}\tilde\tau_2 = \frac{1}{k_2}\xi_2^{3-2q_1}(-l_3)\xi_2^{3q_1-2} = -\frac{l_3}{k_2}\xi_2^{1+q_1} \le 0,$$

$$\text{and } \tilde\tau_2 = 0 \Rightarrow \xi_2 = 0 \Rightarrow \frac{\partial V_3}{\partial u_1} = \frac{1}{k_2}\xi_2^{3-2q_1} = 0.$$

Therefore, according to *Theorem 1*, subsystem of (6)

$$\begin{cases} \dot x_1 = \alpha_2 x_2, \\ \dot x_2 = u_1, \\ \dot u_1 = \gamma_2(t)\tau_2 + \varphi_2(t), \end{cases}$$

can be stabilized to zero in a finite time T_2 by the controller τ_2 in *Step 2*.

Finally, by using the similar proof method, it's simple to prove that the subsystem of (6)

$$\begin{cases} \dot x_0 = \alpha_1 u_0, \\ \dot u_0 = \gamma_1(t)\tau_1 + \varphi_1(t), \end{cases}$$

can be stabilized to zero in a finite time T_3 by the controller τ_1 in *Step 3*. For brevity, here omit the detailed process in this step.

And this completes the proof of *Theorem 2*. □

Remark 3: Note that, if we choose sufficiently large k_1 and k_2 in (15)-(17), then it's possible to find a group of feasible solutions for the design parameters although it's difficult to obtain all the solutions of these parameters as pointed out in [46]. Here, we propose a simple search algorithm for finding a group of feasible solutions of the parameters step by step as follows:

First, choose $q_1 = \frac{p}{q} \in (\frac{2}{3}, 1), \bar k_{ij} > 0, (i = 1, 2; j = 1, 2, 3)$, then select k_1, k_2, l_1, l_2, l_3 in turn.

Step a: Given a sufficiently large $\bar l > 0$, select $k_1 > 0$ satisfying (44)-(46):

$$\frac{\alpha_2^{\max}(2-q_1)2^{1-q_1}}{k_1(1+q_1)}\bar l^{1+\frac{1}{q_1}} q_1 < \bar k_{11}, \tag{44}$$

$$\frac{\alpha_2^{\max}(2-q_1)}{k_1}\bar l^{\frac{1}{q_1}}\big(\frac{2^{1-q_1}\bar l}{1+q_1} + 2^{2-2q_1}\big)$$

$$+\frac{(2-q_1)2^{2-2q_1}}{k_1(1+q_1)} < \bar k_{12}, \tag{45}$$

$$\frac{(2q_1-1)2^{2-2q_1}}{k_1(1+q_1)} < \bar k_{13}. \tag{46}$$

Select $k_2 > 0$ satisfying (47)-(49):

$$\frac{(3-2q_1)(1-q_1)\bar l^{\frac{1}{q_1}-1}}{k_2 q_1(1+q_1)}\big(2^{2-2q_1}\bar l^{\frac{2q_1}{2q_1-1}} + 2^{4-4q_1}\bar l^{\frac{1}{2q_1-1}}\big)$$

$$+\frac{\alpha_2^{\max}(3-2q_1)q_1}{k_2(1+q_1)}2^{2-2q_1}\bar l^{1+\frac{1}{q_1}}\bar l^{\frac{1}{2q_1-1}} < \bar k_{21}, \tag{47}$$

$$\frac{(3-2q_1)q_1}{k_2(1+q_1)}2^{3-3q_1}\bar l^{\frac{1}{q_1}}\bar l^{\frac{1}{2q_1-1}} + \frac{(3-2q_1)\bar l^{\frac{1}{2q_1-1}}}{k_2 q_1(1+q_1)}$$

$$\cdot\Big(2^{2-2q_1}\bar l^{\frac{1}{q_1}-1}\bar l(2q_1-1) + 2^{2-q_1}q_1\bar l$$

$$+2^{4-3q_1}(1-q_1)\Big) < \bar k_{22}, \tag{48}$$

$$\frac{(3-2q_1)\bar l^{\frac{1}{2q_1-1}}}{k_2 q_1(1+q_1)}\Big(2^{2-2q_1}\bar l^{\frac{1}{q_1}-1}\bar l + 2^{2-q_1}\bar l + 2^{5-3q_1}q_1$$

$$+2^{5-4q_1}q_1\bar l^{\frac{1}{q_1}-1}\Big) + \frac{\alpha_2^{\max}(3-2q_1)\bar l^{\frac{1}{q_1}}\bar l^{\frac{1}{2q_1-1}}}{k_2(1+q_1)}$$

$$(2^{2-2q_1}\bar{l} + 2^{3-3q_1}) < \bar{k}_{23}. \quad (49)$$

If $\max\{\bar{l}_1, \bar{l}_2\} \leq \bar{l}$, then go to *Step b*. Else if $\max\{\bar{l}_1, \bar{l}_2\} > \bar{l}$, let $\bar{l} := \max\{\bar{l}_1, \bar{l}_2\}$, go to *Step a*. Where

$$\bar{l}_1 = \frac{2^{1-q_1}}{1+q_1} + \bar{k}_{11} + \bar{k}_{21}, \ \ \bar{l}_2 = k_1\Big(\frac{2^{1-q_1}q_1}{1+q_1} + \bar{k}_{12} + \bar{k}_{22}\Big).$$

Step b: Select l_1, l_2, l_3 such that

$$\bar{l} > l_1 > \bar{l}_1, \ \ \bar{l} > l_2 > \bar{l}_2, \ \ l_3 > k_2(\bar{k}_{13} + \bar{k}_{23}).$$

Remark 4: As for (18)-(19), we can choose $q_2 = \frac{p_1}{q_1} \in (0,1)$, $\tilde{k}_1 > 0$, then choose $\tilde{k}_2 > 0$ such that

$$\frac{\tilde{k}_1^{2+q_2}\alpha_1^{\max}}{4\tilde{k}_2}2^{1-\frac{1}{q_2}} < \frac{3}{4}\tilde{k}_1.$$

Finally, we select $\tilde{k}_3 > 0$ such that

$$\frac{\tilde{k}_3}{\tilde{k}_2} > \frac{2^{1-\frac{1}{q_2}}}{1+q_2}\Big(\frac{2^{3-\frac{1}{q_2}}q_2}{(1+q_2)\tilde{k}_1}\Big)^{q_2} + \frac{\tilde{k}_1^{q_2}\alpha_1^{\max}}{\tilde{k}_2}2^{1-\frac{1}{q_2}}$$
$$+ \frac{\tilde{k}_1^{1+q_2}q_2\alpha_1^{\max}2^{1-\frac{1}{q_2}}}{(1+q_2)\tilde{k}_2}\Big(\frac{4}{(1+q_2)\tilde{k}_1}\Big)^{\frac{1}{q_2}}.$$

Remark 5: In this paper, we present a finite-time switching controller for the uncertain robot systems, but it is challenging to estimate the bounds for the settled time, because this time depends on the bounds of uncertain parameters, the external disturbance and the initial state value.

4. Simulations

In this section, the switching controller proposed in *Theorem 2* is used to show how to stabilize the uncertain visual feedback system (6) in a finite time. We will demonstrate the effectiveness of our methods by an example.

In the following simulation, we assume that: $\alpha_i^{\min} = 0.5, \alpha_i^{\max} = 1.5, \gamma_i^m = 1, \gamma_i^M = 2, \varphi_i^M = 2.5, (i = 1,2)$. According to the discussion of selecting parameters in *Remark 2* and *Remark 3*, for given $\bar{l} = 4.5, \bar{k}_{ij} = 0.6, (i = 1,2; j = 1,2,3)$, choosing $q_0 = \frac{1}{2}, q_1 = \frac{7}{9}, q_2 = \frac{1}{3}$, $\lambda_1 = 1, k_1 = 2.4, k_2 = 1.7$, $l_1 = 2, l_2 = 4.2, l_3 = 2.1$, $\tilde{k}_1 = 1.5, \tilde{k}_2 = 2.6, \tilde{k}_3 = 6.5$. The initial condition of (6) is $(-0.2, 0.3, -0.5, 5.6, 0.8)$.

Figures 3-5 show some simulation results with MATLAB. Figure 3 shows that the state variable (x_0, u_0) goes to zero in a finite time $t \leq 35s$. From which, one can observe that u_0 is stabilized to a constant ($u_0 = 1$) and keep it in *Step 2* until it is driven to zero together with x_0 in the last step.

Note that in Figure 3, for u_0, during the control process, we design a controller τ_1 to make it converge to the constant 1 in Step 1 - Step 2, while in Step 3, we give another robust controller τ_1 such that (x_0, u_0) can be stabilized to zero in a finite time, and thus, we find that there is a peak at t=25s because of using this discontinuous switching controller.

Figure 4 shows the state (x_1, x_2, u_1) can be stabilized to zero step by step within the finite-time interval $[0, 35s]$ under this switching controller (τ_1, τ_2) demonstrated in Figure 5.

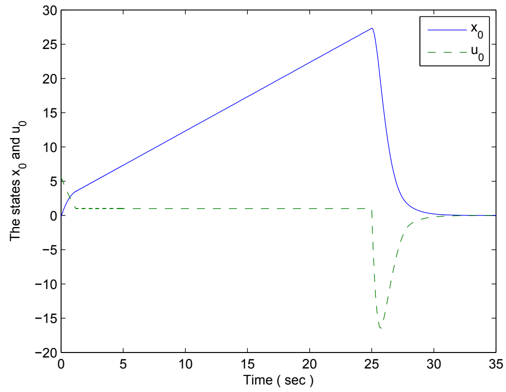

Figure 3. The response of the state variable (x_0, u_0) with respect to time

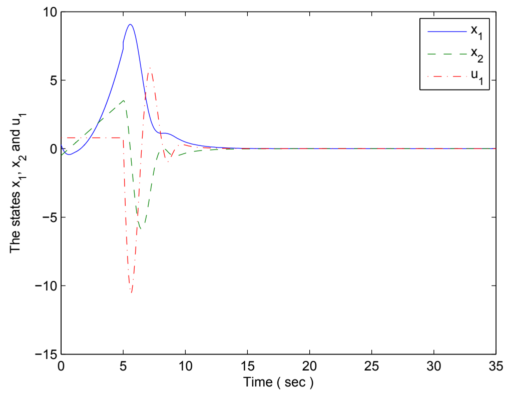

Figure 4. The response of state variable (x_1, x_2, u_1) with respect to time

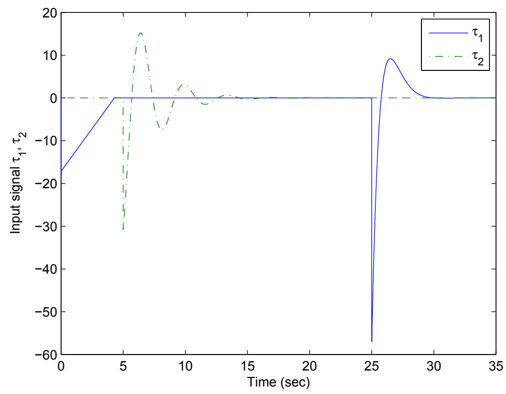

Figure 5. The response of the control input (τ_1, τ_2) with respect to time

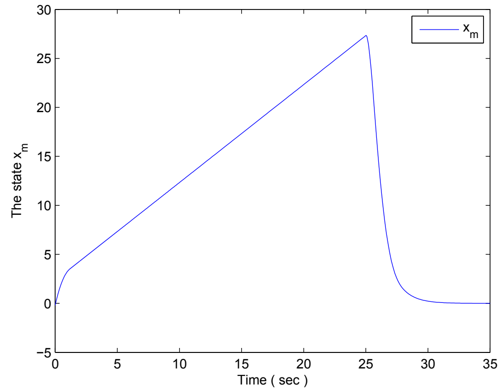

Figure 6. The response of the state variable x_m with respect to time

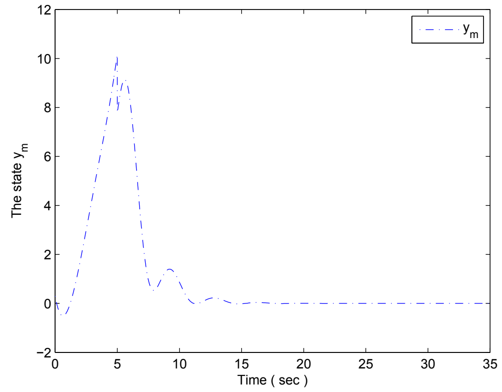

Figure 7. The response of the state variable y_m with respect to time

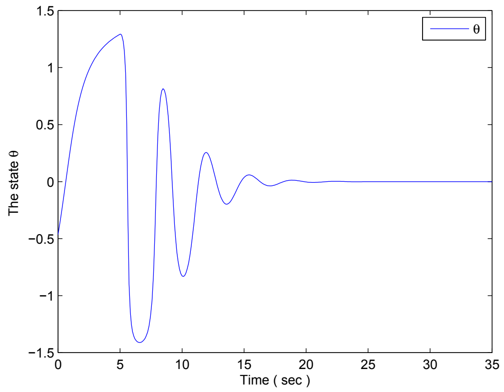

Figure 8. The response of the state variable θ with respect to time

Additionally, the following simulation results Figures 6-8 are about the original mobile robot system with visual servoing (4), from which, we can observe that the system state (x_m, y_m, θ) can also be stabilized to zero in a finite time.

5. Conclusion

In this article, a new switching controller is presented for solving the global finite-time stabilization problem of the nonholonomic mobile robots based on visual serving with uncalibrated camera parameters and external perturbation. The best innovation of this paper is that the discontinuous controller design is based on applying the stability theorem of finite-time and a new switching design method such that the states of closed-loop system can be stabilized to origin point in a finite time. In the near future, we will discuss the corresponding finite-time stabilization problem with saturated control inputs.

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Appendix A.

Adopt the similar proof method as it in [45], the time derivative of the Lyapunov function $\tilde{V}$ along a non-trivial trajectory of system (11) is given as

$$\begin{aligned}\dot{\tilde{V}} &= \sum_{i=1}^{r-1}\frac{\partial \tilde{V}}{\partial z_i}\beta_i z_{i+1} + \frac{\partial \tilde{V}}{\partial z_r}(\gamma(t)u + \varphi(t)) \\ &\le \sum_{i=1}^{r-1}\frac{\partial \tilde{V}}{\partial z_i}\beta_i z_{i+1} + \frac{\partial \tilde{V}}{\partial z_r}\bar{u}_0(z) \\ &\quad + \frac{\partial \tilde{V}}{\partial z_r}\Big(\frac{\gamma(t)}{\gamma_m}\bar{\varphi} sign(\bar{u}_0(z)) + \varphi(t)\Big)\end{aligned}$$

Because $\frac{\partial \tilde{V}}{\partial z_r}(z)\bar{u}_0(z) \le 0$, if $\bar{u}_0(z) \ge 0$, then $\frac{\partial \tilde{V}}{\partial z_r} \le 0$,

$$\frac{\partial \tilde{V}}{\partial z_r}\Big(\frac{\gamma(t)}{\gamma_m}\bar{\varphi} sign(\bar{u}_0(z)) + \varphi(t)\Big) \le \frac{\partial \tilde{V}}{\partial z_r}\big(\bar{\varphi} - |\varphi(t)|\big) \le 0.$$

On the other hand, if $\bar{u}_0(z) < 0$, then $\frac{\partial \tilde{V}}{\partial z_r} \ge 0$ and

$$\begin{aligned}\frac{\partial \tilde{V}}{\partial z_r}\Big(\frac{\gamma(t)}{\gamma_m}\bar{\varphi} sign(\bar{u}_0(z)) + \varphi(t)\Big) &= \frac{\partial \tilde{V}}{\partial z_r}\big(\frac{-\gamma(t)}{\gamma_m}\bar{\varphi} + \varphi(t)\big) \\ &\le \frac{\partial \tilde{V}}{\partial z_r}\big(|\varphi(t)| - \bar{\varphi}\big) \le 0.\end{aligned}$$

Therefore, we have

$$\dot{\tilde{V}} \le \sum_{i=1}^{r-1}\frac{\partial \tilde{V}}{\partial z_i}\beta_i z_{i+1} + \frac{\partial \tilde{V}}{\partial z_r}\bar{u}_0(z) \le -\tilde{c}\tilde{V}^{\tilde{\alpha}}(z).$$

According to *Lemma 1*, any a non-trivial trajectory z of system (11) reaches zero and stays there in a finite time.