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The microstructure of Lipschitz solutions for a one-dimensional logarithmic diffusion equation

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Abstract: We consider the initial-boundary-value problem for the one-dimensional fast diffusion equation $u_t = [\mathrm{sign}(u_x)\log|u_x|]_x$ on $Q_T = [0,T] \times [0,l]$. For monotone initial data the existence of classical solutions is known. The case of non-monotone initial data is delicate since the equation is singular at $u_x = 0$. We 'explicitly' construct infinitely many weak Lipschitz solutions to non-monotone initial data following an approach to the Perona-Malik equation. For this construction we rephrase the problem as a differential inclusion which enables us to use methods from the description of material microstructures. The Lipschitz solutions are constructed iteratively by adding ever finer oscillations to an approximate solution. These fine structures account for the fact that solutions are not continuously differentiable in any open subset of Q_T and that the derivative u_x is not of bounded variation in any such open set. We derive a characterization of the derivative, namely $u_x = d^+ \ {}_A + d^- \ {}_B$ with continuous functions $d^+ > 0$ and $d^- < 0$ and dense sets A and B, both of positive measure but with infinite perimeter. This characterization holds for any Lipschitz solution constructed with the same method, in particular for the 'microstructured' Lipschitz solutions to the one-dimensional Perona-Malik equation.

Keywords: logarithmic diffusion, one-dimensional, differential inclusion, microstructured Lipschitz solutions

AMS Subject Classification: 34A05, 35B05, 35B65

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