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Summation equations with sign changing kernels and applications to discrete fractional boundary value problems

Abstract: We consider the summation equation, for $t \in [\mu - 2, \mu + b]_{\mathbb{N}_{\mu-2}}$,

$$y(t) = \gamma_1(t)H_1\left(\sum_{i=1}^{n} a_i y\left(\xi_i\right)\right) + \gamma_2(t)H_2\left(\sum_{i=1}^{m} b_i y\left(\zeta_i\right)\right) + \lambda \sum_{s=0}^{b} G(t,s)f(s + \mu - 1, y(s + \mu - 1))$$

in the case where the map $(t,s) \mapsto G(t,s)$ may change sign; here $\mu \in (1,2]$ is a parameter, which may be understood as the order of an associated discrete fractional boundary value problem. In spite of the fact that G is allowed to change sign, by introducing a new cone we are able to establish the existence of at least one positive solution to this problem by imposing some growth conditions on the functions H_1 and H_2. Finally, as an application of the abstract existence result, we demonstrate that by choosing the maps $t \mapsto \gamma_1(t)$, $\gamma_2(t)$ in particular ways, we can recover the existence of at least one positive solution to various discrete fractional- or integer-order boundary value problems possessing Green's functions that change sign.

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