Wen and Hu *Fixed Point Theory and Applications* 2012, **2012**:98
http://www.fixedpointtheoryandapplications.com/content/2012/1/98

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

Strong convergence of an new iterative method for a zero of accretive operator and nonexpansive mapping

Meng Wen and Changsong Hu*

* Correspondence: huchang1004@yahoo.com.cn
Department of Mathematics, Hubei Normal University, Huangshi 435002, P. R. China

Abstract

Let E be a Banach space and A an m-accretive operator with a zero. Consider the iterative method that generates the sequence $\{x_n\}$ by the algorithm $x_{n+1} = \alpha_n\gamma\phi(x_n) + (I - \alpha_n F)J_{r_n}x_n$, where $\{a_n\}$ and $\{r_n\}$ are two sequences satisfying certain conditions, J_{r_n} denotes the resolvent $(I + r_nA)^{-1}$ for $r_n > 0$, F be a strongly positive bounded linear operator on E is $0 < \gamma < \bar{\gamma}$, and φ be a MKC on E. Strong convergence of the algorithm $\{x_n\}$ is proved assuming E either has a weakly continuous duality map or is uniformly smooth.

MSC: 47H09; 47H10

Keywords: MKC, accretive operators, the resolvent operator, iterative method, weakly continuous duality map

1 Introduction

Let E be a real Banach space, C a nonempty closed convex subset of E, and $T : C \to C$ a mapping. Recall that T is nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. A point $x \in C$ is a fixed point of T provided $Tx = x$. Denote by $F(T)$ the set of fixed points of T, that is, $F(T) = \{x \in C, Tx = x\}$.

It is assumed throughout the paper that T is a nonexpansive mapping such that $F(t) \neq \emptyset$. The normalized duality mapping J from a Banach space E into 2^{E^*} is given by $J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}$, $x \in E$, where E^* denotes the dual space of E and $\langle .,. \rangle$ denotes the generalized duality pairing.

Theorem 1.1. (Banach [1]). *Let (X, d) be a complete metric space and let f be a contraction on X, that is, there exists $r \in (0, 1)$ such that $d(f(x), f(y)) \le rd(x, y)$ for all $x, y \in X$. Then f has a unique fixed point.*

Theorem 1.2. (Meir and Keeler [2]). Let (X, d) be a complete metric space and let φ be a Meir-Keeler contraction (MKC, for short) on X, that is, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $d(x, y) < \varepsilon + \delta$ implies $d(\varphi(x), \varphi(y)) < \varepsilon$ for all $x, y \in X$. Then φ has a unique fixed point.

This theorem is one of generalizations of Theorem 1.1, because contractions are Meir-Keeler contractions.

Let F be a strongly positive bounded linear operator on E, that is, there exists a constant $\tilde{\gamma} > 0$ such that

$$\langle Fx, J(x)\rangle \geq \tilde{\gamma}\|x\|^2, \|aI - bF\| = \sup_{\|x\|\leq 1}\left\{\left|\langle (aI - bF)x, J(x)\rangle\right| : a \in [0, 1], b \in [0, 1]\right\},$$

where I is the identity mapping and J is the normalized duality mapping.

Let D be a subset of C. Then $Q : C \to D$ is called a retraction from C onto D if $Q(x) = x$ for all $x \in D$. A retraction $Q : C \to D$ is said to be sunny if $Q(x + t(x - Q(x))) = Q(x)$ for all $x \in C$ and $t \geq 0$ whenever $x + t(x - Q(x)) \in C$. A subset D of C is said to be a sunny nonexpansive retract of C if there exists a sunny nonexpansive retraction of C onto D. In a smooth Banach space E, it is known (cf. [[3], p. 48]) that $Q : C \to D$ is a sunny nonexpansive retraction if and only if the following condition holds:

$$\langle x - Q(x), J(z - Q(x))\rangle \leq 0, \quad x \in C,\ z \in D. \tag{1.1}$$

Recall that an operator A with domain $D(A)$ and range $R(A)$ in E is said to be accretive, if for each $x_i \in D(A)$ and $y_i \in Ax_i$, $i = 1, 2$, there is a $j \in J(x_2 - x_1)$ such that

$$\langle y_2 - y_1, j\rangle \geq 0.$$

An accretive operator A is m-accretive if $R(I + \lambda A) = E$ for all $\lambda > 0$. Denote by $N(A)$ the zero set of A; i.e.,

$$N(A) := A^{-1}0 = \{x \in D(A) : Ax = 0\}.$$

Throughout the rest of this paper it is always assumed that A is m-accretive and $N(A)$ is nonempty. Denote by J_r the resolvent of A for $r > 0$:

$$J_r = (I + rA)^{-1}.$$

Note that if A is m-accretive, then $J_r : E \to E$ is nonexpansive and $F(J_r) = N(A)$ for all $r > 0$. We also denote by A_r the Yosida approximation of A, i.e., $A_r = \frac{1}{r}(I - J_r)$. It is well known that J_r is a nonexpansive mapping from E to $C := D(A)$.

Recall that a gauge is a continuous strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ such that $\phi(0) = 0$ and $\phi(t) \to \infty$ as $t \to \infty$. Associated to a gauge ϕ is the duality mapping $J_\phi : E \to E^*$ defined by

$$J_\varphi(x) = \left\{x^* \in E^* : \langle x, x^*\rangle = \|x\|\,\varphi(\|x\|),\ \left\|x^*\right\| = \varphi(\|x\|)\right\}, \quad x \in E.$$

Following Browder [4], we say that a Banach space E has a weakly continuous duality map if there exists a gauge ϕ for which the duality map J_ϕ is single-valued and weak-to-weak* sequentially continuous(i.e., if $\{x_n\}$ is a sequence in E weakly convergent to a point x, then the sequence $J_\phi(x_n)$ converges weakly* to $J_\phi(x)$). It is known that l^p has a weakly continuous duality map for all $1 < p < \infty$, with gauge $\phi(t) = t^{p-1}$. Set

$$\Phi(t) = \int_0^t \varphi(\tau)d\tau, \quad t \geq 0. \tag{1.2}$$

Then

$$J_\varphi(x) = \partial\Phi(\|x\|), \quad x \in E,$$

where ∂ denotes the subdifferential in the sense of convex analysis.

Recently, Hong-Kun Xu [5] introduced the following iterative scheme: for $x_1 = x \in C$,

$$x_{n+1} = \alpha_n u + (1 + \alpha_n)J_{r_n}x_n, \quad \forall n \geq 1, \tag{1.3}$$

where $\{a_n\}$ and $\{r_n\}$ are two sequences satisfying certain conditions, and J_{r_n} denotes the resolvent $(I + r_nA)^{-1}$ for $r_n > 0$. He proved the strong convergence of the algorithm $\{x_n\}$ assuming E either has a weakly continuous duality map or is uniformly smooth.

Motivated and inspired by the results of Hong-Kun Xu, we introduce the following iterative scheme: for any $x_0 \in E$,

$$x_{n+1} = \alpha_n \gamma \phi(x_n) + (I - \alpha_n F)J_{r_n}x_n, \quad \forall n \geq 0, \tag{1.4}$$

where $\{a_n\}$ and $\{r_n\}$ are two sequences satisfying certain conditions, J_{r_n} denotes the resolvent $(I + r_nA)^{-1}$ for $r_n > 0$, F be a strongly positive bounded linear operator on E is $0 < \gamma < \bar{\gamma}$, and φ be a MKC on E. Strong convergence of the algorithm $\{x_n\}$ is proved assuming E either has a weakly continuous duality map or is uniformly smooth. Our results extend and improve the corresponding results of Hong-Kun Xu [5] and many others.

2 Preliminaries

In order to prove our main results, we need the following lemmas.

Lemma 2.1. [5]. *Assume that E has a weakly continuous duality map J_ϕ with gauge ϕ,*

(i) *For all $x, y \in E$, there holds the inequality*

$$\Phi\left(\|x + y\|\right) \leq \Phi\left(\|x\|\right) + \left\langle y, J_\varphi(x + y)\right\rangle.$$

(ii) *Assume a sequence $\{x_n\}$ in E is weakly convergent to a point x, then there holds the equality*

$$\limsup_{n\to\infty} \Phi\left(\|x_n - y\|\right) = \limsup_{n\to\infty} \Phi\left(\|x_n - x\|\right) + \Phi\left(\|y - x\|\right), \quad x, y \in E.$$

Lemma 2.2. [6,7]. *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1 - \lambda_n)s_n + \lambda_n\delta_n + \gamma_n, \quad n \geq 0,$$

where $\{\lambda_n\}$, $\{\delta_n\}$ and $\{\gamma_n\}$ satisfy the following conditions:

(i) $\{\lambda_n\} \subset [0,1]$ *and* $\sum_{n=0}^{\infty} \lambda_n = \infty$,

(ii) $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=0}^{\infty} \lambda_n\delta_n < \infty$ *(iii)* $\gamma_n \geq 0 (n \geq 0), \sum_{n=0}^{\infty} \gamma_n < \infty$. *Then* $\lim_{n\to\infty} s_n = 0$.

Lemma 2.3. (The Resolvent Identity [8,9]). *For $\lambda > 0$ and $\nu > 0$ and $x \in E$,*

$$J_\lambda x = J_\nu\left(\frac{\nu}{\lambda} + \left(1 - \frac{\nu}{\lambda}\right)J_\lambda x\right).$$

Lemma 2.4. (see [[10], Lemma 2.3]). *Assume that F is a strongly positive linear bounded operator on a smooth Banach space E with coefficient* $\bar{\gamma} > 0$ *and* $0 < \rho \le \|F\|^{-1}$. *Then,*

$$\|I - \rho F\| \le 1 - \rho\bar{\gamma}.$$

Lemma 2.5. (see [[11], Lemma 2.3]). *Let* φ *be a MKC on a convex subset C of a Banach space E. Then for each* $\varepsilon > 0$, *there exists* $r \in (0,1)$ *such that*

$$\|x - y\| \ge \varepsilon \text{ implies } \|\phi x - \phi y\| \le r\|x - y\| \quad \forall x, y \in C.$$

Lemma 2.6. *Let E be a reflexive Banach space which admits a weakly continuous duality map* J_ϕ *with gauge* ϕ. *Let* $T : E \to E$ *be a nonexpansive mapping. Now given* $\varphi : E \to E$ *be a MKC, F be a strongly positive linear bounded operator with coefficient* $\bar{\gamma} > 0$. *Assume that* $0 < \gamma < \bar{\gamma}$, *the sequence* $\{x_t\}$ *defined by* $x_t = t\gamma\varphi(x_t) + (I - tF)Tx_t$. *Then T has a fixed point if and only if* $\{x_t\}$ *remains bounded as* $t \to 0^+$, *and in this case,* $\{x_t\}$ *converges as* $t \to 0^+$ *strongly to a fixed point of T. If* $\tilde{x} := \lim_{t\to 0} x_t$, *then* $\tilde{x}$ *uniquely solves the variational inequality*

$$\langle (F - \gamma\phi)\tilde{x}, J(\tilde{x} - p)\rangle \le 0, \quad P \in F(T).$$

Proof. The definition of $\{x_t\}$ is well defined. Indeed, from the definition of MKC, we can see MKC is also a nonexpansive mapping. Consider a mapping S_t on E defined by

$$S_t(x) = t\gamma\phi(x) + (I - tF)Tx, \quad x \in E.$$

It is easy to see that S_t is a contraction. Indeed, by Lemma 2.4, we have

$$\begin{aligned}\|S_t x - S_t y\| &\le t\gamma\|\phi(x) - \phi(y)\| + \|(I - tF)(Tx - Ty)\| \\ &\le t\gamma\|x - y\| + (1 - t\bar{\gamma})\|Tx - Ty\| \\ &\le [1 - t(\bar{\gamma} - \gamma)]\|x - y\|,\end{aligned}$$

for all $x, y \in E$. Hence S_t has a unique fixed point, denoted as x_t, which uniquely solves the fixed point equation

$$x_t = t\gamma\phi(x_t) + (I - tF)Tx_t, \quad x_t \in E. \tag{2.1}$$

We next show the sequence $\{x_t\}$ is bounded. Indeed, we may assume $F(t) \ne \emptyset$ and with no loss of generality $t < \|F\|^{-1}$. Take $p \in F(T)$ to deduce that, for $t \in (0, 1)$,

$$\begin{aligned}\|x_t - p\| &= \|t\gamma\phi(x_t) + (I - tF)Tx_t - p\| \\ &= \|t(\gamma\phi(x_t) - Fp) + (I - tF)(Tx_t - p)\| \\ &\le (1 - t\bar{\gamma})\|x_t - p\| + t\gamma\|x_t - p\| + t\|\gamma\phi(p) - Fp\| \\ &\le [1 - t(\bar{\gamma} - \gamma)]\|x_t - p\| + t\|\gamma\phi(p) - Fp\|.\end{aligned}$$

Hence

$$\|x_t - p\| \le \frac{1}{\bar{\gamma} - \gamma}\|\gamma\phi(p) - Fp\|$$

and $\{x_t\}$ is bounded.

Next assume that $\{x_t\}$ is bounded as $t \to 0^+$. Assume $t_n \to 0^+$ and $\{x_{t_n}\}$ is bounded. Since E is reflexive, we may assume that $x_{t_n} \rightharpoonup z$ for some $z \in E$. Since J_ϕ is weakly continuous, we have by Lemma 2.1,

$$\limsup_{n\to\infty} \Phi\left(\|x_{t_n} - x\|\right) = \limsup_{n\to\infty} \Phi\left(\|x_{t_n} - z\|\right) + \Phi\left(\|x - z\|\right), \quad \forall x \in E.$$

Put

$$f(x) = \limsup_{n\to\infty} \Phi\left(\|x_{t_n} - x\|\right), \quad x \in E.$$

It follows that

$$f(x) = f(z) + \Phi\left(\|x - z\|\right), \quad x \in E.$$

Since

$$\|x_{t_n} - Tx_{t_n}\| = t_n \|\gamma\phi(x_{t_n}) - FTx_{t_n}\| \to 0,$$

we obtain

$$\begin{aligned} f(Tz) &= \limsup_{n\to\infty} \Phi\left(\|x_{t_n} - Tz\|\right) = \limsup_{n\to\infty} \Phi\left(\|Tx_{t_n} - Tz\|\right) \\ &\leq \limsup_{n\to\infty} \Phi\left(\|x_{t_n} - z\|\right) = f(z). \end{aligned} \tag{2.2}$$

On the other hand, however,

$$f(Tz) = f(z) + \Phi\left(\|Tz - z\|\right) \tag{2.3}$$

Combining Equations (2.2) and (2.3) yields

$$\Phi\left(\|Tz - z\|\right) \leq 0.$$

Hence, $Tz = z$ and $z \in F(T)$.

Finally, we prove that $\{x_t\}$ converges strongly to a fixed point of T provided it remains bounded when $t \to 0$.

Let $\{t_n\}$ be a sequence in (0, 1) such that $t_n \to 0$ and $x_{t_n} \rightharpoonup z$ as $n \to \infty$. Then the argument above shows that $z \in F(T)$. We next show that $x_{t_n} \to z$. By contradiction, there is a number $\varepsilon_0 > 0$ such that $\|x_{t_n} - z\| \geq \varepsilon_0$. Then by Lemma 2.8, there is a number $r \in (0, 1)$ such that

$$\begin{aligned} \|\phi(x_{t_n}) - \phi(z)\| &\leq r\|x_{t_n} - z\|, \\ \|x_{t_n} - z\|\varphi\left(\|x_{t_n} - z\|\right) &= \langle x_{t_n} - z, J_\varphi(x_{t_n} - z)\rangle \\ &= \langle t_n(\gamma\phi(x_{t_n}) - Fz) + (I - t_nF)(Tx_{t_n} - z), J_\varphi(x_{t_n} - z)\rangle \\ &\leq t_n\langle \gamma\phi(x_{t_n}) - Fz, J_\varphi(x_n - z)\rangle + \|(I - t_nF)(Tx_{t_n} - z)\|\varphi\left(\|x_{t_n} - z\|\right) \\ &\leq (1 - t_n\bar{\gamma})\|x_{t_n} - z\|\varphi\left(\|x_{t_n} - z\|\right) + t_n\langle\gamma\phi(x_{t_n}) - Fz, J_\varphi(x_n - z)\rangle. \end{aligned}$$

It follows that

$$\begin{aligned} \|x_{t_n} - z\|\varphi\left(\|x_{t_n} - z\|\right) &\leq \frac{1}{\bar{\gamma}}\langle\gamma\phi(x_{t_n}) - Fz, J_\varphi(x_{t_n} - z)\rangle \\ &= \frac{1}{\bar{\gamma}}\left[\langle\gamma\phi(x_{t_n}) - \gamma\phi(z), J_\varphi(x_{t_n} - z)\rangle + \langle\gamma\phi(z) - Fz, J_\varphi(x_{t_n} - z)\rangle\right] \\ &\leq \frac{\gamma}{\bar{\gamma}}r\|x_{t_n} - z\|\varphi\left(\|x_{t_n} - z\|\right) + \frac{1}{\bar{\gamma}}\langle\gamma\phi(z) - Fz, J_\varphi(x_{t_n} - z)\rangle\bigg]. \end{aligned}$$

Therefore,

$$\left\|x_{t_n}-z\right\|\varphi\left(\left\|x_{t_n}-z\right\|\right)\leq\frac{1}{\bar{\gamma}-\gamma r}\left\langle\gamma\phi(z)-Fz,J_{\phi}(x_{t_n}-z)\right\rangle.$$

Now observing that $x_{t_n}\rightharpoonup z$ implies $J_{\varphi}(x_{t_n}-z)\rightharpoonup 0$, we conclude from the last inequality that

$$\lim_{n\to\infty}\left\|x_{t_n}-z\right\|\varphi\left(\left\|x_{t_n}-z\right\|\right)=0.$$

It contradicts $\left\|x_{t_n}-z\right\|\varphi\left(\left\|x_{t_n}-z\right\|\right)\geq\varepsilon_0\varphi(\varepsilon_0)>0$. Hence $x_{t_n}\to z$.

We finally prove that the entire net $\{x_t\}$ converges strongly. Towards this end, we assume that two null sequences $\{t_n\}$ and $\{s_n\}$ in (0, 1) are such that

$$t_n\to 0,\ x_{t_n}\to z\quad\text{and}\quad s_n\to 0,\ \ x_{s_n}\to\acute{z}.$$

We have to show $z=\acute{z}$. Indeed, for $p\in F(T)$. Since

$$x_t=t\gamma\phi(x_t)+(I-tf)Tx_t,$$

we derive that

$$(F-\gamma\phi)x_t=-\frac{1}{t}(I-tF)(I-T)x_t. \tag{2.4}$$

Notice

$$\begin{aligned}\left\langle(I-T)x_t-(I-T)p,J_{\varphi}(x_t-p)\right\rangle&=\left\|x_t-p\right\|\varphi\left(\left\|x_t-p\right\|\right)+\left\langle Tp-Tx_t,J_{\varphi}(x_t-p)\right\rangle\\&\geq\left\|x_t-p\right\|\varphi\left(\left\|x_t-p\right\|\right)-\left\|Tp-Tx_t\right\|\left\|J_{\varphi}\left(x_t-p\right)\right\|\\&\geq\left\|x_t-p\right\|\left[\varphi\left(\left\|x_t-p\right\|\right)-\varphi\left(\left\|x_t-p\right\|\right)\right]\\&=0.\end{aligned}$$

It follows that,

$$\begin{aligned}\left\langle(F-\gamma\phi)x_t,J_{\varphi}(x_t-p)\right\rangle&=-\tfrac{1}{t}\left\langle(I-tF)(I-T)x_t,J_{\varphi}(x_t-p)\right\rangle\\&\leq\left\langle(F(I-T)x_t,J_{\varphi}(x_t-p)\right\rangle.\end{aligned} \tag{2.5}$$

Now replacing t in (2.5) with t_n and letting $n\to\infty$, noticing $(I-T)x_{t_n}\to(I-T)z=0$ for $z\in F(T)$, we obtain $\langle(F-\gamma\varphi)z,J_{\phi}(z-p)\rangle\leq 0$. In the same way, we have $\left\langle(F-\gamma\phi)\acute{z},J_{\varphi}(\acute{z}-p)\right\rangle\leq 0$.

Thus, we have

$$\left\langle(F-\gamma\phi)z,J_{\varphi}(z-\acute{z})\right\rangle\leq 0\quad\text{and}\quad\left\langle(F-\gamma\phi)\acute{z},J_{\varphi}(\acute{z}-z)\right\rangle\leq 0. \tag{2.6}$$

Adding up (2.6) gets

$$\left\langle(F-\gamma\phi)z-(F-\gamma\phi)\acute{z},\quad J_{\varphi}(z-\acute{z})\right\rangle\leq 0.$$

On the other hand, without loss of generality, we may assume there is a number ε such that $\left\|z-\acute{z}\right\|\geq\varepsilon$, then by Lemma 2.5 there is a number r_1 such that $\left\|\phi(z)-\phi(\acute{z})\right\|\leq r_1\left\|z-\acute{z}\right\|$. Noticing that

$$\begin{aligned} J_\varphi &= \left(\varphi\left(\|x\|\right)/\|x\|\right)J(x) \quad x \neq 0,\\ \left\langle (F-\gamma\phi)z-(F-\gamma\phi)\acute{z}, J_\varphi(z-\acute{z})\right\rangle &= \left\langle F(z-\acute{z}), J_\varphi(z-\acute{z})\right\rangle - \left\langle \gamma\phi z-\gamma\phi\acute{z}, J_\varphi(z-\acute{z})\right\rangle\\ &\geq \bar{\gamma}\left\|z-\acute{z}\right\|\left\|J_\varphi(z-\acute{z})\right\| - \gamma r_1\left\|z-\acute{z}\right\|\left\|J_\varphi(z-\acute{z})\right\|\\ &\geq (\bar{\gamma}-\gamma r_1)\left\|z-\acute{z}\right\|\varphi\left(\left\|z-\acute{z}\right\|\right)\\ &> 0. \end{aligned}$$

Hence $z = \acute{z}$ and $\{x_t\}$ converges strongly. Thus we may assume $x_t \to \tilde{x}$. Since we have proved that, for all $t \in (0, 1)$ and $p \in F(T)$,

$$\left\langle (F-\gamma\phi)x_t, J_\varphi(x_t-p)\right\rangle \leq \left\langle (F(I-T)x_t, J_\varphi(x_t-p)\right\rangle,$$

letting $t \to 0$, we obtain that

$$\left\langle (F-\gamma\phi)\tilde{x}, J_\varphi(\tilde{x}-p)\right\rangle \leq 0.$$

This implies that

$$\left\langle (F-\gamma\phi)\tilde{x}, J(\tilde{x}-p)\right\rangle \leq 0.$$

Lemma 2.7. (see [12]). *Assume that* $C_2 \geq C_1 > 0$. *Then* $\left\|J_{C_1}x - x\right\| \leq 2\left\|J_{C_2}x - x\right\|$ *for all* $x \in E$.

Lemma 2.8. [13]. *Let C be a nonempty closed convex subset of a reflexive Banach space E which satisfies Opial's condition, and suppose* $T : C \to E$ *is a nonexpansive mapping. Then the mapping* $I - T$ *is demiclosed at zero, that is* $x_n \rightharpoonup x$ *and* $\|x_n - Tx_n\| \to 0$, *then* $x = Tx$.

Lemma 2.9. *In a smooth Banach space E there holds the inequality*

$$\left\|x+y\right\|^2 \leq \|x\|^2 + 2\left\langle y, J(x+y)\right\rangle, \quad x, y \in E.$$

3 Main result

Theorem 3.1. *Suppose that E is reflexive which admits a weakly continuous duality map* J_ϕ *with gauge* ϕ *and A is an m-accretive operator in E such that* $F^* = N(A) \neq \emptyset$. *Now given* $\varphi : E \to E$ *be a MKC, and let F be a strongly positive linear bounded operator on E with coefficient* $\bar{\gamma} > 0, 0 < \gamma < \bar{\gamma}$. *Assume*

(i) $\lim_{n\to\infty}\alpha_n = 0, \sum_{n=0}^{\infty}\alpha_n = \infty$;
(ii) $r_n \to \infty$.

Then $\{x_n\}$ *defined by (1.4) converges strongly to a point in* F^*.

Proof. First notice that $\{x_n\}$ is bounded. Indeed, take $p \in F^*$ to get

$$\begin{aligned} \left\|x_{n+1}-p\right\| &= \left\|\alpha_n\phi(x_n) + (I-\alpha_nF)J_{r_n}x_n - p\right\|\\ &= \left\|\alpha_n\gamma\phi(x_n) - \alpha_nFp + \alpha_nFp + (I-\alpha_nF)J_{r_n}x_n - p\right\|\\ &= \left\|\alpha_n(\gamma\phi(x_n) - Fp) + (I-\alpha_nF)(J_{r_n}x_n - p)\right\|\\ &\leq (1-\alpha_n\bar{\gamma})\left\|x_n-p\right\| + \alpha_n\gamma\left\|x_n-p\right\| + \alpha_n\left\|\gamma\phi(p)-Fp\right\|\\ &\leq [1-\alpha_n(\bar{\gamma}-\gamma)]\left\|x_n-p\right\| + \alpha_n\left\|\gamma\phi(p)-Fp\right\|\\ &\leq [1-\alpha_n(\bar{\gamma}-\gamma)]\left\|x_n-p\right\| + \alpha_n(\bar{\gamma}-\gamma)\frac{\left\|\gamma\phi(p)-Fp\right\|}{\bar{\gamma}-\gamma}\\ &\leq \max\left\{\left\|x_n-p\right\|, \frac{\left\|\gamma\phi(p)-Fp\right\|}{\bar{\gamma}-\gamma}\right\}, \quad n \geq 0. \end{aligned}$$

By induction, we have

$$\|x_n - p\| \le \max\left\{\|x_0 - p\|, \frac{\|\gamma\phi(p) - Fp\|}{\bar{\gamma} - \gamma}\right\}, \quad n \ge 0.$$

This implies that $\{x_n\}$ is bounded and hence

$$\begin{aligned}\|x_{n+1} - J_{r_n}x_n\| &= \|\alpha_n\gamma\phi(x_n) + (I - \alpha_n F)J_{r_n}x_n - J_{r_n}x_n\| \\ &= \alpha_n\|\gamma\phi(x_n) - FJ_{r_n}x_n\| \to 0.\end{aligned}$$

We next prove that $\lim\sup_{n\to\infty}\langle\gamma\varphi(p) - Fp, J_\phi(x_n - p)\rangle \le 0$, where $p = \lim_{t\to 0} x_t$ with $x_t = t\gamma\phi(x_t) + (I - tF)J_{r_n}x_t$.

Since $\{x_n\}$ is bounded, take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle\gamma\phi(p) - Fp, J_\phi(x_n - p)\rangle = \lim_{k\to\infty}\langle\gamma\phi(p) - Fp, J_\varphi(x_{n_k} - p)\rangle. \tag{3.1}$$

Since E is reflexive, we may further assume that $x_{n_k} \rightharpoonup \tilde{x}$. Moreover, since

$$\|x_{n+1} - J_{r_n}x_n\| \to 0,$$

we obtain

$$J_{r_{n_k}-1}x_{n_k} - 1 \rightharpoonup \tilde{x}.$$

Taking the limit as $k \to \infty$ in the relation

$$\left[J_{r_{n_k}-1}x_{n_k-1}, A_{r_{n_k}-1}x_{n_k} - 1\right] \in A,$$

we get $[\tilde{x}, 0] \in A$. That is, $\tilde{x} \in F^*$. Hence by (3.1) and Lemma 2.6 we have

$$\limsup_{n\to\infty}\langle\gamma\phi(p) - Fp, J_\varphi(x_n - p)\rangle = \lim_{k\to\infty}\langle\gamma\phi(p) - Fp, J_\varphi(x_{n_k} - p)\rangle = \langle\gamma\phi(p) - Fp, J_\varphi(\tilde{x} - p)\rangle \le 0.$$

Finally to prove that $x_n \to p$, we apply Lemma 2.1 to get

$$\begin{aligned}\Phi(\|x_{n+1} - p\|) &= \Phi(\|\alpha_n\gamma\phi(x_n) + (I - \alpha_n F)J_{r_n}x_n - p\|) \\ &= \Phi(\|(I - \alpha_n F)(J_{r_n}x_n - p) + \alpha_n(\gamma\phi(x_n) - Fp)\|) \\ &= \Phi(\|(I - \alpha_n F)(J_{r_n}x_n - p) + \alpha_n(\gamma\phi(x_n) - \gamma\phi(p)) + \alpha_n(\gamma\phi(p) - Fp)\|) \\ &\le \Phi(\|(I - \alpha_n F)(J_{r_n}x_n - p) + \alpha_n(\gamma\phi(x_n) - \gamma\phi(p))\|) + \alpha_n\langle\gamma\phi(p) - Fp, J_\varphi(x_{n+1} - p)\rangle \\ &\le [1 - \alpha_n(\bar{\gamma} - \gamma)]\Phi(\|x_n - p\|) + \alpha_n\langle\gamma\phi(p) - Fp, J_\varphi(x_{n+1} - p)\rangle.\end{aligned}$$

An application of Lemma 2.2 yields that $\Phi(\|x_n - p\|) \to 0$. That is, $\|x_n - p\| \to 0$, i.e., $x_n \to p$. The proof is complete.

Theorem 3.2. *Suppose that E is reflexive which admits a weakly continuous duality map J_ϕ with gauge ϕ and A is an m-accretive operator in E such that $F^* = N(A) \neq \emptyset$. Now given $\varphi : E \to E$ be a MKC, and let F be a strongly positive linear bounded operator on E with coefficient $\bar{\gamma} > 0, 0 < \gamma < \bar{\gamma}$. Assume*

(i) $\lim_{n\to\infty}\alpha_n = 0, \sum_{n=0}^{\infty}\alpha_n = \infty$, *and* $\sum_{n=1}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty(e.g., \alpha_n = \frac{1}{n})$;

(ii) $r_n \ge \varepsilon$ *for all n and* $\sum_{n=1}^{\infty}|r_{n+1} - r_n| < \infty(e.g., r_n = 1 + \frac{1}{n})$.

Then $\{x_n\}$ *defined by (1.4) converges strongly to a point in* F^*.

Proof. We only include the differences. We have

$$x_{n+1} = \alpha_n \gamma \phi(x_n) + (I - \alpha_n F) J_{r_n} x_n, \quad x_n = \alpha_{n-1} \gamma \phi(x_{n-1}) + (I - \alpha_{n-1} F) J_{r_{n-1}} x_{n-1}.$$

Thus,

$$x_{n+1} - x_n = (I - \alpha_n F)(J_{r_n} x_n - J_{r_{n-1}} x_{n-1}) + \alpha_n \gamma \phi(x_n) + \alpha_{n-1} \gamma \phi(x_{n-1}) + (\alpha_n - \alpha_{n-1}) F J_{r_{n-1}} x_{n-1}. \tag{3.2}$$

If $r_{n-1} \leq r_n$, using the resolvent identity

$$J_{r_n} x_n = J_{r_{n-1}} \left(\frac{r_{n-1}}{r_n} x_n + \left(1 - \frac{r_{n-1}}{r_n}\right) J_{r_n} x_n \right),$$

we obtain

$$\begin{aligned} \left\| J_{r_n} x_n - J_{r_{n-1}} x_{n-1} \right\| &\leq \tfrac{r_{n-1}}{r_n} \|x_n - x_{n-1}\| + \left(1 - \tfrac{r_{n-1}}{r_n}\right) \left\| J_{r_n} x_n - x_{n-1} \right\| \\ &\leq \|x_n - x_{n-1}\| + \left(\tfrac{r_n - r_{n-1}}{r_n}\right) \left\| J_{r_n} x_n - x_{n-1} \right\| \\ &\leq \|x_n - x_{n-1}\| + \tfrac{1}{\varepsilon} |r_{n-1} - r_n| \left\| J_{r_n} x_n - x_{n-1} \right\|. \end{aligned} \tag{3.2a}$$

It follows from (3.2) that

$$\|x_{n+1} - x_n\| \leq \left(1 - \alpha_n(\bar{\gamma} - \gamma)\right) \|x_n - x_{n-1}\| + M\left(|\alpha_n - \alpha_{n-1}| + |r_{n-1} - r_n|\right), \tag{3.3}$$

where $M > 0$ is some appropriate constant. Similarly we can prove (3.3) if $r_{n-1} \geq r_n$. By assumptions (i) and (ii) and Lemma 2.2, we conclude that

$$\|x_{n+1} - x_n\| \to 0.$$

This implies that

$$\left\| x_n - J_{r_n} x_n \right\| \leq \|x_{n+1} - x_n\| + \left\| x_{n+1} - J_{r_n} x_n \right\| \to 0, \tag{3.4}$$

since $\left\| x_{n+1} - J_{r_n} x_n \right\| = \alpha_n \left\| \gamma \phi(x_n) - F J_{r_n} x_n \right\| \to 0$. It follows that

$$\left\| A_{r_n} x_n \right\| = \frac{1}{r_n} \left\| x_n - J_{r_n} x_n \right\| \leq \frac{1}{\varepsilon} \left\| x_n - J_{r_n} x_n \right\| \to 0.$$

Now if $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ converging weakly to a point $\tilde{x}$, then taking the limit as $k \to \infty$ in the relation

$$[J_{r_{n_k}} x_{n_k}, A_{r_{n_k}} x_{n_k}] \in A,$$

we get $[\tilde{x}, 0] \in A$; i.e., $\tilde{x} \in F^*$. We therefore conclude that all weak limit points of $\{x_n\}$ are zeros of A.

The rest of the proof follows that of Theorem 3.1.

Finally, we consider the framework of uniformly smooth Banach spaces. Assume $r_n \geq \varepsilon$ for some $\varepsilon > 0$ (not necessarily $r_n \to \infty$), A is an m-accretive operator in E. Moreover let $\varphi : E \to E$ be a MKC and F be a strongly positive linear bounded operator on E. Since J_{r_n} is nonexpansive, the map $S : x \in E \mapsto t\gamma\phi(x) + (I - tF)J_{r_n} x$ is a contraction and for each integer $n \geq 1$ it has a unique fixed $z_{t,n} \in E$. Hence the scheme

$$z_{t,n} = t\gamma\phi(z_{t,n}) + (I - tF) J_{r_n} z_{t,n} \tag{3.5}$$

is well defined.

Note that $\{z_{t,n}\}$ is uniformly bounded; indeed, $\|z_{t,n} - p\| \leq \frac{1}{\bar{\gamma}-\gamma} \|\gamma\phi(p) - Fp\|$ for all $t \in (0, 1)$, $n \geq 1$ and $p \in F^*$. A key component of the proof of the next theorem is the following lemma.

Lemma 3.1. *The limit* $\hat{z} = \lim_{t\to 0} z_{t,n}$ *is uniform for all* $n \geq 1$.

Proof. It suffices to show that for any positive integer n_t (which may depend on $t \in (0, 1)$), if $z_{t,n_t} \in E$ is the unique point in E that satisfies the property

$$z_{t,n_t} = t\gamma\phi(z_{t,n_t}) + (I - tF)J_{r_{n_t}} z_{t,n_t}, \tag{3.6}$$

then $\{z_{t,n_t}\}$ converges as $t \to 0$ to a point in F^*. For simplicity put

$$w_t = z_{t,n_t} \quad \text{and} \quad V_t = J_{r_{n_t}}.$$

It follows that

$$w_t = t\gamma\phi(w_t) + (I - tF)V_t w_t. \tag{3.7}$$

Note that $Fix(V_t) = F^*$ for all t. Note also that $\{w_t\}$ is bounded; indeed, we have $\|w_t - p\| \leq \frac{1}{\bar{\gamma}-\gamma} \|\gamma\phi(p) - Fp\|$ for all $t \in (0, 1)$ and $p \in F^*$. Since $\{V_t\, w_t\}$ is bounded, it is easy to see that

$$\|w_t - V_t w_t\| = t \left\|\gamma\phi(w_t) - FV_t w_t\right\| \to 0, \quad \text{as} \quad t \to 0.$$

Since $r_n \geq \varepsilon$ for all n, by Lemma 2.7, we have

$$\|w_t - J_\varepsilon w_t\| \leq 2 \left\|w_t - J_{r_{n_t}} w_t\right\| = 2\, \|w_t - V_t w_t\| \to 0. \tag{3.8}$$

Let $\{t_k\}$ be a sequence in (0,1) such that $t_k \to 0$ as $k \to \infty$. Define a function f on E by

$$f(w) = \mathrm{LIM}_k \frac{1}{2} \left\|w_{t_k} - w\right\|^2, \quad w \in E,$$

where LIM denotes a Banach limit on l^∞. Let

$$K := \left\{w \in E : f(w) = \min\{f(y) : y \in E\}\right\}.$$

Then K is a nonempty closed convex bounded subset of E. We claim that K is also invariant under the nonexpansive mapping J_ε. Indeed, noting (3.8), we have for $w \in K$,

$$\begin{aligned} f(J_\varepsilon w) &= \mathrm{LIM}_k \frac{1}{2} \left\|w_{t_k} - J_\varepsilon w\right\|^2 = \mathrm{LIM}_k \frac{1}{2} \left\|J_\varepsilon w_{t_k} - J_\varepsilon w\right\|^2 \\ &\leq \mathrm{LIM}_k \frac{1}{2} \left\|w_{t_k} - w\right\|^2 = f(w). \end{aligned}$$

Since a uniformly smooth Banach space has the fixed point property for nonexpansive mappings and since J_ε is a nonexpansive self-mapping of E, J_ε has a fixed point in K, say w'. Now since w' is also a minimizer of f over E, it follows that, for $w \in E$,

$$\begin{aligned} 0 &\leq \frac{f(w' + \lambda(w - w')) - f(w')}{\lambda} \\ &= \mathrm{LIM}_k \frac{\frac{1}{2}\left\|w_{t_k} - w' + \lambda(w' - w)\right\|^2 - \frac{1}{2}\left\|w_{t_k} - w'\right\|^2}{\lambda}. \end{aligned}$$

Since E is uniformly smooth, the duality map J is uniformly continuous on bounded sets, letting $\lambda \to 0^+$ in the last equation yields

$$0 \le \mathrm{LIM}_k \left\langle w' - w, J(w_{t_k} - w') \right\rangle, \quad w \in E. \tag{3.9}$$

Since

$$w_{t_k} - w' = (I - t_k F)(V_{t_k} w_{t_k} - w') + t_k(\gamma \phi(w_{t_k}) - Fw'),$$

we obtain

$$\begin{aligned}
\left\| w_{t_n} - w' \right\|^2 &= t_k \left\langle \gamma\phi(w_{t_k}) - Fw', J(w_{t_k} - w') \right\rangle + \left\langle (I - t_k F)(V_{t_k} w_{t_k} - w'), J(w_{t_k} - w') \right\rangle \\
&\le t_k \left\langle \gamma\phi(w_{t_k}) - Fw', J(w_{t_k} - w') \right\rangle + (1 - t_k \bar{\gamma}) \left\| w_{t_k} - w' \right\|^2 \\
&\le t_k \left\langle \gamma\phi(w_{t_k}) - \gamma\phi(w'), J(w_{t_k} - w') \right\rangle + t_k \left\langle \gamma\phi(w') - Fw', J(w_{t_k} - w') \right\rangle + (1 + t_k \bar{\gamma}) \left\| w_{t_k} - w' \right\|^2 \\
&\le [1 - t_k(\bar{\gamma} - \gamma)] \left\| w_{t_k} - w' \right\|^2 + t_k \left\langle \gamma\phi(w') - Fw', J(w_{t_k} - w') \right\rangle.
\end{aligned}$$

It follows that

$$\left\| w_{t_k} - w' \right\|^2 \le \frac{1}{\bar{\gamma} - \gamma} \left\langle \gamma\phi(w') - Fw', J(w_{t_k} - w') \right\rangle. \tag{3.10}$$

Upon letting $w = \gamma\varphi(w') - Fw' + w'$ in (3.9), we see that the last equation implies

$$\mathrm{LIM}_k \left\| w_{t_k} - w' \right\|^2 \le 0. \tag{3.11}$$

Therefore, $\{w_{t_k}\}$ contains a subsequence, still denoted $\{w_{t_k}\}$, converging strongly to w_1 (say). By virtue of (3.8), w_1 is a fixed point of J_ε; i.e., a point in F^*.

To prove that the entire net $\{w_t\}$ converges strongly, assume $\{s_k\}$ is another null subsequence in (0, 1) such that $w_{s_k} \to w_2$ strongly. Then $w_2 \in F^*$.

Repeating the argument of (3.10) we obtain

$$\left\| w_t - w' \right\|^2 \le \frac{1}{\bar{\gamma} - \gamma} \left\langle \gamma\phi(w') - Fw', J(w_t - w') \right\rangle, \quad \forall w' \in F^*.$$

In particular,

$$\|w_1 - w_2\|^2 \le \frac{1}{\bar{\gamma} - \gamma} \left\langle \gamma\phi(w_1) - Fw_1, J(w_2 - w_1) \right\rangle \tag{3.12}$$

and

$$\|w_2 - w_1\|^2 \le \frac{1}{\bar{\gamma} - \gamma} \left\langle \gamma\phi(w_2) - Fw_2, J(w_1 - w_2) \right\rangle. \tag{3.13}$$

Adding up the last two equations gives

$$\|w_1 - w_2\|^2 \le 0.$$

That is, $w_1 = w_2$. This concludes the proof.

Theorem 3.3. *Suppose that E is a uniformly smooth Banach space and A is an m-accretive operator in E such that $F^* = N(A) \neq \emptyset$. Now given $\varphi : E \to E$ be a MKC, and let F be a strongly positive linear bounded operator on E with coefficient $\bar{\gamma} > 0, 0 < \gamma < \bar{\gamma}$. Assume*

(i) $\lim_{n\to\infty}\alpha_n = 0, \sum_{n=0}^{\infty}\alpha_n = \infty$, *and* $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty(e.g., \alpha_n = \frac{1}{n})$;

(ii) $\lim_{n\to\infty} = r_n = r, r \in R^+$, $r_n \geq \varepsilon$ *for all* n *and* $\sum_{n=1}^{\infty}|r_{n+1}-r_n| < \infty(e.g., r_n = 1+\frac{1}{n})$.

Then $\{x_n\}$ *defined by (1.4) converges strongly to a point in* F^*.

Proof. Since

$$\begin{aligned}\left\|J_{r_n}x_n - J_r x_n\right\| &\leq \left\|J_r\left(\frac{r}{r_n}x_n + \left(1-\frac{r}{r_n}\right)J_{r_n}x_n\right) - J_r x_n\right\| \\ &\leq \left\|\left(\frac{r}{r_n}x_n + \left(1-\frac{r}{r_n}\right)J_{r_n}x_n\right) - x_n\right\| \\ &\leq \left|1-\frac{r}{r_n}\right|\left\|J_{r_n}x_n - x_n\right\| \to 0 \text{ as } (n\to\infty).\end{aligned} \tag{3.14}$$

Thus

$$\|x_n - J_r x_n\| \leq \left\|x_n - J_{r_n}x_n\right\| + \left\|J_{r_n}x_n - J_r x_n\right\| \to 0. \tag{3.15}$$

We next claim that $\limsup_{n\to\infty}\langle\gamma\phi(\hat{z}) - F\hat{z}, J(x_n-\hat{z})\rangle \leq 0$, where $\hat{z} = \lim_{t\to 0}z_{t,n}$ with $z_{t,n} = t\gamma\varphi(z_{t,n}) + (I - tF)J_r z_{t,n}$.

For this purpose, let $\{x_{n_k}\}$ be a subsequence chosen in such a way that $\limsup_{n\to\infty}\langle\gamma\phi(\hat{z}) - F\hat{z}, J(x_n-\hat{z})\rangle = \lim_{k\to\infty}\langle\gamma\phi(\hat{z}) - F\hat{z}, J(x_{n_k}-\hat{z})\rangle$ and $x_{n_k} \rightharpoonup \tilde{x}$. Moreover, since $\|x_n - J_r x_n\| \to 0$, using Lemma 2.8, we know $\tilde{x} \in F(J_r)$. Hence by Lemma 2.6, we have

$$\limsup_{n\to\infty}\langle\gamma\phi(\hat{z}) - F\hat{z}, J(x_n-\hat{z})\rangle = \lim_{k\to\infty}\langle\gamma\phi(\hat{z}) - F\hat{z}, J(x_{n_k}-\hat{z})\rangle = \langle\gamma\phi(\hat{z}) - F\hat{z}, \tilde{x}-\hat{z}\rangle \leq 0. \tag{3.16}$$

Finally to prove that $x_n \to \hat{z}$ strongly, we write

$$x_{n+1} - \hat{z} = (I - \alpha_n F)(J_{r_n}x_n - \hat{z}) + \alpha_n(\gamma\phi(x_n) - F\hat{z}).$$

Apply Lemma 2.9 to get

$$\begin{aligned}\left\|x_{n+1}-\hat{z}\right\|^2 &\leq (1-\alpha_n\bar{\gamma})^2\left\|x_n-\hat{z}\right\|^2 + 2\alpha_n\langle\gamma\phi(x_n) - F\hat{z}, J(x_{n+1}-\hat{z})\rangle \\ &\leq (1-\alpha_n\bar{\gamma})^2\left\|x_n-\hat{z}\right\|^2 + 2\alpha_n\langle\gamma\phi(x_n) - \gamma\phi(\hat{z}), J(x_{n+1}-\hat{z})\rangle \\ &\quad + 2\alpha_n\langle\gamma\phi(\hat{z}) - F\hat{z}, J(x_{n+1}-\hat{z})\rangle \\ &\leq (1-\alpha_n\bar{\gamma})^2\left\|x_n-\hat{z}\right\|^2 + \alpha_n\gamma\left(\left\|x_n-\hat{z}\right\|^2 + \left\|x_{n+1}-\hat{z}\right\|^2\right) \\ &\quad + 2\alpha_n\langle\gamma\phi(\hat{z}) - F\hat{z}, J(x_{n+1}-\hat{z})\rangle\end{aligned}$$

It follows that

$$\begin{aligned}\left\|x_{n+1}-\hat{z}\right\|^2 &\leq \frac{(1-\alpha_n\bar{\gamma})^2+\alpha_n\gamma}{1-\alpha_n\gamma}\left\|x_n-\hat{z}\right\|^2 + \frac{2\alpha_n}{1-\alpha_n\gamma}\langle\gamma\phi(\hat{z}) - F\hat{z}, J(x_{n+1}-\hat{z})\rangle \\ &\leq \left(1-\frac{2\alpha_n(\bar{\gamma}-\gamma)}{1-\alpha_n\gamma}\right)\left\|x_n-\hat{z}\right\|^2 + \frac{2\alpha_n(\bar{\gamma}-\gamma)}{1-\alpha_n\gamma}\left[\frac{1}{\bar{\gamma}-\gamma}\langle\gamma\phi(\hat{z}) - F\hat{z}, J(x_{n+1}-\hat{z})\rangle \right. \\ &\quad \left. + \frac{\alpha_n\bar{\gamma}^2}{2(\bar{\gamma}-\gamma)}M_1\right],\end{aligned}$$

where $M_1 = \sup_{n\geq 1}\left\|x_n-\hat{z}\right\|^2$. By Lemma 2.2 and (3.16), we see that $x_n \to \hat{z}$.

Remark 3.4. If $\gamma = 1$, F is the identity operator and $\varphi(x_n) = u$ in our results, we can obtain Theorems 3.1, 4.1, 4.2, 4.4 and Lemma 4.3 of Hong-Kun Xu [5].

Authors' contributions
The main idea of this paper is proposed by Meng Wen. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

Received: 28 November 2011 Accepted: 15 June 2012 Published: 15 June 2012

References

1. Banach, S: Surles opérations dans les ensembles abstraits et leur application aux equations intégrales. Fund Math. **3**, 133–181 (1922)
2. Meir, A, Keeler, E: A theorem on contraction mappings. J Math Anal Appl. **28**, 326–329 (1969). doi:10.1016/0022-247X(69)90031-6
3. Goebel, K, Reich, S: Uniform Convexity. In: Hyperbolic Geometry and Nonexpansive Mappings. Marcel Dekker, New York (1984)
4. Browder, FE: Convergence of approximations to points of non-expansive maps in Banach spaces. Arch Ration Mech Anal. **24**, 82–90 (1967)
5. Xu, HK: Strong convergence of an iterative method for nonexpansive and accretive operators. J Math Anal Appl. **314**, 631–643 (2006). doi:10.1016/j.jmaa.2005.04.082
6. Liu, LS: Iterative processes with errors for nonlinear strongly accretive mappings in Banach spaces. J Math Anal Appl. **194**, 114–125 (1995). doi:10.1006/jmaa.1995.1289
7. Xu, HK: Iterative algorithms for nonlinear operators. J Lond Math Sov. **66**, 240–256 (2002). doi:10.1112/S0024610702003332
8. Bruck, RE, Passty, GB: Almost convergence of the infinite product of resolvents in Banach spaces. Nonlinear Anal. **3**, 279–282 (1979). doi:10.1016/0362-546X(79)90083-X
9. Bruck, RE, Reich, S: Nonexpansive projections and resolvents in Banach spaces. Houst J Math. **3**, 459–470 (1977)
10. Cai, G, Hu, CS: Strong convergence theorems of a general iterative process for a finite family of λ_i-strict pseudo-contractions in q-uniformly smooth Banach spaces. Comput Math Appl. **59**(1), 149–160 (2010). doi:10.1016/j.camwa.2009.07.068
11. Suzuki, T: Moudafi's viscosity approximations with Meir-Keeler contractions. J Math Anal Appl. **325**(1), 342–352 (2007). doi:10.1016/j.jmaa.2006.01.080
12. Marino, G, Xu, HK: Convergence of generalized proximal point algorithms. Commun Pure Appl Anal. **3**, 791–808 (2004)
13. Jung, JS: Iterative approaches to common fixed points of nonexpansive mappings in Banach spaces. J Math Anal Appl. **302**(2), 509–520 (2005). doi:10.1016/j.jmaa.2004.08.022

doi:10.1186/1687-1812-2012-98
Cite this article as: Wen and Hu: Strong convergence of an new iterative method for a zero of accretive operator and nonexpansive mapping. *Fixed Point Theory and Applications* 2012 **2012**:98.