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Positive solutions for a class of semipositone Neumann problems

Tianlan Chen and Ruyun Ma*

*Correspondence:
mary@nwnu.edu.cn
Department of Mathematics,
Northwest Normal University,
Lanzhou, 730070, P.R. China

Abstract

In this paper, by using the quadrature method, we show how changes in the sign of f lead to multiple positive solutions for the semipositone Neumann problems

$$-u''(x) = \lambda f(u(x)), \qquad u'(0) = 0 = u'(1),$$

when the parameter λ belongs to some intervals.

MSC: 34B18

Keywords: semipositone; Neumann problem; positive solution; quadrature method

1 Introduction

In this paper, we are concerned with the existence of multiple positive solutions for the semipositone problem

$$-u''(x) = \lambda f(u(x)), \quad x \in (0,1), \tag{1.1}$$

$$u'(0) = 0 = u'(1), \tag{1.2}$$

where $\lambda > 0$ is a parameter, $f(0) < 0$ (semipositone).

Semipositone problems arise in many different areas of applied mathematics and physics, such as the buckling of mechanical systems, the design of suspension bridges, chemical reactions, and population models with harvesting effort; see [1–4].

The study of semipositone problems was formally introduced by Castro and Shivaji in [5]. In general, studying positive solutions for semipositone problems is more difficult than that for positone problems. The difficulty is due to the fact that in the semipositone case, solutions have to live in regions where the nonlinear term is negative as well as positive. Due to its importance, one-dimensional semipositone problems have been widely studied by many authors; see [5–8] and the references within. For higher-dimensional results of semipositone problems, see [9–12].

However, the study of positive solutions for the semipositone Neumann problems (1.1)-(1.2) is relatively rare; see Miciano and Shivaji in [8], by using the quadrature method (time map method), they obtained the following result when f has the only unique positive zero.

Theorem A *Let* $f \in C^2[0,\infty)$, $f(0) < 0$, $f'(u) > 0$ *for* $u > 0$, $\lim_{u\to+\infty} f(u) > 0$, β *and* θ $(>\beta)$ *be the unique positive zero of* f *and* $F(s) = \int_0^s f(t)\,dt$, *respectively. Further, let* $S = (\frac{\pi^2}{f'(\beta)}, \frac{\theta^2}{-2F(\beta)})$ *be nonempty. Then for* $\lambda \in S$, *there exist at least three positive solutions of* (1.1) *and* (1.2).

The quadrature method (time map method) introduced by Laetsch in [13] has been used to find positive solutions to various types of equations; see [5, 7, 8] and the references therein. Moreover, applying the quadrature method, Brown and Budin [14] obtained multiple positive solutions for (1.1) with Dirichlet boundary conditions when f has n positive zeros and $f(0) > 0$, the detailed result can be found in [14], Theorem 3.8.

Motivated by the above papers [8, 14], this paper is devoted to studying how changes in the sign of f lead to multiple positive solutions of (1.1)-(1.2). We consider the following assumptions:

(f1) $f : [0,\infty) \to \mathbb{R}$ has continuous derivative.

(f2) $f(0) < 0$ (semipositone).

(f3) There exist $\beta_1, \beta_2, \ldots, \beta_n \in \mathbb{R}$ such that $\beta_1 < \beta_2 < \cdots < \beta_n$ and $f(\beta_i) = 0$ for $i = 1, 2, \ldots, n$.

(f4) There exist $\theta_1, \theta_2, \ldots, \theta_n \in \mathbb{R}$ such that $\theta_0 := 0 < \beta_1 < \theta_1 < \beta_2 < \cdots < \beta_n < \theta_n$ and $F(\theta_i) = 0$, for $i = 1, 2, \ldots, n$.

(f5) $f'(s) > 0$ for $s \in (\theta_{2k}, \theta_{2k+1})$, $k \in \{0, 1, \ldots, [\frac{n-1}{2}]\}$, where $[x]$ denotes the integer part of $x \in \mathbb{R}$.

We prove the following result.

Theorem 1.1 *Assume that* (f1)-(f5) *hold. For each* $k \in \{0, 1, \ldots, [\frac{n-1}{2}]\}$, *let* $S_k = (\frac{\pi^2}{f'(\beta_{2k+1})}, \frac{(\theta_{2k+1}-\theta_{2k})^2}{-2F(\beta_{2k+1})})$ *be nonempty. Then for each* $\lambda \in S_k$, (1.1)-(1.2) *has at least two nonconstant positive solutions and* $2k+1$ *constant positive solutions.*

Remark 1.1 Note that Theorem 1.1 is Theorem A in the special case $n = 1$. It is worthy to point out that in our study problem (1.1)-(1.2) admits n positive zeros, it may have some interest to investigate positive solutions.

In addition, Hung in [7] studied the exact multiplicity of positive solutions of (1.1) with Dirichlet boundary conditions when f satisfies (f2) and the following hypotheses:

(H1) $f \in C[0,\infty) \cap C^2(0,\infty)$.

(H2) f is concave-convex on $(0,\infty)$, *i.e.* f has a unique positive inflection point η such that

$$f''(s) \begin{cases} < 0, & \text{on } s \in (0,\eta), \\ = 0, & \text{when } s = \eta, \\ > 0, & \text{on } s \in (\eta,\infty). \end{cases}$$

(H3) f is asymptotic superlinear, that is, $\lim_{s\to\infty} \frac{f(s)}{s} = \infty$.

(H4) f has three distinct positive zeros $\beta_1 < \beta_2 < \beta_3$.

Hung in [7] obtained some significant results according to $F(\beta_2) = 0$, $F(\beta_2) > 0$ and $F(\beta_2) < 0$, the detailed results can be seen in [7], Theorem 2.2. The proofs of the main results are based on the quadrature method and variational techniques.

Naturally, what is really interesting is to find the conditions which permit positive solutions of Neumann problems (1.1)-(1.2). Motivated by the above papers [7, 8], by using the quadrature method, we obtain the following result.

Theorem 1.2 *Assume that* (f2), (H1), *and* (H4) *hold, and* f *satisfies* (H2) *with* $\beta_2 < \eta < \beta_3$. *Let* $F(\beta_2) < 0$ *and* θ *be the unique positive zero of* F. *Given* $\beta_* \in (0, \beta_1)$ *satisfying* $F(\beta_*) > F(\beta_2)$. *Then there exist* $0 < \lambda_* < \lambda^* < \infty$ ($\lambda_* = \lambda_*(\beta_1, \beta_2, \beta_3)$ *is relevant to* $\beta_1, \beta_2, \beta_3$, *and* $\lambda^* = \lambda^*(\beta_*)$ *is relevant to* β_*) *such that for* $\lambda \in [\lambda_*, \lambda^*]$, (1.1)-(1.2) *has exactly two nonconstant positive solutions and three constant positive solutions*.

Remark 1.2 Note that the case $F(\beta_2) > 0$ has been treated by Theorem 1.1. Moreover, we only prove in Theorem 1.1 that there exist multiple positive solutions for (1.1)-(1.2), and by virtue of [7], under appropriate conditions of the nonlinearity f, we also obtain the exact multiplicity of positive solutions of (1.1)-(1.2) in Theorem 1.2. However, the critical case $F(\beta_2) = 0$ is still open.

Remark 1.3 Let us consider the function

$$f(u) = (u-1)(u-2)(u-4).$$

Obviously f satisfies all of the conditions in Theorem 1.2 if we take $\beta_1 = 1$, $\beta_2 = 2$, $\beta_3 = 4$, $F(\beta_2) = F(2) = -\frac{8}{3} < 0$, $f''(\frac{7}{3}) = 0$, and $\eta = \frac{7}{3}$, *i.e.* $\beta_2 < \eta < \beta_3$. Thus Theorem 1.2 guarantees that (1.1)-(1.2) has exactly five positive solutions for $\lambda_* \le \lambda \le \lambda^*$.

The rest of the paper is arranged as follows: in Section 2, we state and prove several preliminary results related the use of quadrature method. Finally in Section 3, we prove the main results of this paper.

2 Preliminary results

Lemma 2.1 ([8]) *If* $u(x)$ *is a solution of* (1.1)-(1.2), *then* $u(1-x)$ *is also a solution of* (1.1)-(1.2).

Lemma 2.2 ([8]) *Any zero of* f *is a solution of* (1.1)-(1.2).

For each $k \in \{0, 1, \dots, [\frac{n-1}{2}]\}$. Now consider positive solutions $u(x)$ obtained by Theorem 1.1. Here $u(0) = \alpha_k$, $u(1) = \gamma_k$ ($\gamma_k = \gamma_k(\alpha_k)$) such that $F(\alpha_k) = F(\gamma_k)$, $\theta_{2k} \le \alpha_k < \beta_{2k+1} < \gamma_k \le \theta_{2k+1}$, and $u'' > 0$ on $(0, t_k)$ and $u'' < 0$ on $(t_k, 1)$, where $t_k \in (0,1)$ is such that $u(t_k) = \beta_{2k+1}$.

We now apply the quadrature technique to (1.1)-(1.2) when f has n positive zeros. Multiplying (1.1) by $u'(x)$ and integrating, we obtain

$$\frac{(u'(x))^2}{2} + \lambda F\bigl(u(x)\bigr) = C, \tag{2.1}$$

where C is a constant. Applying the boundary conditions and the assumption that $u(0) = \alpha_k$ and $u(1) = \gamma_k$ ($\gamma_k = \gamma_k(\alpha_k)$), we see that $\lambda F(\alpha_k) = C = \lambda F(\gamma_k)$. Hence for each $k \in$

$\{0,1,\ldots,[\frac{n-1}{2}]\}$, $\alpha_k \in [\theta_{2k},\beta_{2k+1})$ such that $u(0)=\alpha_k$, $\gamma_k \in (\beta_{2k+1},\theta_{2k+1}]$ is the unique solution of $F(\alpha_k)=F(\gamma_k)$. Thus if $u(0)=\alpha_k$, (2.1) becomes

$$u'(x)=\sqrt{2\lambda\bigl[F(\alpha_k)-F\bigl(u(x)\bigr)\bigr]},\quad x\in[0,1], \tag{2.2}$$

integrating (2.1) on $[0,x]$ and applying the boundary conditions, we have

$$\int_{\alpha_k}^{u(x)}\frac{ds}{\sqrt{F(\alpha_k)-F(s)}}=\sqrt{2\lambda}x,\quad x\in[0,1]. \tag{2.3}$$

Substituting $x=1$ in (2.3), it follows that

$$\sqrt{\lambda}=\frac{1}{\sqrt{2}}\int_{\alpha_k}^{\gamma_k}\frac{ds}{\sqrt{F(\alpha_k)-F(s)}}=:G(\alpha_k). \tag{2.4}$$

In fact, the following result is true.

Theorem 2.1 *Assume that* (f1)-(f5) *hold. For each* $k\in\{0,1,\ldots,[\frac{n-1}{2}]\}$. *Given* $\lambda>0$ *if there exists* $\alpha_k\in[\theta_{2k},\beta_{2k+1})$ *such that* $G(\alpha_k)=\sqrt{\lambda}$, *then* (1.1)-(1.2) *has a unique positive solution* $u(x)$ *satisfying* $u(0)=\alpha_k$, $u(1)=\gamma_k$, *where* γ_k *is such that* $F(\alpha_k)=F(\gamma_k)$ *and* $u'>0$ *on* $(0,1)$. *Furthermore,* $G(\alpha_k)$ *is a continuous and differentiable function in* $[\theta_{2k},\beta_{2k+1})$. *Its derivative is given by*

$$\begin{aligned}\frac{dG(\alpha_k)}{d\alpha_k}&=-\frac{1}{2\sqrt{2}}\int_0^1\frac{H(\alpha_k)-H(t(\beta_{2k+1}-\alpha_k)+\alpha_k)}{[F(\alpha_k)-F(t(\beta_{2k+1}-\alpha_k)+\alpha_k)]^{3/2}}\,dt\\&\quad+\left(\frac{d\gamma_k}{d\alpha_k}\right)\frac{1}{2\sqrt{2}}\int_0^1\frac{H(\gamma_k)-H(t(\beta_{2k+1}-\gamma_k)+\gamma_k)}{[F(\gamma_k)-F(t(\beta_{2k+1}-\gamma_k)+\gamma_k)]^{3/2}}\,dt,\end{aligned} \tag{2.5}$$

where $H(s)=2F(s)+(\beta_{2k+1}-s)f(s)$.

Proof The proof of Theorem 2.1 is the similar argument to that in Miciano and Shivaji [8] and Miciano [15] with obvious changes, so we omit it. □

3 Proofs of main results

In the section, we shall use the preliminary results in the previous section to prove Theorems 1.1-1.2.

Proof of Theorem 1.1 For each $k\in\{0,1,\ldots,[\frac{n-1}{2}]\}$. Recall the definition of θ_{2k+1}, we replace f in (1.1) by

$$f_k(u)=\begin{cases}f(u), & \text{if } 0\le u\le\theta_{2k+1},\\ f'(\theta_{2k+1})(u-\theta_{2k+1})+f(\theta_{2k+1}), & \text{if } u>\theta_{2k+1},\end{cases} \tag{3.1}$$

and $F_k(u)=\int_0^u f_k(s)\,ds$.

If $k=0$, it is shown in Miciano and Shivaji [8] that

$$\frac{\theta_1^2}{-2F_0(\beta_1)}\le\bigl[G(0)\bigr]^2\le\frac{2\theta_1^2}{-F_0(\beta_1)} \tag{3.2}$$

and

$$\frac{4}{f_0'(\beta_1)} \le \Bigl[\lim_{\alpha_0\to\beta_1^-} G(\alpha_0)\Bigr]^2 \le \frac{16}{f_0'(\beta_1)}, \tag{3.3}$$

where $\lim_{\alpha_0\to 0^+} G(\alpha_0) =: G(0)$.

Since the only possible bifurcation points on the curve of solution (λ,β_1) are $\frac{\pi^2 m^2}{f_0'(\beta_1)}$, $m = 0,1,\ldots$, by (3.3), it follows that $[\lim_{\alpha_0\to\beta_1^-} G(\alpha_0)]^2 = \frac{\pi^2 m^2}{f_0'(\beta_1)}$. From the definition of f_0, we know that $f_0'(\beta_1) = f'(\beta_1)$ and $F_0(\beta_1) = F(\beta_1)$. One deduces from the hypothesis $\frac{\pi^2}{f'(\beta_1)} < \frac{\theta_1^2}{-2F(\beta_1)}$ that the range of $[G(\alpha_0)]^2$ contains $S_0 = (\frac{\pi^2}{f'(\beta_1)}, \frac{\theta_1^2}{-2F(\beta_1)})$. Consequently, from Theorem 2.1, Lemma 2.1, and Lemma 2.2, the problem (1.1)-(1.2) with $k = 0$ has three distinct positive solutions as $\lambda \in S_0$.

Next, we prove that, for each $k \in \{1,\ldots,[\frac{n-1}{2}]\}$,

$$\frac{(\theta_{2k+1}-\theta_{2k})^2}{-2F_k(\beta_{2k+1})} \le \Bigl[\lim_{\alpha_k\to\theta_{2k}^+} G(\alpha_k)\Bigr]^2 \le \frac{2(\theta_{2k+1}-\theta_{2k})^2}{-F_k(\beta_{2k+1})} \tag{3.4}$$

and

$$\frac{4}{f_k'(\beta_{2k+1})} \le \Bigl[\lim_{\alpha_k\to\beta_{2k+1}^-} G(\alpha_k)\Bigr]^2 \le \frac{16}{f_k'(\beta_{2k+1})}. \tag{3.5}$$

Without loss of generality, we consider $k = 1$, and $\alpha_1 \in [\theta_2,\beta_3)$, $\gamma_1 \in (\beta_3,\theta_3]$, from (2.4),

$$G(\alpha_1) = \frac{1}{\sqrt{2}}\int_{\alpha_1}^{\gamma_1} \frac{ds}{\sqrt{F_1(\alpha_1)-F_1(s)}}. \tag{3.6}$$

Then

$$\lim_{\alpha_1\to\theta_2^+} G(\alpha_1) = \frac{1}{\sqrt{2}}\int_{\theta_2}^{\theta_3} \frac{ds}{\sqrt{-F_1(s)}} =: G(\theta_2). \tag{3.7}$$

By (f5), $F_1''(s) = f_1'(s) > 0$ for $s \in (\theta_2,\theta_3)$, we have

$$-F_1(s) \ge \begin{cases} \frac{-F_1(\beta_3)}{\beta_3-\theta_2}(s-\theta_2), & \theta_2 < s \le \beta_3, \\ \frac{-F_1(\beta_3)}{\theta_3-\beta_3}(\theta_3-s), & \beta_3 < s < \theta_3, \end{cases}$$

and so

$$\frac{1}{\sqrt{-F_1(s)}} \le \begin{cases} \sqrt{\frac{\beta_3-\theta_2}{-F_1(\beta_3)(s-\theta_2)}}, & \theta_2 < s \le \beta_3, \\ \sqrt{\frac{\theta_3-\beta_3}{-F_1(\beta_3)(\theta_3-s)}}, & \beta_3 < s < \theta_3. \end{cases}$$

Thus, from (3.7), it follows that

$$\begin{aligned} G(\theta_2) &= \frac{1}{\sqrt{2}}\int_{\theta_2}^{\beta_3} \frac{ds}{\sqrt{-F_1(s)}} + \frac{1}{\sqrt{2}}\int_{\beta_3}^{\theta_3} \frac{ds}{\sqrt{-F_1(s)}} \\ &\le \frac{1}{\sqrt{2}}\sqrt{\frac{\beta_3-\theta_2}{-F_1(\beta_3)}}\,2\sqrt{\beta_3-\theta_2} + \frac{1}{\sqrt{2}}\sqrt{\frac{\theta_3-\beta_3}{-F_1(\beta_3)}}\,2\sqrt{\theta_3-\beta_3} \\ &\le \frac{\sqrt{2}(\theta_3-\theta_2)}{\sqrt{-F_1(\beta_3)}}. \end{aligned} \tag{3.8}$$

Also, we see that

$$-F_1(s) \le -F_1(\beta_3), \quad s \in [\theta_2, \theta_3].$$

Hence

$$G(\theta_2) = \frac{1}{\sqrt{2}} \int_{\theta_2}^{\theta_3} \frac{ds}{\sqrt{-F_1(s)}} \ge \frac{1}{\sqrt{2}} \int_{\theta_2}^{\theta_3} \frac{ds}{\sqrt{-F_1(\beta_3)}} = \frac{1}{\sqrt{2}} \frac{\theta_3 - \theta_2}{\sqrt{-F_1(\beta_3)}}. \tag{3.9}$$

Then from (3.8) and (3.9), it follows that

$$\frac{(\theta_3 - \theta_2)^2}{-2F_1(\beta_3)} \le \big[G(\theta_2)\big]^2 \le \frac{2(\theta_3 - \theta_2)^2}{-F_1(\beta_3)}.$$

Thus (3.4) holds as $k = 1$.

On the other hand, it is easy to verify that

$$G(\alpha_1) = \frac{1}{\sqrt{2}} \int_{\alpha_1}^{\gamma_1} \frac{ds}{\sqrt{F_1(\alpha_1) - F_1(s)}} \le \frac{1}{\sqrt{2}} \int_{\alpha_1}^{\gamma_1} \frac{ds}{\sqrt{Z_1(s)}},$$

where $Z_1(s)$ is defined as

$$Z_1(s) = \begin{cases} \frac{F_1(\alpha_1) - F_1(\beta_3)}{\beta_3 - \alpha_1}(s - \alpha_1), & \alpha_1 < s < \beta_3, \\ \frac{F_1(\gamma_1) - F_1(\beta_3)}{\gamma_1 - \beta_3}(\gamma_1 - s), & \beta_3 < s < \gamma_1. \end{cases}$$

Furthermore,

$$\begin{aligned} \int_{\alpha_1}^{\gamma_1} \frac{ds}{\sqrt{Z_1(s)}} &= \sqrt{\frac{\beta_3 - \alpha_1}{F_1(\alpha_1) - F_1(\beta_3)}} \int_{\alpha_1}^{\beta_3} \frac{ds}{\sqrt{s - \alpha_1}} + \sqrt{\frac{\gamma_1 - \beta_3}{F_1(\gamma_1) - F_1(\beta_3)}} \int_{\beta_3}^{\gamma_1} \frac{ds}{\sqrt{\gamma_1 - s}} \\ &= \sqrt{\frac{\beta_3 - \alpha_1}{F_1(\alpha_1) - F_1(\beta_3)}} 2\sqrt{\beta_3 - \alpha_1} + \sqrt{\frac{\gamma_1 - \beta_3}{F_1(\gamma_1) - F_1(\beta_3)}} 2\sqrt{\gamma_1 - \beta_3} \\ &= \frac{2(\beta_3 - \alpha_1)}{\sqrt{F_1(\alpha_1) - F_1(\beta_3)}} + \frac{2(\gamma_1 - \beta_3)}{\sqrt{F_1(\gamma_1) - F_1(\beta_3)}}. \end{aligned} \tag{3.10}$$

Now as $\alpha_1 \to \beta_3^-$, $\gamma_1 \to \beta_3^+$,

$$\lim_{\alpha_1 \to \beta_3^-} \frac{4(\beta_3 - \alpha_1)^2}{F_1(\alpha_1) - F_1(\beta_3)} = \frac{8}{f_1'(\beta_3)} \quad \text{and} \quad \lim_{\gamma_1 \to \beta_3^+} \frac{4(\gamma_1 - \beta_3)^2}{F_1(\gamma_1) - F_1(\beta_3)} = \frac{8}{f_1'(\beta_3)}. \tag{3.11}$$

Hence (3.10) and (3.11) imply that

$$\lim_{\alpha_1 \to \beta_3^-} G(\alpha_1) \le \lim_{\alpha_1 \to \beta_3^-} \frac{1}{\sqrt{2}} \int_{\alpha_1}^{\gamma_1} \frac{ds}{\sqrt{Z_1(s)}} = \frac{4}{\sqrt{f_1'(\beta_3)}}. \tag{3.12}$$

Also from (2.4), we have

$$\begin{aligned} G(\alpha_1) &= \frac{1}{\sqrt{2}} \int_{\alpha_1}^{\gamma_1} \frac{ds}{\sqrt{F_1(\alpha_1) - F_1(s)}} \\ &= \frac{1}{\sqrt{2}} \int_{\alpha_1}^{\beta_3} \frac{ds}{\sqrt{F_1(\alpha_1) - F_1(s)}} + \frac{1}{\sqrt{2}} \int_{\beta_3}^{\gamma_1} \frac{ds}{\sqrt{F_1(\gamma_1) - F_1(s)}} \end{aligned}$$

$$\geq \frac{1}{\sqrt{2}}\int_{\alpha_1}^{\beta_3}\frac{ds}{\sqrt{F_1(\alpha_1)-F_1(\beta_3)}}+\frac{1}{\sqrt{2}}\int_{\beta_3}^{\gamma_1}\frac{ds}{\sqrt{F_1(\gamma_1)-F_1(\beta_3)}}$$
$$=\frac{1}{\sqrt{2}}\frac{\beta_3-\alpha_1}{\sqrt{F_1(\alpha_1)-F_1(\beta_3)}}+\frac{1}{\sqrt{2}}\frac{\gamma_1-\beta_3}{\sqrt{F_1(\gamma_1)-F_1(\beta_3)}},$$

and as $\alpha_1 \to \beta_3^-$, using (3.11), we obtain

$$\lim_{\alpha_1\to\beta_3^-} G(\alpha_1) \geq \frac{2}{\sqrt{f_1'(\beta_3)}}. \tag{3.13}$$

Then from (3.12) and (3.13), we get

$$\frac{2}{\sqrt{f_1'(\beta_3)}} \leq \lim_{\alpha_1\to\beta_3^-} G(\alpha_1) \leq \frac{4}{\sqrt{f_1'(\beta_3)}},$$

and so

$$\frac{4}{f_1'(\beta_3)} \leq \Big[\lim_{\alpha_1\to\beta_3^-} G(\alpha_1)\Big]^2 \leq \frac{16}{f_1'(\beta_3)}.$$

Thus (3.5) holds as $k = 1$.

By a similar argument to the case $k = 1$ with obvious changes, we can deduce that, for each $k \in \{2,\dots,[\frac{n-1}{2}]\}$, (3.4) and (3.5) hold.

Moreover, for each $k \in \{1,\dots,[\frac{n-1}{2}]\}$, since the only possible bifurcation points on the curve of solutions (λ,β_{2k+1}) are $\frac{\pi^2 m^2}{f_k'(\beta_{2k+1})}$, $m = 0,1,\dots$, by (3.5), it follows that

$$\Big[\lim_{\alpha_k\to\beta_{2k+1}^-} G(\alpha_k)\Big]^2 = \frac{\pi^2 m^2}{f_k'(\beta_{2k+1})}.$$

From the definition of f_k, we know that $f_k'(\beta_{2k+1}) = f'(\beta_{2k+1})$ and $F_k(\beta_{2k+1}) = F(\beta_{2k+1})$. One deduces from our hypothesis $\frac{\pi^2}{f'(\beta_{2k+1})} < \frac{(\theta_{2k+1}-\theta_{2k})^2}{-2F(\beta_{2k+1})}$ that the range of $[G(\alpha_k)]^2$ contains $S_k = (\frac{\pi^2}{f'(\beta_{2k+1})}, \frac{(\theta_{2k+1}-\theta_{2k})^2}{-2F(\beta_{2k+1})})$. Consequently, by Lemma 2.2, f_k has $2k+1$ positive zeros such that (1.1)-(1.2) has $2k+1$ positive solutions, moreover, by Theorem 2.1 and Lemma 2.1, for $\lambda \in S_k$, $u(x)$ and $u(1-x)$ are two other positive solutions for (1.1)-(1.2).

The proof of Theorem 1.1 is now complete. □

Remark 3.1 Note that since f is autonomous, every solution of (1.1)-(1.2) is symmetric about its critical points (see Miciano and Shivaji [8], Lemma 2.2). Hence if positive solutions v with $m-1$ interior critical points at $\frac{i}{m}$, $i = 1,2,\dots,m-1$, it suffices to study solutions $v_m(x) := v|_{x\in[0,\frac{1}{m}]}$, let $v_m(x) = u(mx)$ for $x \in [0,\frac{1}{m}]$. Together with Theorem 1.1, we can easily get similar results to Theorem A, here we omit them.

Next we consider positive solutions $u(x)$ obtained by Theorem 1.2. Here $u(0) = \alpha$, $u(1) = \gamma$ ($\gamma = \gamma(\alpha)$) such that $F(\alpha) = F(\gamma)$, $0 \leq \alpha \leq \beta_* < \gamma \leq \theta$, $u'' > 0$ on $(0,t_1)\cup(t_2,t_3)$ and $u'' < 0$ on $(t_1,t_2)\cup(t_3,1)$, where $0 < t_1 < t_2 < t_3 < 1$ are such that $u(t_k) = \beta_k$, $k = 1,2,3$.

Before proving Theorem 1.2, we now show some preliminary results. Given $\beta_* \in (0,\beta_1)$ satisfying $F(\beta_*) > F(\beta_2)$, there exists $\beta^* \in (\beta_3,\theta)$ such that $F(\beta_*) = F(\beta^*)$. Let $\alpha \in [0,\beta_*]$,

$\gamma \in [\beta^*, \theta]$ such that $u(0) = \alpha$, $u(1) = \gamma$. From (2.1), we see that $\lambda F(\alpha) = C = \lambda F(\gamma)$, by a similar argument to (2.4) with obvious changes, we have

$$\sqrt{\lambda} = \frac{1}{\sqrt{2}} \int_\alpha^\gamma \frac{ds}{\sqrt{F(\alpha) - F(s)}} =: G(\alpha). \tag{3.14}$$

Next, we give an important result.

Theorem 3.1 *Let f satisfy all hypotheses of Theorem* 1.2. *Given $\lambda > 0$ if there exists $\alpha \in [0, \beta_*]$ such that $G(\alpha) = \sqrt{\lambda}$, then* (1.1)-(1.2) *has a unique positive solution $u(x)$ satisfying $u(0) = \alpha$, $u(1) = \gamma$, where γ is such that $F(\alpha) = F(\gamma)$ and $u' > 0$ on $(0,1)$. Moreover, $G(\alpha)$ is a continuous and differentiable function in $[0, \beta_*]$, its derivative is given by*

$$\begin{aligned} \frac{dG(\alpha)}{d\alpha} = &-\frac{1}{2\sqrt{2}} \int_0^1 \frac{H_1(\alpha) - H_1(t(\beta_1 - \alpha) + \alpha)}{[F(\alpha) - F(t(\beta_1 - \alpha) + \alpha)]^{3/2}}\, dt \\ &- \frac{1}{2\sqrt{2}} \int_{\beta_1}^{\beta_3} \frac{f(\alpha)}{[F(\alpha) - F(t)]^{3/2}}\, dt \\ &+ \frac{d\gamma}{d\alpha} \frac{1}{2\sqrt{2}} \int_0^1 \frac{H_2(\gamma) - H_2(t(\beta_3 - \gamma) + \gamma)}{[F(\gamma) - F(t(\beta_3 - \gamma) + \gamma)]^{3/2}}\, dt > 0, \end{aligned} \tag{3.15}$$

where $H_1(s) = 2F(s) + (\beta_1 - s)f(s)$ and $H_2(s) = 2F(s) + (\beta_3 - s)f(s)$.

Proof Obviously, for given $\lambda > 0$, there exists $\alpha \in [0, \beta_*]$ such that $G(\alpha) = \sqrt{\lambda}$, it follows that (1.1)-(1.2) has a unique positive solution $u(x)$ given by

$$\int_\alpha^{u(x)} \frac{ds}{\sqrt{F(\alpha) - F(s)}} = \sqrt{2\lambda} x, \quad x \in [0,1], \tag{3.16}$$

and $u(0) = \alpha$, $u(1) = \gamma$. Moreover, it is easy to see that $G(\alpha)$ is a continuous and differentiable function in $[0, \beta_*]$. Now put

$$\begin{aligned} \int_\alpha^\gamma \frac{ds}{\sqrt{F(\alpha) - F(s)}} = &\int_\alpha^{\beta_1} \frac{ds}{\sqrt{F(\alpha) - F(s)}} + \int_{\beta_1}^{\beta_3} \frac{ds}{\sqrt{F(\alpha) - F(s)}} \\ &+ \int_{\beta_3}^\gamma \frac{ds}{\sqrt{F(\gamma) - F(s)}}. \end{aligned} \tag{3.17}$$

Thus

$$\frac{d}{d\alpha} \int_{\beta_1}^{\beta_3} \frac{ds}{\sqrt{F(\alpha) - F(s)}} = -\frac{1}{2} \int_{\beta_1}^{\beta_3} \frac{f(\alpha)\, ds}{[F(\alpha) - F(s)]^{3/2}} > 0. \tag{3.18}$$

In fact, $f(\alpha) < 0$ for $\alpha \in [0, \beta_*]$.

Let $s = t(\beta_1 - \alpha) + \alpha$, then

$$\int_\alpha^{\beta_1} \frac{ds}{\sqrt{F(\alpha) - F(s)}} = \int_0^1 \frac{(\beta_1 - \alpha)\, dt}{\sqrt{F(\alpha) - F(t(\beta_1 - \alpha) + \alpha)}},$$

and

$$\frac{d}{d\alpha} \int_0^1 \frac{(\beta_1 - \alpha)\, dt}{\sqrt{F(\alpha) - F(t(\beta_1 - \alpha) + \alpha)}} = -\frac{1}{2} \int_0^1 \frac{H_1(\alpha) - H_1[t(\beta_1 - \alpha) + \alpha]}{[F(\alpha) - F(t(\beta_1 - \alpha) + \alpha)]^{3/2}}\, dt \geq 0, \tag{3.19}$$

where $H_1(s) = 2F(s) + (\beta_1 - s)f(s)$ for $s \in (\alpha, \beta_1)$. Indeed, $H_1'(s) = f(s) + (\beta_1 - s)f'(s)$, $H_1''(s) = (\beta_1 - s)f''(s) < 0$ since $f''(s) < 0$, $s \in (\alpha, \beta_1)$, and $H_1'(\beta_1) = 0$, so $H_1'(s) \geq 0$, that is, $H_1(\alpha) - H_1(t(\beta_1 - \alpha) + \alpha) \leq 0$, for $t \in (0,1)$. Then (3.19) holds.

Let $s = t(\beta_3 - \gamma) + \gamma$, then

$$\int_{\beta_3}^{\gamma} \frac{ds}{\sqrt{F(\gamma) - F(s)}} = \int_0^1 \frac{(\gamma - \beta_3)\,dt}{\sqrt{F(\gamma) - F(t(\beta_3 - \gamma) + \gamma)}}$$

and

$$\begin{aligned} &\frac{d}{d\alpha} \int_0^1 \frac{(\gamma - \beta_3)\,dt}{\sqrt{F(\gamma) - F[t(\beta_3 - \gamma) + \gamma]}} \\ &\quad = \frac{d\gamma}{d\alpha} \frac{1}{2} \int_0^1 \frac{H_2(\gamma) - H_2[t(\beta_3 - \gamma) + \gamma]}{[F(\gamma) - F(t(\beta_3 - \gamma) + \gamma)]^{3/2}}\,dt \geq 0, \end{aligned} \tag{3.20}$$

where $H_2(s) = 2F(s) + (\beta_3 - s)f(s)$ for $s \in (\beta_3, \gamma)$. In fact, $H_2'(s) = f(s) + (\beta_3 - s)f'(s)$, $H_2''(s) = (\beta_3 - s)f''(s) < 0$ since $f''(s) > 0$, $s \in (\beta_3, \gamma)$, and $H_2'(\beta_3) = 0$, so $H_2'(s) \leq 0$, that is, $H_2(\gamma) - H_2(t(\beta_3 - \gamma) + \gamma) \leq 0$, for $t \in (0,1)$. But we know that $\frac{d\gamma}{d\alpha} < 0$. Then (3.20) holds.

Combining (3.14) with (3.17)-(3.20), it follows that

$$\frac{d}{d\alpha} G(\alpha) = \frac{d}{d\alpha} \frac{1}{\sqrt{2}} \int_\alpha^\gamma \frac{ds}{\sqrt{F(\alpha) - F(s)}} > 0.$$

This completes the proof of the theorem. □

Proof of Theorem 1.2 From (3.14), we have

$$\lim_{\alpha \to 0^+} G(\alpha) = \frac{1}{\sqrt{2}} \int_0^\theta \frac{ds}{\sqrt{-F(s)}} =: G(0) \tag{3.21}$$

and

$$-F(s) \leq \max\{-F(\beta_1), -F(\beta_3)\}, \quad s \in [0, \theta].$$

Hence

$$G(0) = \frac{1}{\sqrt{2}} \int_0^\theta \frac{ds}{\sqrt{-F(s)}} \geq \frac{1}{\sqrt{2}} \frac{\theta}{\sqrt{\max\{-F(\beta_1), -F(\beta_3)\}}}. \tag{3.22}$$

Since $F''(s) = f'(s) > 0$, for $0 < s \leq \beta_1$ and $\beta_3 < s < \theta$, we get

$$-F(s) \geq \begin{cases} \frac{-F(\beta_1)}{\beta_1} s, & 0 < s \leq \beta_1, \\ -F(\beta_2), & \beta_1 < s \leq \beta_3, \\ \frac{-F(\beta_3)}{\theta - \beta_3}(\theta - s), & \beta_3 < s < \theta. \end{cases}$$

Thus

$$\frac{1}{\sqrt{-F(s)}} \leq \begin{cases} \frac{\sqrt{\beta_1}}{\sqrt{-F(\beta_1)}} \frac{1}{\sqrt{s}}, & 0 < s \leq \beta_1, \\ \frac{1}{\sqrt{-F(\beta_2)}}, & \beta_1 < s \leq \beta_3, \\ \frac{\sqrt{\theta - \beta_3}}{\sqrt{-F(\beta_3)}} \frac{1}{\sqrt{\theta - s}}, & \beta_3 < s < \theta. \end{cases}$$

By virtue of (3.21), we obtain

$$\begin{aligned} G(0) &\le \frac{1}{\sqrt{2}} \int_0^{\beta_1} \frac{\sqrt{\beta_1}}{\sqrt{-F(\beta_1)}} \frac{ds}{\sqrt{s}} + \frac{1}{\sqrt{2}} \int_{\beta_1}^{\beta_3} \frac{ds}{\sqrt{-F(\beta_2)}} + \frac{1}{\sqrt{2}} \int_{\beta_3}^{\theta} \frac{\sqrt{\theta-\beta_3}}{\sqrt{-F(\beta_3)}} \frac{ds}{\sqrt{\theta - s}} \\ &= \frac{\sqrt{2}\beta_1}{\sqrt{-F(\beta_1)}} + \frac{1}{\sqrt{2}} \frac{\beta_3 - \beta_1}{\sqrt{-F(\beta_2)}} + \frac{\sqrt{2}(\theta - \beta_3)}{\sqrt{-F(\beta_3)}} \\ &< \frac{\sqrt{2}\theta}{\sqrt{-F(\beta_2)}}. \end{aligned} \tag{3.23}$$

Then from (3.22)-(3.23) we have

$$\frac{\theta^2}{2\max\{-F(\beta_1), -F(\beta_3)\}} \le \bigl[G(0)\bigr]^2 < \frac{2\theta^2}{-F(\beta_2)}. \tag{3.24}$$

Since

$$\lim_{\alpha \to \beta_*^-} G(\alpha) = \frac{1}{\sqrt{2}} \int_{\beta_*}^{\beta^*} \frac{ds}{\sqrt{F(\beta_*) - F(s)}}, \tag{3.25}$$

where β^* satisfies $F(\beta^*) = F(\beta_*)$. Hence

$$\begin{aligned} \frac{1}{\sqrt{2}} \int_{\beta_*}^{\beta^*} \frac{ds}{\sqrt{F(\beta_*) - F(s)}} &= \frac{1}{\sqrt{2}} \int_{\beta_*}^{\beta_2} \frac{ds}{\sqrt{F(\beta_*) - F(s)}} + \frac{1}{\sqrt{2}} \int_{\beta_2}^{\beta^*} \frac{ds}{\sqrt{F(\beta_*) - F(s)}} \\ &\ge \frac{1}{\sqrt{2}} \frac{\beta_2 - \beta_*}{\sqrt{F(\beta_*) - F(\beta_1)}} + \frac{1}{\sqrt{2}} \frac{\beta^* - \beta_2}{\sqrt{F(\beta_*) - F(\beta_3)}} \\ &\ge \frac{1}{\sqrt{2}} \frac{\beta^* - \beta_*}{\sqrt{F(\beta_*) + \max\{-F(\beta_1), -F(\beta_3)\}}}. \end{aligned} \tag{3.26}$$

On the other hand,

$$\begin{aligned} \int_{\beta_*}^{\beta^*} \frac{ds}{\sqrt{F(\beta_*) - F(s)}} &= \int_{\beta_*}^{\underline{\beta}} \frac{ds}{\sqrt{F(\beta_*) - F(s)}} \\ &\quad + \int_{\underline{\beta}}^{\overline{\beta}} \frac{ds}{\sqrt{F(\beta_*) - F(s)}} + \int_{\overline{\beta}}^{\beta^*} \frac{ds}{\sqrt{F(\beta^*) - F(s)}}, \end{aligned} \tag{3.27}$$

where $\beta_* < \underline{\beta} < \beta_2 < \overline{\beta} < \beta^*$ such that $F(\underline{\beta}) = F(\overline{\beta}) = F(\beta_2)$. We can easily get

$$\int_{\underline{\beta}}^{\overline{\beta}} \frac{ds}{\sqrt{F(\beta_*) - F(s)}} \le \frac{\overline{\beta} - \underline{\beta}}{\sqrt{F(\beta_*) - F(\beta_2)}}. \tag{3.28}$$

Since $F''(s) = f'(s) > 0$ for $s \in (\beta_*, \underline{\beta})$ and $s \in (\overline{\beta}, \beta^*)$, we have

$$F(\beta_*) - F(s) \ge \frac{F(\beta_*) - F(\underline{\beta})}{\underline{\beta} - \beta_*}(s - \beta_*), \quad s \in (\beta_*, \underline{\beta}), \tag{3.29}$$

and

$$F\bigl(\beta^*\bigr) - F(s) \ge \frac{F(\beta^*) - F(\overline{\beta})}{\beta^* - \overline{\beta}}\bigl(\beta^* - s\bigr), \quad s \in \bigl(\overline{\beta}, \beta^*\bigr). \tag{3.30}$$

Thus

$$\begin{aligned}
&\int_{\beta_*}^{\underline{\beta}} \frac{ds}{\sqrt{F(\beta_*)-F(s)}} + \int_{\overline{\beta}}^{\beta^*} \frac{ds}{\sqrt{F(\beta^*)-F(s)}} \\
&\quad \le \frac{\sqrt{\underline{\beta}-\beta_*}}{\sqrt{F(\beta_*)-F(\underline{\beta})}} \int_{\beta_*}^{\underline{\beta}} \frac{1}{\sqrt{s-\beta_*}} + \frac{\sqrt{\beta^*-\overline{\beta}}}{\sqrt{F(\beta^*)-F(\overline{\beta})}} \int_{\overline{\beta}}^{\beta^*} \frac{1}{\sqrt{\beta^*-s}} \\
&\quad = \frac{2(\underline{\beta}-\beta_*)}{\sqrt{F(\beta_*)-F(\underline{\beta})}} + \frac{2(\beta^*-\overline{\beta})}{\sqrt{F(\beta^*)-F(\overline{\beta})}}. \qquad (3.31)
\end{aligned}$$

Consequently, from (3.27), (3.28), and (3.31), we obtain

$$\begin{aligned}
\frac{1}{\sqrt{2}} \int_{\beta_*}^{\beta^*} \frac{ds}{\sqrt{F(\beta_*)-F(s)}} &= \frac{1}{\sqrt{2}} \int_{\beta_*}^{\underline{\beta}} \frac{ds}{\sqrt{F(\beta_*)-F(s)}} + \frac{1}{\sqrt{2}} \int_{\underline{\beta}}^{\overline{\beta}} \frac{ds}{\sqrt{F(\beta_*)-F(s)}} \\
&\quad + \frac{1}{\sqrt{2}} \int_{\overline{\beta}}^{\beta^*} \frac{ds}{\sqrt{F(\beta^*)-F(s)}} \le \frac{1}{\sqrt{2}} \frac{2(\underline{\beta}-\beta_*)}{\sqrt{F(\beta_*)-F(\underline{\beta})}} \\
&\quad + \frac{1}{\sqrt{2}} \frac{\overline{\beta}-\underline{\beta}}{\sqrt{F(\beta_*)-F(\beta_2)}} + \frac{1}{\sqrt{2}} \frac{2(\beta^*-\overline{\beta})}{\sqrt{F(\beta^*)-F(\overline{\beta})}} \\
&= \frac{2(\beta^*-\beta_*)-(\overline{\beta}-\underline{\beta})}{\sqrt{2}\sqrt{F(\beta_*)-F(\beta_2)}}. \qquad (3.32)
\end{aligned}$$

Combining this with (3.26), we get

$$\frac{(\beta^*-\beta_*)^2}{2[F(\beta_*)+\max\{-F(\beta_1),-F(\beta_3)\}]} \le \Bigl[\lim_{\alpha\to\beta_*^-} G(\alpha)\Bigr]^2 \le \frac{[2(\beta^*-\beta_*)-(\overline{\beta}-\underline{\beta})]^2}{2[F(\beta_*)-F(\beta_2)]}. \qquad (3.33)$$

Finally, by Theorem 3.1, it follows that $[\lim_{\alpha\to0^+} G(\alpha)]^2 < [\lim_{\alpha\to\beta_*^-} G(\alpha)]^2$, we let

$$\lambda_* = \Bigl[\lim_{\alpha\to0^+} G(\alpha)\Bigr]^2, \qquad \lambda^* = \Bigl[\lim_{\alpha\to\beta_*^-} G(\alpha)\Bigr]^2,$$

and λ_*, λ^* satisfy the inequality of (3.24) and (3.33), respectively. By Lemma 2.1 and Lemma 2.2, (1.1)-(1.2) has exactly five distinct positive solutions.

The proof of Theorem 1.2 is now complete. □

Remark 3.2 To the best of our knowledge, positive solutions obtained in Theorem 1.2 are of a new shape since its convexity and concavity changes more than once.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally to this paper. All authors read and approved the final manuscript.

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