Liang et al. *Boundary Value Problems* (2015) 2015:73
DOI 10.1186/s13661-015-0338-0

Boundary Value Problems
a SpringerOpen Journal

RESEARCH

Open Access

# On second order nonlinear boundary value problems and the distributional Henstock-Kurzweil integral

Bing Liang*, Guoju Ye, Wei Liu and Hongmei Wang

*Correspondence:
bingliang0212@163.com
College of Science, Hohai University, Nanjing, 210098, P.R. China

**Abstract**
In the present paper, we investigate the existence of solutions to second order nonlinear boundary value problems (BVPs) involving the distributional Henstock-Kurzweil integral. The present results in this article are generalizations of previous results in the literature.

**Keywords:** distributional Henstock-Kurzweil integral; nonlinear boundary value problems; distributional derivative; Schauder's fixed point theorem

## 1 Introduction

New existence results are derived for solutions of the second order differential equation

$$-x''(t) = f\bigl(t, x(t)\bigr), \quad t \in [0,1], \tag{1.1}$$

subject to the boundary conditions

$$x'(0) = 0, \qquad \beta x'(1) + x(\eta) = 0, \tag{1.2}$$

where $x''$, $x'$ stand for the distributional derivative of the function $x \in C[0,1]$, $C[0,1]$ denotes the space where the functions $x : [0,1] \to \mathbb{R}$ are continuous, $f$ is a distribution (generalized function), $\beta$ is a positive constant and $\eta \in [0,1]$. The space $C[0,1]$ is considered with the uniform norm $\|\cdot\|_\infty$.

In recent years, the existence of solutions of boundary value problems have been studied by many authors [1–4]. Chew and Flordeliza, in [5], generalized the classical Carathéodory's existence theorem on the Cauchy problem $x' = f(t,x)$ with $x(0) = 0$. Particularly, in [6, 7], the differential equations involving the approximate derivatives are considered. BVP (1.1)-(1.2), which has been studied in [4] by using ordinary derivatives, came from the steady-state of a heat bar model. The boundary conditions model the behavior of a thermostat where the sensor measures the temperature. The heat bar is insulated at $t = 0$, and the controller releases heat at $t = 1$ depending on the temperature at $t = \eta$. However, it is well known that the notion of a distributional derivative is very general, including ordinary derivatives and approximate derivatives. Without loss of generality, we use the distributional derivatives to discuss (1.1)-(1.2) in a general form. The existence result obtained under weaker conditions extends the previous results in the literatures.

This paper is organized as follows. In Section 2, we introduce fundamental concepts and basic results of the distributional Henstock-Kurzweil integral. In Section 3, we apply Schauder's fixed point theorem to verify the existence of BVP (1.1) and (1.2). In Section 4, we give an example to illustrate Theorem 3.1 in this paper.

## 2 Distributional Henstock-Kurzweil integral

In this section, the definition of distributional Henstock-Kurzweil integral and its main properties needed in this paper are presented.

Define the space

$$C_c^\infty = \{\phi : \mathbb{R} \to \mathbb{R} \mid \phi \in C^\infty \text{ and } \phi \text{ has compact support in } \mathbb{R}\},$$

where the *support* of a function $\phi$ is the closure of the set on which $\phi$ does not equal zero. A sequence $\{\phi_n\} \subset C_c^\infty$ converges to $\phi \in C_c^\infty$ if there is a compact set $K$ such that all $\phi_n$ have support in $K$ and the sequence of derivatives $\phi_n^{(m)}$ converges to $\phi^{(m)}$ uniformly for every $m \in \mathbb{N} \cup \{0\}$. Denote $C_c^\infty$ endowed with this convergence property by $\mathcal{D}$. Also, if $\phi \in \mathcal{D}$, we call $\phi$ a *test function*. The dual space to $\mathcal{D}$ is denoted by $\mathcal{D}'$ and if $f \in \mathcal{D}'$ then $f : \mathcal{D} \to \mathbb{R}$, $\langle f, \phi\rangle \in \mathbb{R}$ for $\phi \in \mathcal{D}$.

For all $f \in \mathcal{D}'$, we define the distributional derivative $f'$ of $f$ to be a distribution satisfying $\langle f', \phi\rangle = -\langle f, \phi'\rangle$, where $\phi$ is a test function and $\phi'$ is the ordinary derivative of $\phi$. With this definition, it is easy to get that all distributions have derivatives of all orders and each derivative is a distribution.

Let $(a, b)$ be an open interval in $\mathbb{R}$. We define

$$\mathcal{D}\big((a,b)\big) = \{\phi : (a,b) \to \mathbb{R} \mid \phi \in C_c^\infty \text{ and } \phi \text{ has compact support in } (a,b)\}.$$

$\mathcal{D}'((a,b))$ denotes the dual space of $\mathcal{D}((a,b))$.

Let $C[a,b]$ be the space of continuous functions on $[a,b]$ and $B_C = \{F \in C[a,b] \mid F(a) = 0\}$. Note that $B_C$ is a Banach space with the uniform norm

$$\|F\|_\infty = \max_{t\in[a,b]} \big|F(t)\big|.$$

We give an introduction about the definition of the $D_{HK}$-integral.

**Definition 2.1** *A distribution $f$ is distributionally Henstock-Kurzweil integrable or briefly $D_{HK}$-integrable on $[a,b]$ iff is the distributional derivative of a continuous function $F \in B_C$.*

The $D_{HK}$-integral of $f$ on $[a,b]$ is defined by $(D_{HK})\int_a^b f = F(b)$, where $F \in B_C$ is the primitive of $f$ and '$(D_{HK})\int$' denotes the $D_{HK}$-integral. For succinctness, we refer to '$(D_{HK})\int$' as simply '$\int$'. Moreover, the space of $D_{HK}$-integrable distributions is defined by

$$D_{HK} = \{f \in \mathcal{D}'\big((a,b)\big) \mid f = F' \text{ for some } F \in B_C\}.$$

With this definition, if $f \in D_{HK}$ then, for all $\phi \in \mathcal{D}((a,b))$,

$$\langle f, \phi\rangle = \langle F', \phi\rangle = -\langle F, \phi'\rangle = -\int_a^b F\phi'.$$

Now we give an example showing that the $D_{HK}$-integral includes the $HK$-integral, and hence includes Lebesgue and Riemann integrals (see [8–10] for details).

**Example 2.1** In [8], Lee points out that if $F$ is a continuous function and pointwise differentiable nearly everywhere on $[a, b]$, then $F$ is $ACG^*$ (generalized absolutely continuous), and if $F$ is a continuous function which is differentiable nowhere on $[a, b]$, then $F$ is not $ACG^*$. A primitive $F$ of the $HK$-integrable function $f$ is $ACG^*$ (see [8, 10] for details). Therefore, if $F \in C[a, b]$ but is differentiable nowhere on $[a, b]$, the distributional derivative of $F$ exists and is $D_{HK}$-integrable but not $HK$-integrable. On another aspect, if $F$ is $ACG^*$ then it belongs to $C[a, b]$. Hence $F'$ is not only $HK$-integrable but also $D_{HK}$-integrable.

Let us introduce some basic results of the distributional Henstock-Kurzweil integral needed later.

**Lemma 2.1** ([11], Theorem 4, fundamental theorem of calculus)

(a) *Let $f \in D_{HK}$, and define $F(t) = \int_a^t f$. Then $F \in B_C$ and $F' = f$.*

(b) *Let $F \in C[a, b]$. Then $\int_a^t F' = F(t) - F(a)$ for all $t \in [a, b]$.*

For $f \in D_{HK}$ and $F \in B_C$ with $F' = f$, we define the *Alexiewicz* norm by

$$\|f\| = \|F\|_\infty.$$

We say a sequence $\{f_n\} \subset D_{HK}$ converges strongly to $f \in D_{HK}$ if $\|f_n - f\| \to 0$ as $n \to \infty$. Then the following result holds.

**Lemma 2.2** ([11], Theorem 2) *With the Alexiewicz norm, $D_{HK}$ is a Banach space.*

Now we impose a partial ordering on $D_{HK}$: for $f, g \in D_{HK}$, we say that $f \succeq g$ (or $g \preceq f$) if and only if $f - g$ is a positive measure on $[a, b]$. By the definition, if $f, g \in D_{HK}$, then $\int_I f \succeq \int_I g$ for every $I = [c, d] \subset [a, b]$, whenever $f \succeq g$. See [12] for details.

It is shown that the following results hold.

**Lemma 2.3** ([11], Definition 6, integration by parts) *Let $f \in D_{HK}$, and $g$ is a function of bounded variation. Define $fg = DH$, where $H(t) = F(t)g(t) - \int_a^t F\,dg$. Then $fg \in D_{HK}$ and*

$$(D_{HK}) \int_a^b fg = F(b)g(b) - (D_{HK}) \int_a^b F\,dg.$$

**Lemma 2.4** ([12], Corollary 5, dominated convergence theorem for the $D_{HK}$-integral) *Let $\{f_n\}_{n=0}^\infty$ be a sequence in $D_{HK}$ such that $f_n \to f$ as $n \to \infty$ in $\mathcal{D}'$. Suppose that there exist $f_-, f_+ \in D_{HK}$ satisfying $f_- \preceq f_n \preceq f_+$ for $\forall n \in \mathbb{N}$. Then $f \in D_{HK}$ and $\lim_{n\to\infty} \int_a^b f_n = \int_a^b f$.*

The next statement is modified from [13] and [11].

**Lemma 2.5** *Let $f \in D_{HK}$ and $\{f_n\}_{n=0}^\infty$ be a sequence in $D_{HK}$ such that $f_n \to f$ as $n \to \infty$ in $\mathcal{D}'$. Define $F_n(x) = \int_a^x f_n$ and $F(x) = \int_a^x f$. If $g$ is a function of bounded variation and $F_n \to F$ as $n \to \infty$ uniformly on $[a, b]$, then $\int_a^b f_n g \to \int_a^b fg$ as $n \to \infty$.*

## 3 Main results

In this section, we firstly assume that $f$ satisfies the following assumptions:

($C_1$) $f(t,x)$ is $D_{HK}$-integrable with respect to $t$ for all $x \in C[0,1]$;
($C_2$) $f(t,x)$ is continuous with respect to $x$ for all $t \in [0,1]$, *i.e.*, for each $t \in [0,1]$, $\|f(t,x_n) - f(t,x)\| \to 0$ as $\|x_n - x\|_\infty \to 0$ for $\{x_n\} \subset C[0,1]$;
($C_3$) There exist $f_-, f_+ \in D_{HK}$ such that $f_-(\cdot) \preceq f(\cdot,x) \preceq f_+(\cdot)$ for all $x \in C[0,1]$.

**Lemma 3.1** *BVP* (1.1)-(1.2) *is equivalent to the integral equation*

$$\begin{aligned} x(t) = {} & \int_0^1 \beta f\bigl(s,x(s)\bigr)\,ds + \int_0^\eta (\eta - s) f\bigl(s,x(s)\bigr)\,ds \\ & - \int_0^t (t-s) f\bigl(s,x(s)\bigr)\,ds, \quad t \in [0,1], \end{aligned} \tag{3.1}$$

*where* $\eta$ *is a constant with* $0 \le \eta \le 1$.

*Proof* In view of Eq. (1.1), condition ($C_1$) and Lemma 2.1, we have $x' \in C[0,1]$, and for all $t \in [0,1]$, $s \in [0,1]$,

$$\begin{aligned} \int_\eta^t s x''(s)\,ds &= t x'(t) - x(t) - \bigl(\eta x'(\eta) - x(\eta)\bigr) \\ &= -x(t) + x(\eta) + t x'(t) - \eta x'(\eta) \\ &= -x(t) + x(\eta) + t \int_0^t x''(s)\,ds - \eta \int_0^\eta x''(s)\,ds. \end{aligned}$$

Then

$$\begin{aligned} x(t) &= x(\eta) + t \int_0^t x''(s)\,ds - \eta \int_0^\eta x''(s)\,ds - \int_\eta^t s x''(s)\,ds \\ &= x(\eta) + t \int_0^t x''(s)\,ds - \eta \int_0^\eta x''(s)\,ds - \int_\eta^0 s x''(s)\,ds - \int_0^t s x''(s)\,ds \\ &= x(\eta) - \int_0^\eta (\eta - s) x''(s)\,ds + \int_0^t (t-s) x''(s)\,ds. \end{aligned}$$

According to the boundary conditions, one has

$$x(t) = -\beta \int_0^1 x''(s)\,ds - \int_0^\eta (\eta - s) x''(s)\,ds + \int_0^t (t-s) x''(s)\,ds, \quad t \in [0,1].$$

Then

$$x(t) = \int_0^1 \beta f\bigl(s,x(s)\bigr)\,ds + \int_0^\eta (\eta - s) f\bigl(s,x(s)\bigr)\,ds - \int_0^t (t-s) f\bigl(s,x(s)\bigr)\,ds. \tag{3.2}$$

It is easy to calculate that BVP (1.1)-(1.2) holds by taking the derivative of both sides of (3.2). This completes the proof. □

**Lemma 3.2** ([14], Theorem 6.15) *Let $M$ be a convex, closed subset of a normed space $X$. Let $T$ be a continuous map of $M$ into a compact subset $K$ of $M$. Then $T$ has a fixed point.*

With the help of the preceding two lemmas, we can now prove the existence of solutions of BVP (1.1)-(1.2).

**Theorem 3.1** *Under assumptions* ($C_1$)-($C_3$), *there exists at least one solution of BVP* (1.1)-(1.2).

*Proof* Suppose that

$$M = \max_{t\in[0,1]}\left|\int_0^t f_-(s)\,ds\right| + \max_{t\in[0,1]}\left|\int_0^t f_+(s)\,ds\right|.$$

Then, for each $t \in [0,1]$, we have

$$-M \le \int_0^t f_-(s)\,ds \le M, \qquad -M \le \int_0^t f_+(s)\,ds \le M. \tag{3.3}$$

Let $B = \{x \in C[0,1] : \|x\|_\infty \le l, l = (\beta + 6)M > 0\}$. For each $x \in B$, $t \in [0,1]$, define

$$\mathcal{A}x(t) := \int_0^1 \beta f\bigl(s,x(s)\bigr)\,ds + \int_0^\eta (\eta - s)f\bigl(s,x(s)\bigr)\,ds - \int_0^t (t - s)f\bigl(s,x(s)\bigr)\,ds. \tag{3.4}$$

Now we prove this theorem in three steps.

Step 1: $\mathcal{A} : B \to B$.

For all $x \in B$, by (3.4), one has

$$\begin{aligned}\|\mathcal{A}x\|_\infty &= \max_{t\in[0,1]}\left|\int_0^1 \beta f\bigl(s,x(s)\bigr)\,ds + \int_0^\eta (\eta - s)f\bigl(s,x(s)\bigr)\,ds - \int_0^t (t - s)f\bigl(s,x(s)\bigr)\,ds\right| \\ &\le \max_{t\in[0,1]} (\beta + t + \eta)\left|\int_0^t f\bigl(s,x(s)\bigr)\,ds\right| + \max_{t\in[0,1]}\left|\int_\eta^t sf\bigl(s,x(s)\bigr)\,ds\right| \\ &\le (\beta + 2)M + \max_{t\in[0,1]}\left|\int_0^t sf\bigl(s,x(s)\bigr)\,ds\right| + \max_{t\in[0,1]}\left|\int_0^\eta sf\bigl(s,x(s)\bigr)\,ds\right|. \end{aligned} \tag{3.5}$$

Furthermore, let $F(t) = \int_0^t f(s,x(s))\,ds$ for $t \in [0,1]$, and

$$\|F\|_\infty = \max_{t\in[0,1]}\left|\int_0^t f\bigl(s,x(s)\bigr)\,ds\right|.$$

Then, for all $t \in [0,1]$, one has

$$\begin{aligned}\left|\int_0^t sf\bigl(s,x(s)\bigr)\,ds\right| &= \left|\int_0^t s\,dF(s)\right| = \left|sF(s)|_0^t - \int_0^t F(s)\,ds\right| \\ &\le \left|sF(s)|_0^t\right| + \left|\int_0^t F(s)\,ds\right| \le \left|F(s)|_0^t\right| + \left|\int_0^t F(s)\,ds\right| \\ &\le 2\|F\|_\infty. \end{aligned} \tag{3.6}$$

In particular, for $t = \eta$, we have

$$\left|\int_0^\eta sf\big(s,x(s)\big)\,ds\right| \le 2\|F\|_\infty. \tag{3.7}$$

In view of (3.5)-(3.7), one has

$$\begin{aligned}\|\mathcal{A}x\|_\infty &\le (\beta+2)M + \max_{t\in[0,1]}\left(\left|\int_0^t sf\big(s,x(s)\big)\,ds\right| + \left|\int_0^\eta sf\big(s,x(s)\big)\,ds\right|\right)\\ &\le (\beta+2)M + 2\|F\|_\infty + 2\|F\|_\infty\\ &= (\beta+2)M + 4\|F\|_\infty.\end{aligned}$$

By (3.3), we also obtain $\|F\|_\infty \le M$, then $\|\mathcal{A}x\|_\infty \le (\beta+6)M = l$. Hence, $\mathcal{A}(B) \subseteq B$.

Step 2: $\mathcal{A}(B)$ is equicontinuous.

Let $t_1, t_2 \in [0,1]$, $x \in B$

$$\begin{aligned}&\big|\mathcal{A}x(t_1) - \mathcal{A}x(t_2)\big|\\ &\quad= \left|\int_0^{t_1}(t_1-s)f\big(s,x(s)\big)\,ds - \int_0^{t_2}(t_2-s)f\big(s,x(s)\big)\,ds\right|\\ &\quad= \left|t_1\int_0^{t_1} f\big(s,x(s)\big)\,ds - t_2\int_0^{t_2} f\big(s,x(s)\big)\,ds - \int_0^{t_1} sf\big(s,x(s)\big)\,ds + \int_0^{t_2} sf\big(s,x(s)\big)\,ds\right|\\ &\quad= \left|t_1\int_0^{t_2} f\big(s,x(s)\big)\,ds + t_1\int_{t_2}^{t_1} f\big(s,x(s)\big)\,ds - t_2\int_0^{t_2} f\big(s,x(s)\big)\,ds + \int_{t_1}^{t_2} sf\big(s,x(s)\big)\,ds\right|\\ &\quad= \left|(t_1-t_2)\int_0^{t_2} f\big(s,x(s)\big)\,ds + t_1\int_{t_2}^{t_1} f\big(s,x(s)\big)\,ds + \int_{t_2}^{t_1} sf\big(s,x(s)\big)\,ds\right|\\ &\quad\le |t_2-t_1|\left(\left|\int_0^{t_2} f_-(s)\,ds\right| + \left|\int_0^{t_2} f_+(s)\,ds\right|\right) + t_1\left(\left|\int_{t_1}^{t_2} f_-(s)\,ds\right| + \left|\int_{t_1}^{t_2} f_+(s)\,ds\right|\right)\\ &\qquad+ \left|\int_{t_1}^{t_2} sf\big(s,x(s)\big)\,ds\right|.\end{aligned}$$

For every $t \in [0,1]$, we let $F_+(t) = \int_0^t f_+(s)\,ds$, $F_-(t) = \int_0^t f_-(s)\,ds$. By ($C_3$), we obtain

$$F_-(t) \le F(t) \le F_+(t), \quad t \in [0,1],$$

*i.e.*,

$$\int_0^t f_-(s)\,ds \le \int_0^t f\big(s,x(s)\big)\,ds \le \int_0^t f_+(s)\,ds, \quad t \in [0,1].$$

Moreover,

$$\begin{aligned}\left|\int_{t_1}^{t_2} sf\big(s,x(s)\big)\right| &= \left|sF(s)\big|_{t_1}^{t_2} - \int_{t_1}^{t_2} F(s)\,ds\right| = \left|t_2F(t_2) - t_1F(t_1) - \int_{t_1}^{t_2} F(s)\,ds\right|\\ &= \left|(t_2-t_1)F(t_2) - t_1\big(F(t_1) - F(t_2)\big) - \int_{t_1}^{t_2} F(s)\,ds\right|\end{aligned}$$

$$
\begin{aligned}
&\leq \left|(t_2 - t_1)F(t_2)\right| + t_1\left|F(t_1) - F(t_2)\right| + \left|\int_{t_1}^{t_2} F(s)\,ds\right| \\
&\leq |t_2 - t_1|\left(\left|\int_0^{t_2} f_-(s)\,ds\right| + \left|\int_0^{t_2} f_+(s)\,ds\right|\right) \\
&\quad + t_1\left(\left|\int_{t_1}^{t_2} f_-(s)\,ds\right| + \left|\int_{t_1}^{t_2} f_+(s)\,ds\right|\right) \\
&\quad + \left(\left|\int_{t_1}^{t_2} F_+(s)\,ds\right| + \left|\int_{t_1}^{t_2} F_-(s)\,ds\right|\right).
\end{aligned}
$$

So,

$$
\begin{aligned}
\left|\mathcal{A}x(t_1) - \mathcal{A}x(t_2)\right| &\leq 2|t_2 - t_1|\left(\left|\int_0^{t_2} f_-(s)\,ds\right| + \left|\int_0^{t_2} f_+(s)\,ds\right|\right) \\
&\quad + 2t_1\left(\left|\int_{t_1}^{t_2} f_-(s)\,ds\right| + \left|\int_{t_1}^{t_2} f_+(s)\,ds\right|\right) \\
&\quad + \left(\left|\int_{t_1}^{t_2} F_+(s)\,ds\right| + \left|\int_{t_1}^{t_2} F_-(s)\,ds\right|\right).
\end{aligned}
\tag{3.8}
$$

Since $f_-(s), f_+(s), F_-(s), F_+(s) \in D_{HK}$, then their primitives are continuous on $[0,1]$ and hence uniformly continuous on $[0,1]$. Then by (3.8), $\mathcal{A}(B)$ is equiuniformly continuous on $[0,1]$ for all $x \in B$.

In view of Step 1, Step 2 and the Ascoli-Arzelà theorem, $\mathcal{A}(B)$ is relatively compact.

Step 3: $\mathcal{A}$ is a continuous mapping.

Let $x \in B$, $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in $B$ and $x_n \to x$ as $n \to \infty$.

By ($C_2$), one has

$$f(\cdot, x_n) \to f(\cdot, x) \quad \text{as } n \to \infty.$$

According to assumption ($C_3$) and Lemma 2.4, we have

$$\lim_{n\to\infty} \int_0^t f\bigl(s, x_n(s)\bigr)\,ds = \int_0^t f\bigl(s, x(s)\bigr)\,ds, \quad t \in [0,1].$$

It is easy to verify, by Lemma 2.5, that

$$\lim_{n\to\infty} \mathcal{A}(x_n) = \mathcal{A}(x).$$

Hence, $\mathcal{A}$ is continuous.

Thus, $\mathcal{A}$ satisfies the hypotheses of Lemma 3.2, then there exists a fixed point of $\mathcal{A}$ which is a solution of (3.1). By Lemma 3.1, BVP (1.1)-(1.2) has at least one solution. □

## 4 Example

In this section, we give an example for the application of Theorem 3.1.

**Example 4.1** Consider the initial value problem

$$
\begin{cases}
-x'' = t^2 \sin x + r, & t \in [0,1], \\
x'(0) = 0, \\
\beta x'(1) + x(0) = 0,
\end{cases}
\tag{4.1}
$$

where $r$ is the distributional derivative of the Riemann function $R(t) = \sum_{n=1}^{\infty} \frac{\sin n^2 \pi t}{n^2}$ in [15].

It is easy to see that $R(t) \in C[0,1]$ and $R(0) = 0$, hence $r$ is $D_{HK}$-integrable. Let $f(t,x) = t^2 \sin x + r$, then $f(t,x)$ satisfies ($C_1$), ($C_2$). Moreover, let $f_-(t) = -t^2 + r$ and $f_+(t) = t^2 + r$, then

$$f_-(\cdot) \preceq f(\cdot,x) \preceq f_+(\cdot),$$

*i.e.*, ($C_3$) holds. Therefore, the initial value problem (4.1) has a solution.

**Remark 4.1** It is well known that the function $R(t)$ given by Riemann is continuous but pointwise differentiable nowhere on $[0,1]$, then the distributional derivative $r$ in (4.1) is neither $HK$ nor Lebesgue integrable. Hence, this example is not covered by any result using HK or Lebesgue integral. Thus, Theorem 3.1 is more extensive.

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

**Acknowledgements**
The authors are highly grateful for the referees' careful reading and comments on this paper that led to the improvement of the original manuscript.

Received: 16 December 2014 Accepted: 21 April 2015 Published online: 01 May 2015

**References**

1. Leela, S: Monotone method for second order periodic boundary value problems. Nonlinear Anal. **7**, 349-355 (1983)
2. Lakshmikantham, V, Sree Hari Rao, V, Vatsala, AS: Monotone method for a system of second order periodic boundary value problems. Appl. Math. Comput. **15**, 71-83 (1984)
3. Zhou, X, Ye, G: Second order periodic boundary value problems involving the distributional Henstock-Kurzweil integral. Adv. Pure Math. **2**, 330-336 (2012)
4. Infante, G, Webb, JRL: Loss of positivity in a nonlinear scalar heat equation. Nonlinear Differ. Equ. Appl. **13**, 249-261 (2006)
5. Chew, TS, Flordeliza, F: On $x' = f(t,x)$ and Henstock-Kurzweil integrals. Differ. Integral Equ. **4**, 861-868 (1991)
6. Bugajewska, D: On the equation of *n*th order and the Denjoy integral. Nonlinear Anal. **34**, 1111-1115 (1998)
7. Bullen, PS, Sarkhel, DN: On the solution of $(\frac{dy}{dx})_{ap} = f(x,y)$. J. Math. Anal. Appl. **127**, 365-376 (1987)
8. Lee, PY: Lanzhou Lectures on Henstock Integration. World Scientific, Singapore (1989)
9. Kurzweil, J: Henstock-Kurzweil Integration, Its Relation to Topological Vector Spaces. World Scientific, Singapore (1999)
10. Schwabik, Š, Ye, G: Topics in Banach Space Integration. World Scientific, Singapore (2005)
11. Talvila, E: The distributional Denjoy integral. Real Anal. Exch. **33**, 51-82 (2008)
12. Ang, DD, Schmitt, K, Vy, LK: A multidimensional analogue of the Denjoy-Perron-Henstock-Kurzweil integral. Bull. Belg. Math. Soc. Simon Stevin **4**, 355-371 (1997)
13. Talvila, E: Limits and Henstock integrals of products. Real Anal. Exch. **25**, 907-918 (1999/2000)
14. Schechter, M: An Introduction to Nonlinear Analysis. Cambridge University Press, Cambridge (2004)
15. Hardy, GH: Weierstrass's non-differential function. Trans. Am. Math. Soc. **3**, 301-325 (1916)