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Multiplicity results on discrete boundary value problems with double resonance via variational methods

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Abstract

The existence of solutions for a class of difference equations with double resonance is studied via variational methods, and multiplicity results are derived.

Keywords: discrete boundary value problem; double resonance; existence and multiplicity; variational method

1 Introduction

Let $\mathbf{Z}$ denote the set of integers, and for $a, b \in \mathbf{Z}$ with $a < b$, define $\mathbf{Z}[a, b] = \{a, a+1, \dots, b\}$. For a given positive integer $N \geq 2$, consider the following discrete boundary value problem:

$$\text{(BP)} \qquad \begin{cases} -\Delta^2 x(k-1) = f(k, x(k)), & k \in \mathbf{Z}[1, N], \\ x(0) = x(N+1) = 0, \end{cases}$$

where Δ is the forward difference operator defined by $\Delta x(k) = x(k+1) - x(k)$ and $\Delta^2 x(k) = \Delta(\Delta x(k))$ for $k \in \mathbf{Z}$. Throughout this paper, we always assume that $f : \mathbf{Z}[1, N] \times \mathbf{R} \to \mathbf{R}$ is C^1-differentiable with respect to the second variable and satisfies $f(k, 0) \equiv 0$ for $k \in \mathbf{Z}[1, N]$, which implies that (BP) has a trivial solution $x(k) = 0$, $k = 0, 1, \dots, N+1$. We investigate the existence of nontrivial solutions of (BP).

In different fields of research, such as computer science, mechanical engineering, control systems, population biology, economics and many others, the mathematical modeling of important questions leads naturally to the consideration of nonlinear difference equations. The dynamic behaviors of nonlinear difference equations have been studied extensively in [1, 2]. Recently, many authors considered the solvability of nonlinear difference equations via variational methods. For example, on the second-order difference equations, the boundary value problems are studied in [3–7] and the existence of periodic solutions is investigated in [8–10].

As a natural phenomenon, resonance exists in the real world from macrocosm to microcosm. In a system described by a mathematical model, the feature of resonance lies in the interaction between the linear spectrum and the nonlinearity. It is known from [1] that the eigenvalue problem

$$\begin{cases} -\Delta^2 x(k-1) = \lambda x(k), & k \in \mathbf{Z}[1,N], \\ x(0) = x(N+1) = 0, \end{cases}$$

possesses N distinct eigenvalues $\lambda_l = 4\sin^2(l\pi/2(N+1))$, $l = 1,2,\ldots,N$. Many authors considered the complete resonance situation in the sense that for some $h \in \mathbf{Z}[1,N]$,

$$\lim_{|t|\to\infty} \frac{f(k,t)}{t} = \lambda_h, \quad k \in \mathbf{Z}[1,N]$$

via different methods in critical point theory such as Morse theory [6], index theory [9] and minimax methods [7]. The assumption that

(f_∞^*) there exists some $h \in \mathbf{Z}[1,N-1]$ such that

$$\lambda_h \le \liminf_{|t|\to\infty} \frac{f(k,t)}{t} \le \limsup_{|t|\to\infty} \frac{f(k,t)}{t} \le \lambda_{h+1} \quad \text{for } k \in \mathbf{Z}[1,N]$$

characterizes problem (BP) as double resonance between two consecutive eigenvalues at infinity. In the case of resonance, one needs to impose various conditions on the nonlinearity of f near infinity to ensure the global compactness. In fact, many results on differential equations with double resonance have been obtained (see [11–13]). As to discrete boundary value problems with double resonance, however, there are few results published. In [10], the existence of periodic solutions to a second-order difference equation with double resonance, as is described in (f_∞^*), is investigated.

Motivated by the study in [10], we consider problem (BP) with double resonance indicated in (f_∞^*). To control the double resonance, a selectable restriction on the nonlinearity of f is that

($\mathrm{f}_\infty^\dagger$) there exists some $h \in \mathbf{Z}[1,N-1]$ such that

$$\left.\begin{aligned} &\text{(i)} \quad \underline{r}_h(f) := \liminf_{|t|\to\infty} |t| \left(\frac{f(k,t)}{t} - \lambda_h \right) > 0 \\ &\text{(ii)} \quad \overline{r}_h(f) := \limsup_{|t|\to\infty} |t| \left(\frac{f(k,t)}{t} - \lambda_{h+1} \right) < 0 \end{aligned}\right. \quad \text{for } k \in \mathbf{Z}[1,N],$$

which has completely the same form as its counterpart in [10]. However, instead of ($\mathrm{f}_\infty^\dagger$), in this paper we assume that

(f_∞) there exists some $h \in \mathbf{Z}[1,N-1]$ such that

$$\left.\begin{aligned} &\text{(i)} \quad \underline{R}_h(f) := \liminf_{|t|\to\infty} t^2 \left(\frac{f(k,t)}{t} - \lambda_h \right) > 0 \\ &\text{(ii)} \quad \overline{R}_h(f) := \limsup_{|t|\to\infty} t^2 \left(\frac{f(k,t)}{t} - \lambda_{h+1} \right) < 0 \end{aligned}\right. \quad \text{for } k \in \mathbf{Z}[1,N].$$

Remark 1.1 It is easy to see that, as a restriction on the nonlinearity of f, (f_∞) is more relaxed than ($\mathrm{f}_\infty^\dagger$) (see Examples 1.1-1.3 and Remark 1.3). In addition, (f_∞), as well as ($\mathrm{f}_\infty^\dagger$), implies ($\mathrm{f}_\infty^*$).

A sequence $\{x(0), x(1), \ldots, x(N+1)\}$ is said to be a positive (negative) solution of (BP) if it satisfies (BP) and $x(k) > 0$ (< 0) for $k \in \mathbf{Z}[1, N]$.

Theorem 1.1 *Assume that* (f_∞) *holds. Then* (BP) *has at least four nontrivial solutions in which one is positive and one is negative in each of the following two cases*:

(i) $h \in \mathbf{Z}[2, N-1]$ *and* $f'(k, 0) < \lambda_1$ *for* $k \in \mathbf{Z}[1, N]$;
(ii) $h \in \mathbf{Z}[1, N-2]$ *and* $f'(k, 0) > \lambda_N$ *for* $k \in \mathbf{Z}[1, N]$.

To state the following theorems, we further assume that

(f_0) there exists $t_0 \neq 0$ such that $f(k, t_0) = 0$ for $k \in \mathbf{Z}[1, N]$.

Theorem 1.2 *Assume that* (f_0) *and* (f_∞) *hold with* $h \in \mathbf{Z}[2, N-1]$. *If there exists* $m \in \mathbf{Z}[2, N-1]$ *with* $m \neq h$ *such that* $\lambda_m < f'(k, 0) < \lambda_{m+1}$ *for* $k \in \mathbf{Z}[1, N]$, *then* (BP) *has at least four nontrivial solutions*.

Let $f'(k, t)$ denote the derivative of $f(k, t)$ with respect to the second variable. In the case where (BP) is also resonant at the origin, that is, there exists $m \in \mathbf{Z}[1, N]$ such that $f'(k, 0) \equiv \lambda_m$ for $k \in \mathbf{Z}[1, N]$, we assume that

$$(\mathrm{F}_0^\pm) \qquad \pm \int_0^t \bigl(f(k, s) - \lambda_m s\bigr)\, ds \geq 0 \qquad \text{for } |t| > 0 \text{ small and } k \in \mathbf{Z}[1, N].$$

Theorem 1.3 *Assume that* (f_0) *and* (f_∞) *hold with* $h \in \mathbf{Z}[2, N-1]$. *If there exists* $m \in \mathbf{Z}[1, N]$ *such that* $f'(k, 0) \equiv \lambda_m$ *for* $k \in \mathbf{Z}[1, N]$, *then* (BP) *has at least four nontrivial solutions in each of the following two cases*:

(i) (F_0^+) *with* $m \geq 2$ *and* $m \neq h$;
(ii) (F_0^-) *with* $m \geq 3$ *and* $m \neq h+1$.

Remark 1.2 In view of the proofs in Section 4, we see that if $t_0 > 0$ (< 0) in (f_0), two of the solutions derived in Theorems 1.2, 1.3 are positive (negative).

Set $h \in \mathbf{Z}[1, N-1]$ and define $g : \mathbf{R} \mapsto \mathbf{R}$ by

$$g(t) = \lambda_h t + (\lambda_{h+1} - \lambda_h) t \bigl(1 + t^2\bigr)^{-1} \bigl(\sin^2 t + t^2 \cos^2 t\bigr).$$

By calculation, we get $\underline{R}_h(g) = -\overline{R}_h(g) = \lambda_{h+1} - \lambda_h > 0$ and $g'(0) = \lambda_h$. Define $g_1(t) = g(t) + \alpha t(1+t^4)^{-1}$, $g_2(t) = g(t) + (\alpha t + \beta t^2)(1+t^4)^{-1}$ and $g_3(t) = g(t) + (\alpha t + \beta t^3)(1+t^6)^{-1}$, $t \in \mathbf{R}$, where α and β are constants. Obviously, $\underline{R}_h(g_i) = \underline{R}_h(g) > 0$, $\overline{R}_h(g_i) = \overline{R}_h(g) < 0$ and $g_i'(0) = \lambda_h + \alpha$, $i = 1, 2, 3$. The following examples are presented to illustrate the applications of the above results.

Example 1.1 Consider (BP) with $f(k, t) \equiv g_1(t)$, $(k, t) \in \mathbf{Z}[1, N] \times \mathbf{R}$. We have $f'(k, 0) = \lambda_h + \alpha$ for $k \in \mathbf{Z}[1, N]$. If $h \in \mathbf{Z}[2, N-1]$ and $\alpha < \lambda_1 - \lambda_h$ or $h \in \mathbf{Z}[1, N-2]$ and $\alpha > \lambda_N - \lambda_h$, then by Theorem 1.1, (BP) has at least four nontrivial solutions in which one is positive and one is negative.

Example 1.2 Set $h, m \in \mathbf{Z}[2, N-1]$ with $m \neq h$. Let $\alpha \in (\lambda_m - \lambda_h, \lambda_{m+1} - \lambda_h)$ and $\beta < -(\lambda_h + \lambda_{h+1} + \alpha)$. Consider (BP) with $f(k, t) \equiv g_2(t)$, $(k, t) \in \mathbf{Z}[1, N] \times \mathbf{R}$. We have $f'(k, 0) = \lambda_h + \alpha >$

$\lambda_m > 0$ and $f(k,1) = (\lambda_h + \lambda_{h+1} + \alpha + \beta)/2 < 0$ for $k \in \mathbf{Z}[1,N]$, which implies that there exists $t_0 \in (0,1)$ such that $f(k,t_0) = 0$ for $k \in \mathbf{Z}[1,N]$. By Theorem 1.2 and Remark 1.2, (BP) has at least four nontrivial solutions in which two are positive.

Example 1.3 Set $h \in \mathbf{Z}[2,N-1]$, $\alpha = \lambda_m - \lambda_h$ and $\beta < -\max\{(\lambda_m + \lambda_{h+1}), 2(\lambda_{h+1} - \lambda_h)\}$ for some $m \in \mathbf{Z}[3,N] \setminus \{h+1\}$. Consider (BP) with $f(k,t) \equiv g_3(t)$, $(k,t) \in \mathbf{Z}[1,N] \times \mathbf{R}$. We have $f'(k,0) = \lambda_h + \alpha = \lambda_m > 0$ and $f(k,1) = (\lambda_m + \lambda_{h+1} + \beta)/2 < 0$ for $k \in \mathbf{Z}[1,N]$, which implies that there exists $t_0 \in (0,1)$ such that $f(k,t_0) = 0$ for $k \in \mathbf{Z}[1,N]$. Moreover,

$$\int_0^t \bigl(f(k,s) - \lambda_m s\bigr)\,ds = \left(\frac{\beta}{4} + \frac{\lambda_{h+1}}{2} - \frac{\lambda_h}{2}\right)t^4 + o\bigl(t^4\bigr) \quad (t \to 0).$$

By Theorem 1.3(ii) and Remark 1.2, (BP) has at least four nontrivial solutions in which two are positive.

Remark 1.3 It is easy to see that $\underline{r}_h(g_i) = \overline{r}_h(g_i) = 0$, $i = 1,2,3$, that is, the restriction imposed here is more relaxed than that in [10].

The paper is organized as follows. In Section 2 we give a simple revisit to Morse theory, and in Section 3 we give some lemmas. The main results will be proved in Section 4.

2 Preliminary results on critical groups

Let H be a Hilbert space and $\Phi \in C^2(H,\mathbf{R})$ be a functional satisfying the Palais-Smale condition ((PS) in short), that is, every sequence $\{x_n\} \subset H$ such that $\{\Phi(x_n)\}$ is bounded and $\Phi'(x_n) \to 0$ as $n \to \infty$ has a convergent subsequence. Denote by $H_q(X,Y)$ the qth singular relative homology group of the topological pair (X,Y) with integer coefficients. Let u_0 be an isolated critical point of Φ with $\Phi(u_0) = c$, $c \in \mathbf{R}$, and U be a neighborhood of u_0. For $q \in \mathbf{N} \cup \{0\}$, the group

$$C_q(\Phi,u_0) := H_q\bigl(\Phi^c \cap U, \Phi^c \cap U \setminus \{u_0\}\bigr)$$

is called the qth critical group of Φ at u_0, where $\Phi^c = \{u \in H : \Phi(u) \le c\}$.

If the set of the critical points of Φ, denoted by $\mathcal{K} := \{u \in H : \Phi'(u) = 0\}$, is finite and $a < \inf \Phi(\mathcal{K})$, the critical groups of Φ at infinity are defined by (see [14])

$$C_q(\Phi,\infty) := H_q\bigl(H, \Phi^a\bigr), \quad q \in \mathbf{N} \cup \{0\}.$$

For $q \in \mathbf{N} \cup \{0\}$, we call $\beta_q := \dim C_q(\Phi,\infty)$ the Betti numbers of Φ and define the Morse-type numbers of the pair (H,Φ^a) by

$$M_q := M_q\bigl(H,\Phi^a\bigr) = \sum_{u \in \mathcal{K}} \dim C_q(\Phi,u).$$

With the above notations, we have the following facts (2.a)-(2.f) [15, Chapter 8].

(2.a) If $C_\mu(\Phi,\infty) \ncong 0$ for some $\mu \in \mathbf{N} \cup \{0\}$, then there exists $x_0 \in \mathcal{K}$ such that $C_\mu(\Phi,x_0) \ncong 0$;

(2.b) If $\mathcal{K} = \{x_0\}$, then $C_q(\Phi, \infty) \cong C_q(\Phi, x_0)$;

(2.c) $\sum_{j=0}^{\infty}(-1)^j M_j = \sum_{j=0}^{\infty}(-1)^j \beta_j$.

If $x_0 \in \mathcal{K}$ and $\Phi''(x_0)$ is a Fredholm operator and the Morse index μ_0 and nullity ν_0 of x_0 are finite, then we have

(2.d) $C_q(\Phi, x_0) \cong 0$ for $q \notin \mathbf{Z}[\mu_0, \mu_0 + \nu_0]$;

(2.e) If $C_{\mu_0}(\Phi, x_0) \ncong 0$, then $C_q(\Phi, x_0) \cong \delta_{q,\mu_0}\mathbf{Z}$, and if $C_{\mu_0+\nu_0}(\Phi, x_0) \ncong 0$, then $C_q(\Phi, x_0) \cong \delta_{q,\mu_0+\nu_0}\mathbf{Z}$;

(2.f) If $m := \dim H < +\infty$, then $C_q(\Phi, x_0) \cong \delta_{q,0}\mathbf{Z}$ when x_0 is local minimizer of Φ, while $C_q(\Phi, x_0) \cong \delta_{q,m}\mathbf{Z}$ when x_0 is the local maximizer of Φ.

We say that Φ has a local linking at $x_0 \in \mathcal{K}$ if there exists the direct sum decomposition: $H = H^+ \oplus H^-$ and $\epsilon > 0$ such that

$$\Phi(x) > \Phi(x_0) \quad \text{if } x - x_0 \in H^+, 0 < \|x - x_0\| \le \epsilon,$$

$$\Phi(x) \le \Phi(x_0) \quad \text{if } x - x_0 \in H^-, \|x - x_0\| \le \epsilon.$$

The following results are due to Su [13].

(2.g) Assume that Φ has a local linking at $x_0 \in \mathcal{K}$ with respect to $H = H^+ \oplus H^-$ and $k = \dim H^- < +\infty$. Then

$$C_q(\Phi, x_0) \cong \delta_{q,\mu_0}\mathbf{Z} \quad \text{if } k = \mu_0,$$

$$C_q(\Phi, x_0) \cong \delta_{q,\mu_0+\nu_0}\mathbf{Z} \quad \text{if } k = \mu_0 + \nu_0.$$

We say that Φ satisfies the Cerami condition ((C) in short) if every sequence $\{x_n\} \subset H$ such that $\{\Phi(x_n)\}$ is bounded and $(1 + \|x_n\|)\Phi'(x_n) \to 0$ as $n \to \infty$ has a convergent subsequence. The following lemma derives from [11, Proposition 3.2].

Lemma 2.1 [11] *Let H be a Hilbert space, and $\{\Phi_s \in C^1(H, \mathbf{R}) | s \in [0,1]\}$ are a family of functionals such that Φ_s' and $\partial_s\Phi_s$ are locally Lipschitz continuous. Assume that Φ_0 and Φ_1 satisfy (C). If there exists $M > 0$ such that*

$$\inf_{s\in[0,1],\|x\|>M} \bigl(1 + \|x\|\bigr)\bigl\|\Phi_s'(x)\bigr\| > 0 \quad \textit{and} \quad \inf_{s\in[0,1],\|x\|\le M} \Phi_s(x) > -\infty,$$

then

$$C_q(\Phi_0, \infty) = C_q(\Phi_1, \infty).$$

Remark 2.1 The deformation lemma can be proved with the weaker condition (C) replacing the usual (PS) condition [16]. Therefore, if the (PS) condition is replaced by the (C) condition, (2.a)-(2.g) stated above still hold.

3 Compactness and critical group at infinity

In this section, we are going to prove the compactness of the associated energy functionals and to calculate the critical groups at infinity. First of all, let us introduce the variational structure for problem (BP).

3.1 Variational structure

The class E of functions $x : \mathbf{Z}[0, N+1] \mapsto R$ such that $x(0) = x(N+1) = 0$, equipped with the inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ as follows:

$$\langle x, y\rangle = \sum_{k=1}^{N} x(k)y(k), \qquad \|x\| = \left(\sum_{k=1}^{N} \bigl|x(k)\bigr|^2\right)^{1/2} \quad \text{for } x, y \in E,$$

is linearly homeomorphic to $\mathbf{R}^N$. Denote $\theta = (0, 0, \dots, 0)^T \in \mathbf{R}^N$. Throughout this paper, we always identify $x \in E$ with $x = (x(1), x(2), \dots, x(N))^T \in \mathbf{R}^N$.

Set $\mathbf{f}(x) = (f(1, x(1)), \dots, f(N, x(N)))^T$, $x \in E$ and

$$A = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ & \cdots & \cdots & \cdots & & \cdots & \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}_{N\times N}.$$

Then we can equivalently rewrite (BP) as a nonlinear algebraic system

$$Ax = \mathbf{f}(x), \quad x \in E. \tag{3.1}$$

Denote $E^l = \ker(A - \lambda_l I)$, $l = 1, \dots, N$, where I is the identity operator. Thus $\dim E^l = 1$, $l = 1, 2, \dots, N$. Set

$$E^- = \bigoplus_{l=1}^{h-1} E^l, \qquad E^+ = \left(\bigoplus_{l=1}^{h+1} E^l\right)^{\perp}, \qquad E^\nu = E^- \oplus E^+,$$

then E has the decomposition $E = E^h \oplus E^{h+1} \oplus E^\nu$. In the rest of this paper, the expression $x = x^h + x^{h+1} + x^\nu$ for $x \in E$ always means $x^\dagger \in E^\dagger$, $\dagger = h, h+1, \nu$.

Define a functional $J : E \to \mathbf{R}$ by

$$J(x) = \frac{1}{2}\langle Ax, x\rangle - \sum_{k=1}^{N} F\bigl(k, x(k)\bigr) \quad \text{for } x \in E,$$

where $F(k,t) = \int_0^t f(k,s)\,ds$, $(k,t) \in \mathbf{Z}[1,N] \times \mathbf{R}$. Then the Fréchet derivative of J at $x \in E$, denoted by $J'(x)$, can be described by (see [3])

$$\bigl\langle J'(x), y\bigr\rangle = \langle Ax, y\rangle - \sum_{k=1}^{N} f\bigl(k, x(k)\bigr)y(k) \quad \text{for } y \in E. \tag{3.2}$$

Remark 3.1 From (3.2) we see that $x \in E$ is a critical point of J if and only if x is a solution of (3.1) (or equivalently (BP)). In addition, J is C^2-differentiable with

$$\bigl\langle J''(x)y, z\bigr\rangle = \langle Ay, z\rangle - \sum_{k=1}^{N} f'\bigl(k, x(k)\bigr)y(k)z(k) \quad \text{for } y, z \in E. \tag{3.3}$$

3.2 Compactness of related functionals

Define a family of functionals $J_s : E \to \mathbf{R}$, $s \in [0,1]$ by

$$J_s(x) = \frac{1}{2}\langle Ax, x\rangle - \frac{1-s}{4}(\lambda_h + \lambda_{h+1})\|x\|^2 - s\sum_{k=1}^{N} F\bigl(k, x(k)\bigr) \quad \text{for } x \in E,$$

then the Fréchet derivative of J_s at $x \in E$, denoted by $J_s'(x)$, can be described by (see [3])

$$\bigl\langle J_s'(x), y\bigr\rangle = \langle Ax, y\rangle - \sum_{k=1}^{N} f_s\bigl(k, x(k)\bigr)y(k) \quad \text{for } y \in E, \tag{3.4}$$

where $s \in [0,1]$ and

$$f_s(k,t) = sf(k,t) + \frac{1-s}{2}(\lambda_h + \lambda_{h+1})t \quad \text{for } (k,t) \in \mathbf{Z}[1,N] \times \mathbf{R}.$$

Lemma 3.1 *Assume that* (f_∞) *holds. For any sequences* $\{x_n\} \subset E$ *and* $\{s_n\} \subset [0,1]$, $\{x_n\}$ *is bounded provided that*

$$\bigl(1 + \|x_n\|\bigr)J_{s_n}'(x_n) \to 0 \quad \textit{as } n \to \infty. \tag{3.5}$$

Moreover, for every $\hat{s} \in [0,1]$, $J_{\hat{s}}$ *satisfies* (C).

Proof Assume, for a contradiction, that $\{x_n\}$ is unbounded. Then there exists a subsequence, which we still call $\{x_n\}$, with $K \subset \mathbf{Z}[1,N]$ being nonempty such that

$$\lim_{n\to\infty} x_n(k) = \infty \quad \text{for } k \in K$$

and either $K = \mathbf{Z}[1,N]$ or, for any fixed $k \in K^c \equiv \mathbf{Z}[1,N] \setminus K$, $\{x_n(k)\}$ is a bounded sequence.

Noticing that [10, Lemma 3.7], with its proof being modified slightly, is applicable here, we know that

$$\text{either } \frac{\|x_n^h\|}{\|x_n\|} \to 1 \quad \text{or} \quad \frac{\|x_n^{h+1}\|}{\|x_n\|} \to 1 \quad \text{as } n \to \infty.$$

Set

$$\Gamma_1 := \limsup_{n\to\infty} \sum_{k=1}^{N} \bigl\{f_{s_n}\bigl(k, x_n(k)\bigr) - \lambda_h x_n(k)\bigr\}x_n^h(k),$$

$$\Gamma_2 := \liminf_{n\to\infty} \sum_{k=1}^{N} \bigl\{f_{s_n}\bigl(k, x_n(k)\bigr) - \lambda_{h+1} x_n(k)\bigr\}x_n^{h+1}(k).$$

Thus we have two cases to be considered.

Case 1. $\|x_n^h\|/\|x_n\| \to 1$ as $n \to \infty$. We have $\|x_n^h\| \to \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} \frac{\|x_n^{h+1}\|}{\|x_n\|} = 0, \qquad \lim_{n\to\infty} \frac{\|x_n^{\nu}\|}{\|x_n\|} = 0. \tag{3.6}$$

By (f_∞)(i), there exist $M > 0$ and $\xi > 0$ such that $t^2(f(k,t)/t - \lambda_h) > \xi$ and $t^2(\lambda_{h+1} - \lambda_h) > \xi$ for $|t| > M$ and $k \in \mathbf{Z}[1,N]$. Then, for $|t| > M$, $k \in \mathbf{Z}[1,N]$ and $s \in [0,1]$,

$$\begin{aligned}\left(\frac{f_s(k,t)}{t} - \lambda_h\right)t^2 &= s\left(\frac{f(k,t)}{t} - \lambda_h\right)t^2 + \frac{1-s}{2}(\lambda_{h+1} - \lambda_h)t^2 \\ &\geq s\xi + \frac{1-s}{2}\xi \geq \frac{\xi}{2}.\end{aligned}$$

Choose $N_1 > 0$ such that $|x_n(k)| > M$ for $k \in K$ and $n > N_1$. It follows that

$$\begin{aligned}&\left\{f_{s_n}\bigl(k, x_n(k)\bigr) - \lambda_h x_n(k)\right\}x_n^h(k) \\ &\quad= \left\{\frac{f_{s_n}(k,x_n(k))}{x_n(k)} - \lambda_h\right\}\bigl(x_n(k)\bigr)^2\left(\frac{(x_n(k) - z_n(k))}{x_n(k)}\right) \\ &\quad\geq \left\{\frac{f_{s_n}(k,x_n(k))}{x_n(k)} - \lambda_h\right\}\bigl(x_n(k)\bigr)^2\left(\frac{(|x_n(k)| - |z_n(k)|)}{\|x_n\|}\right) \\ &\quad\geq \frac{\xi(|x_n(k)| - |z_n(k)|)}{2\|x_n\|} \quad \text{for } k \in K \text{ and } n > N_1, \end{aligned} \tag{3.7}$$

where $z_n = x_n^{h+1} + x_n^\nu$. Since E possesses an equivalent norm defined by $\|x\|_1 \equiv \sum_{k=1}^N |x(k)|$ for $x \in E$, there exists a positive constant $C > 0$ such that $\|x\|_1 \geq C\|x\|$, $x \in E$. Thus, by (3.6) and (3.7),

$$\begin{aligned}\Gamma_1 &\geq \limsup_{n\to\infty} \frac{\xi}{2\|x_n\|}\left\{\sum_{k\in K}\bigl|x_n(k)\bigr| - \sum_{k\in K}\bigl|z_n(k)\bigr|\right\} \\ &= \limsup_{n\to\infty} \frac{\xi}{2\|x_n\|}\left\{\sum_{k=1}^N\bigl|x_n(k)\bigr| - \sum_{k=1}^N\bigl|z_n(k)\bigr|\right\} \\ &\geq \limsup_{n\to\infty} \frac{\xi}{2\|x_n\|}\bigl(C\|x_n\| - \sqrt{p}\|z_n\|\bigr) = \frac{C\xi}{2},\end{aligned}$$

where the equality holds because $|x_n(k)|/\|x_n\| \to 0$ as $n \to \infty$ for $k \in K^c$ in case $K^c \neq \emptyset$.

Case 2. $\|x_n^{h+1}\|/\|x_n\| \to 1$ as $n \to \infty$. In this case, by using (f_∞)(ii), we can show that $\Gamma_2 < 0$ in the same way.

On the other hand, it follows from (3.5) that

$$\left\langle \|x_n\| \cdot J'_{s_n}(x_n), \frac{x_n^\dagger}{\|x_n^\dagger\|}\right\rangle \to 0 \quad \text{as } n \to \infty, \dagger = h, h+1,$$

which implies that

$$\bigl\langle J'_{s_n}(x_n), x_n^\dagger\bigr\rangle \to 0 \quad \text{as } n \to \infty, \dagger = h, h+1,$$

that is,

$$\bigl\langle Ax_n, x_n^\dagger\bigr\rangle - \sum_{k=1}^N f_{s_n}\bigl(k, x_n(k)\bigr)x_n^\dagger(k) \to 0 \quad \text{as } n \to \infty, \dagger = h, h+1.$$

Note that $\langle Ax_n, x_n^\dagger\rangle = \langle \lambda_\dagger x_n, x_n^\dagger\rangle$, $\dagger = h, h+1$, it follows that $\Gamma_1 = \Gamma_2 = 0$. This contradiction proves the first conclusion.

By setting $s_n \equiv \hat{s} \in [0,1]$ in the proven conclusion, we see that $J_{\hat{s}}$ satisfies (C). The proof is complete. □

For $x \in E$, set $x^+(k) = \max\{0, x(k)\}$, $k \in \mathbf{Z}[1,N]$ and $x^+ = (x^+(1), \dots, x^+(N))^T$. The following lemma is derived from [3, Lemma 2.1].

Lemma 3.2 [3] *If x is a solution of*

$$Ax = \mathbf{f}(x^+), \quad x \in E,$$

then $x \geq \theta$ and hence it is also a solution of (3.1). *Moreover, either $x > \theta$ or $x = \theta$.*

For $x, y \in E$, we say that $x \geq y$ ($x > y$) if $x(k) \geq y(k)$ ($x(k) > y(k)$) for $k \in \mathbf{Z}[1,N]$.

Lemma 3.3 *Let ς_j be the eigenvector corresponding to λ_j, $j \in \mathbf{Z}[1,N]$, then ς_1 can be chosen to satisfy $\varsigma_1 > 0$. Moreover, for $j \geq 2$, neither $\varsigma_j \geq \theta$ nor $\varsigma_j \leq \theta$.*

Proof First we claim that $\varsigma_1 \geq \theta$ or $\varsigma_1 \leq \theta$. Otherwise, by setting $\bar{\varsigma}_1 = (|\varsigma_1(1)|, \dots, |\varsigma_1(N)|)^T$, we have

$$\sum_{k=1}^{N} \left|\Delta \bar{\varsigma}_1(k)\right|^2 < \sum_{k=1}^{N} \left|\Delta \varsigma_1(k)\right|^2. \tag{3.8}$$

Since $\lambda_1 = \inf_{\|x\|=1} \langle Ax, x\rangle = \langle A\varsigma_1, \varsigma_1\rangle / \|\varsigma_1\|^2$, it follows from (3.8) that

$$\begin{aligned}
\lambda_1 &\leq \frac{\langle A\bar{\varsigma}_1, \bar{\varsigma}_1\rangle}{\|\bar{\varsigma}_1\|} = \frac{\sum_{k=1}^{N} |\Delta \bar{\varsigma}_1(k)|^2}{\|\varsigma_1\|^2} \\
&< \frac{\sum_{k=1}^{N} |\Delta \varsigma_1(k)|^2}{\|\varsigma_1\|^2} = \lambda_1.
\end{aligned}$$

This contradiction proves the above claim. Thus ς_1 can be assumed to satisfy $\varsigma_1 \geq \theta$ and then $A\varsigma_1 = \lambda_1 \varsigma_1^+$. It follows by Lemma 3.2 that $\varsigma_1 > \theta$ and the first conclusion holds. Further, for $k \geq 2$, ς_k and ς_1 are orthogonal to each other, which implies that neither $\varsigma_j \geq \theta$ nor $\varsigma_j \leq \theta$. The proof is complete. □

Lemma 3.4 *Let the function $g \in C(\mathbf{Z}[1,N] \times \mathbf{R}, \mathbf{R})$ be such that $g(k,t) = 0$ for $t < 0$. Assume that there exists $h \in \mathbf{Z}[2,N]$ such that*

$$\lambda_h \leq \liminf_{t \to +\infty} \frac{g(k,t)}{t} \leq \limsup_{t \to +\infty} \frac{g(k,t)}{t} \leq \lambda_{h+1}. \tag{3.9}$$

Then the functional

$$I(x) = \frac{1}{2}\langle Ax, x\rangle - \sum_{k=1}^{N} G\bigl(k, x(k)\bigr)$$

satisfies the (PS) *condition, where $G(k,t) = \int_0^t g(k,s)\,ds$.*

Proof Let $\{x_n\} \subset E$ be such that

$$I'(x_n) \to 0 \quad \text{as } n \to \infty. \tag{3.10}$$

We only need to prove that $\{x_n\}$ is bounded. In fact, if $\{x_n\}$ is unbounded, there exists a subsequence, still called $\{x_n\}$, such that $\|x_n\| \to \infty$ as $n \to \infty$.

Let $w_n = x_n/\|x_n\|$, then $\|w_n\| = 1$. There is a convergent subsequence of $\{w_n\}$, call it $\{w_n\}$ again, such that $w_n \to w \in E$ as $n \to \infty$. For every $y \in E$, we have $\langle I'(x_n), y\rangle/\|x_n\| \to 0$ as $n \to \infty$, that is,

$$\langle Aw_n, y\rangle - \sum_{k=1}^{N} \frac{g(k, x_n(k))}{\|x_n\|} y(k) \to 0 \quad \text{as } n \to \infty. \tag{3.11}$$

Set

$$K_+ = \bigl\{k \in \mathbf{Z}[1,N] | x_n(k) \to +\infty \text{ as } n \to \infty\bigr\}.$$

We claim that $K_+ \neq \emptyset$, since otherwise (3.11) leads to $\langle Aw_n, y\rangle \to 0$ $(n \to \infty)$ for $y \in E$, which leads to $w = 0$, a contradiction. Thus we have by (3.9) that

$$\lambda_h \le \liminf_{n\to\infty} \frac{g(k, x_n(k))}{x_n(k)} \le \limsup_{n\to\infty} \frac{g(k, x_n(k))}{x_n(k)} \le \lambda_{h+1} \quad \text{for } k \in K_+,$$

which implies that there exists a subsequence of $\{x_n\}$, still called $\{x_n\}$, and $\alpha_k \in [\lambda_h, \lambda_{h+1}]$, $k \in K_+$, such that

$$\lim_{n\to\infty} \frac{g(k, x_n(k))}{x_n(k)} = \alpha_k \quad \text{for } k \in K_+. \tag{3.12}$$

If $k \in \mathbf{Z}[1,N] \setminus K_+$, then $g(k, x_n(k))/\|x_n\| \to 0$ as $n \to \infty$. Thus we can rewrite (3.11) as

$$\langle Aw_n, y\rangle - \sum_{k\in K_+} \frac{g(k, x_n(k))}{x_n(k)} \frac{x_n(k)}{\|x_n\|} y(k) \to 0 \quad \text{as } n \to \infty. \tag{3.13}$$

Letting $n \to \infty$ in (3.13) and using (3.12), we get

$$\langle Aw, y\rangle = \sum_{k\in K_+} \alpha_k w(k) y(k) \quad \text{for } y \in E. \tag{3.14}$$

Since $w(k) \ge 0$ for $k \in K_+$, it follows from (3.14) that

$$Aw = \sum_{k\in k_+} \alpha_k w^+(k), \tag{3.15}$$

which, by Lemma 3.2, implies that $w > 0$ and hence $K_+ = \mathbf{Z}[1,N]$. Thus, (3.14) can be rewritten as

$$\langle Aw, y\rangle = \sum_{k=1}^{N} \alpha_k w(k) y(k) \quad \text{for } y \in E.$$

Noticing that [10, Lemma 3.4], with its proof being modified slightly, is applicable here, we know that w is an eigenvector corresponding to λ_h or λ_{h+1}. Since $h \geq 2$, it follows from Lemma 3.3 that $w \not> \theta$. This contradiction completes the proof. □

3.3 Critical group at infinity

Lemma 3.5 *Let f satisfy* (f_∞). *Then*

$$C_q(J,\infty) \cong \delta_{q,h}\mathbf{Z}, \qquad C_q(-J,\infty) \cong \delta_{q,N-h}\mathbf{Z}. \tag{3.16}$$

Proof We claim that there exists $M > 0$ such that

$$\inf\bigl\{\bigl(1+\|x\|\bigr)\bigl\|J_s'(x)\bigr\| : \|x\| > M, s \in [0,1]\bigr\} > 0, \tag{3.17}$$

otherwise there exist $\{x_n\} \subset E$ and $\{s_n\} \subset [0,1]$ such that $\|x_n\| \to \infty$ and $(1+\|x_n\|)J(x_n) \to 0$ as $n \to \infty$, which contradict Lemma 3.1. Moreover, it is easy to see that $\inf\{J_s(x) : s \in [0,1], \|x\| \leq M\} > -\infty$. Thus, by Lemma 2.1, we have

$$C_q(J,\infty) \cong C_q(J_0,\infty).$$

On the other hand,

$$J_0(x) = \frac{1}{2}\langle Ax, x\rangle - \frac{1}{4}(\lambda_h + \lambda_{h+1})\|x\|^2.$$

Note that $x = \theta$ is the unique critical point of J_0 with the Morse index $\mu := \dim(E^- \oplus E^h) = h$ and nullity $\nu = 0$. Then, by (2.b) and (2.e),

$$C_q(J_0,\infty) \cong C_q(J_0, 0) \cong \delta_{q,h}\mathbf{Z}.$$

Similarly, we have $C_q(-J,\infty) \cong C_q(-J_0,0) \cong \delta_{q,N-h}\mathbf{Z}$. The proof is complete. □

4 Proofs of main results

Now we prove the main results of this paper. First, by applying (3.16) and (2.a), we know that J has a critical point x^* satisfying

$$C_h\bigl(J, x^*\bigr) \neq 0.$$

Define $\alpha_k = f'(k, x^*(k))$, $k \in \mathbf{Z}[1,N]$. Then from (3.3) we know by calculation that $\ker J_1''(x^*)$ is the solution space of the system $Bx = 0$, $x \in E$, where

$$B = \begin{pmatrix} 2-\alpha_1 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2-\alpha_2 & -1 & \cdots & 0 & 0 & 0 \\ & \cdots & \cdots & \cdots & & \cdots & \\ 0 & 0 & 0 & \cdots & -1 & 2-\alpha_{N-1} & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 2-\alpha_N \end{pmatrix}_{N\times N}.$$

Thus $\nu_1 = \dim\ker J_1''(x^*) \le 1$ since B possesses non-degenerate $(N-1)$ order submatrixes. By (2.d)-(2.e), we further have

$$C_q\bigl(J,x^*\bigr) \cong \delta_{q,h}\mathbf{Z}. \tag{4.1}$$

Proof of Theorem 1.1 First we give the proof for the case (i). By $f'(k,0) < \lambda_1$ for $k \in \mathbf{Z}[1,N]$, we know that $x = \theta$ is a strict local minimizer of J. Thus, by (2.f), we have correspondingly

$$C_q(J,\theta) \cong \delta_{q,0}\mathbf{Z}. \tag{4.2}$$

Noticing that $h \ge 2$, by comparing (4.2) with (4.1), we have $x^* \ne \theta$.

For $k \in \mathbf{Z}[1,N]$, set $f^+(k,t) = f(k,t)$ for $t \ge 0$, $f^+(k,t) = 0$ for $t < 0$. Let $F^+(k,t) = \int_0^t f^+(k,s)\,ds$. Then the critical points of

$$J^+(x) = \frac{1}{2}\langle Ax,x\rangle - \sum_{k=1}^{N} F^+\bigl(k,x(k)\bigr)$$

are exactly solutions of the problem

$$Ax = \mathbf{f}^+\bigl(x^+\bigr), \tag{4.3}$$

where $\mathbf{f}^+(x) = (f^+(1,x(1)),\dots,f^+(N,x(N)))^T$, $x \in E$. By Lemma 3.4, we see that $J^+ \in C^{2-0}(E,\mathbf{R})$ satisfies the (PS) condition. From the definition of $f^+(k,\cdot)$ and the assumption $f'(k,0) < \lambda_1$, $k \in \mathbf{Z}[1,N]$, we know that there exists $\eta > 0$ such that $(\lambda_1 t - f^+(k,t))t > 0$ for $t \in (-\eta,\eta) \setminus \{0\}$. For any fixed $x \in E$ with $0 < \|x\| < \eta$, define a function $\phi(s) = J^+(sx)$, $s \in [0,1]$. By the Lagrange mean value theorem, there exists $\xi \in (0,1)$ such that

$$\begin{aligned}
J^+(x) &= \phi(1) - \phi(0) = \phi'(\xi) \\
&= \bigl\langle A(\xi x),x\bigr\rangle - \sum_{k=1}^{N} f^+\bigl(k,\xi x(k)\bigr)x(k) \\
&\ge \sum_{k=1}^{N} \bigl\{\lambda_1\xi x(k) - f^+\bigl(k,\xi x(k)\bigr)\bigr\}x(k) > 0,
\end{aligned}$$

which implies that there exist $\rho > 0$ and $\tau > 0$ such that

$$J^+(x) \ge \tau, \quad x \in E \text{ with } \|x\| = \rho.$$

In addition, let ς_1 be the eigenvector of A corresponding to λ_1 with $\varsigma_1 > 0$, then (f_∞), with $h \in \mathbf{Z}[2,N-1]$, implies that

$$J^+(t\varsigma_1) \to -\infty \quad \text{as } t \to +\infty.$$

By Mountain Pass Theorem [17, 18], J^+ has a critical point $x_1 \ne \theta$ with the critical group property for a mountain pass point [17], that is, $C_1(J^+,x_1) \ncong 0$. Noticing that x_1 satisfies

(4.3), we get by Lemma 3.2 that $x_1 > \theta$ and hence x_1 is also a mountain pass point of J, that is, $C_1(J, x_1) \ncong 0$.

The same argument shows that J has a nontrivial critical point $x_2 < \theta$ with $C_1(J, x_2) \ncong 0$. Noticing that $h \geq 2$, by comparing the critical groups, we see that x_1, x_2 and x^* are three nontrivial critical points of J.

By the same argument as that for (4.1), we get $C_q(J, x_i) \cong \delta_{q,1}\mathbf{Z}$, $i = 1, 2$. If x_1, x_2 and x^* are all the nontrivial critical points of J, then $\mathcal{K} = \{\theta, x_1, x_2, x^*\}$ and then (2.c) reads

$$(-1)^0 \times 1 + (-1)^1 \times 2 + (-1)^h \times 1 = (-1)^h \times 1,$$

a contradiction. Thus we claim that there exist at least four nontrivial critical points of J.

In the case (ii), we consider the functional $-J$. By applying (3.16) and (2.a), we know that $-J$ possesses a critical point x_1^* satisfying

$$C_{N-h}\left(-J, x_1^*\right) \neq 0. \tag{4.4}$$

Since $f'(k, 0) > \lambda_N$ for $k \in \mathbf{Z}[1, N]$, $x = \theta$ is a strict local minimizer of $-J$ and

$$C_q(-J, \theta) \cong \delta_{q,0}\mathbf{Z}. \tag{4.5}$$

Noticing that $h \leq N - 2$, we know by comparing (4.5) with (4.4) that $x_1^* \neq \theta$. The rest of the arguments are similar to that in case (i) and will be omitted. The proof is complete. □

Proof of Theorem 1.2 In view of (3.3) and the assumption $\lambda_m < f'(k, 0) < \lambda_{m+1}$, $k \in \mathbf{Z}[1, N]$, we see that $x = \theta$ is a non-degenerate critical point of J with the Morse index $\mu_0 = m$. Thus

$$C_q(J, 0) \cong \delta_{q,m}\mathbf{Z}. \tag{4.6}$$

Noticing that $h \neq m$, we know by comparing (4.6) with (4.1) that $x^* \neq \theta$.

We may assume that $t_0 > 0$ in (f_0). For $k \in \mathbf{Z}[1, N]$, set

$$\tilde{f}(k, t) = \begin{cases} 0, & t < 0, \\ f(k, t), & t \in [0, t_0], \\ 0, & t > t_0. \end{cases}$$

Define

$$\tilde{J}(x) = \frac{1}{2}\langle Ax, x\rangle - \sum_{k=1}^{N} \tilde{F}\left(k, x(k)\right), \quad x \in E,$$

where $\tilde{F}(k, t) = \int_0^t \tilde{f}(k, s)\, ds$. Since $\tilde{J}(x) \to +\infty$ as $\|x\| \to \infty$, there is a minimizer x_0 of $\tilde{J}$. Thus

$$Ax_0 = \tilde{\mathbf{f}}(x_0),$$

where $\tilde{\mathbf{f}}(x) = (\tilde{f}(1,x(1)),\tilde{f}(2,x(2)),\dots,\tilde{f}(N,x(N)))^T$ for $x \in E$. By the definition of $\tilde{f}$, the above equality can be rewritten as

$$Ax_0 = \tilde{\mathbf{f}}(x_0^+).$$

From Lemma 3.4, we know that $x_0 = \theta$ or $x_0 > \theta$. By assumption $f'(k,0) \in (\lambda_m,\lambda_{m+1})$ and $m \geq 2$, we know that θ is not a minimizer. Thus we have $x_0 > \theta$. In the same way as the proof of Lemma 3.2, we can prove that $x_0(k) < t_0$ for $k \in \mathbf{Z}[1,N]$. Thus x_0 is a local minimizer of J, therefore

$$C_q(J,x_0) \cong \delta_{q,0}\mathbf{Z}. \tag{4.7}$$

Define $\hat{f}(k,t) = f(k,t+x_0(k)) - f(k,x_0(k))$, $(k,t) \in \mathbf{Z}[1,N] \times \mathbf{R}$ and consider the functional

$$\hat{J}(z) = \frac{1}{2}\langle Az,z\rangle - \sum_{k=1}^{N} \hat{F}\bigl(k,z(k)\bigr), \quad z \in E,$$

where $\hat{F}(k,t) = \int_0^t \hat{f}(k,s)\,ds$. A simple calculation shows that if z is a positive critical point of $\hat{J}$, then $x_0 + z$ is a critical point of J, and, moreover, $C_q(\hat{J},z) = C_q(J,x_0+z)$.

Furthermore, define

$$\hat{f}^+(k,t) = \begin{cases} \hat{f}(k,t), & t \geq 0, \\ 0, & t < 0, \end{cases} \quad k \in \mathbf{Z}[1,N]$$

and its energy functional

$$\hat{J}^+(z) = \frac{1}{2}\langle Az,z\rangle - \sum_{k=1}^{N} \hat{F}^+\bigl(k,z(k)\bigr), \quad z \in E,$$

where $\hat{F}^+(k,t) = \int_0^t \hat{f}^+(k,s)\,ds$. By (f_∞), we see that $\hat{f}^+$ satisfies

$$\lambda_h \leq \liminf_{t\to+\infty} \frac{\hat{f}^+(k,t)}{t} \leq \liminf_{t\to+\infty} \frac{\hat{f}^+(k,t)}{t} \leq \lambda_{h+1}, \quad k \in \mathbf{Z}[1,N].$$

It follows from Lemma 3.4 that $\hat{J}^+$ satisfies the (PS) condition. If x_0 is not a strict local minimizer of J, then there exists infinitely many critical points near x_0 and the conclusion holds. Now we assume that x_0 is a strict local minimizer of J, then $z = \theta$ is a strict local minimizer of $\hat{J}^+$. In the same way as the proof of Theorem 1.1, we know that $\hat{J}^+$ has a critical point z_1, which is a mountain pass point of $\hat{J}^+$ with $C_1(\hat{J}^+,z_1) \ncong 0$ and $z_1 > \theta$. Thus z_1 is also a critical point of $\hat{J}$ with $C_1(\hat{J},z_1) \ncong 0$. Hence $x_1 = x_0 + z_1$ is a critical point of J with $C_1(J,x_1) \ncong 0$.

In a similar way, we know that J has a critical point $x_2 < x_0$ with $C_1(J,x_2) \ncong 0$. Finally, by comparing the critical groups and by using the condition $m,h \geq 2$ with $m \neq k$, we see that x^*, x_0, x_1 and x_2 are four nontrivial critical points of J in which x_0 and x_1 are positive. The proof is complete. □

The proof of the following lemma is similar to that of [19, Theorem 3.1] and is omitted.

Lemma 4.1 [19] *Let f satisfy* (F_0^+) (*or* (F_0^-)). *Then J has a local linking at $x = \theta$ with respect to the decomposition $E = H^- \oplus E^+$, where $E^- := \bigoplus_{l \le m} E^l$ (or $E^- := \bigoplus_{l < m} E^l$ respectively).*

Proof of Theorem 1.3 In view of (3.3) and the assumption $f'(k,0) = \lambda_m$, $k \in \mathbf{Z}[1,N]$, we see that $x = \theta$ is a degenerate critical point of J with the Morse index $\mu_0 = m-1$ and nullity $\nu_0 = 1$. By Lemma 4.1 and (2.g), we have, corresponding to (F_0^-) or (F_0^+) respectively,

$$C_q(J,0) \cong \delta_{q,m-1}\mathbf{Z} \quad \text{or} \quad C_q(J,0) \cong \delta_{q,m}\mathbf{Z}. \tag{4.8}$$

which, compared with (4.1), implies that $x^* \neq 0$ in both of cases (i) and (ii). The rest of the proof is similar to that of Theorem 1.2 and will be omitted. The proof is complete. □

Competing interests
The author declares that they have no competing interests.

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