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Strong convergent result for quasi-nonexpansive mappings in Hilbert spaces

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Abstract

In this article, we study an iterative method over the class of quasi-nonexpansive mappings which are more general than nonexpansive mappings in Hilbert spaces. Our strong convergent theorems include several corresponding authors' results.

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1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot,\cdot \rangle$, and induced norm $||\cdot||$. A mapping T: $H \to H$ is called nonexpansive if $||Tx - Ty|| \leq ||x - y||$ for all $x,y \in H$. The set of the fixed points of T is denoted by $Fix(T) := \{x \in H: Tx = x\}$.

The viscosity approximation method was first introduced by Moudafi [1] in 2000. Starting with an arbitrary initial $x_0 \in H$, define a sequence $\{x_n\}$ generated by

$$x_{n+1} = \frac{\varepsilon_n}{1+\varepsilon_n} f(x_n) + \frac{1}{1+\varepsilon_n} Tx_n, \quad \forall\, n \geq 0, \tag{1.1}$$

where f is a contraction with a coefficient $\alpha \in [0,1)$ on H, i.e., $||f(x) - f(y)|| \leq \alpha||x - y||$ for all $x,y \in H$, T is nonexpansive, and $\{\varepsilon_n\}$ is a sequence in (0,1) satisfying the following given conditions:

(i1) $\lim_{n\to\infty} \varepsilon_n = 0$;
(i2) $\sum_{n=0}^{\infty} \varepsilon_n = \infty$;
(i3) $\lim_{n\to\infty}(\frac{1}{\varepsilon_n} - \frac{1}{\varepsilon_{n+1}}) = 0$.

It is proved that the sequence $\{x_n\}$ generated by (1.1) converges strongly to the unique solution $x^* \in C(C := Fix(T))$ of the variational inequality:

$$\langle (I-f)x^*, x - x^* \rangle \geq 0, \quad \forall\, x \in Fix(T).$$

In 2003, Xu [2] proved that the sequence $\{x_n\}$ defined by the below process where T is also nonexpansive, started with an arbitrary initial $x_0 \in H$:

$$x_{n+1} = \alpha_n b + (I - \alpha_n A)Tx_n, \quad \forall\, n \geq 0, \tag{1.2}$$

converges strongly to the unique solution of the minimization problem (1.3) when the sequence $\{\alpha_n\}$ satisfies certain conditions:

$$\min_{x\in C} \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle, \tag{1.3}$$

where C is the set of fixed points set of T on H and b is a given point in H.

In 2006, Marino and Xu [3] combined the iterative method (1.2) with the viscosity approximation method (1.1) and considered the following general iterative method:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Tx_n, \quad \forall\, n \geq 0. \tag{1.4}$$

It is proved that if the sequence $\{\alpha_n\}$ satisfies appropriate conditions, the sequence $\{x_n\}$ generated by (1.4) converges strongly to the unique solution of the variational inequality:

$$\langle(\gamma f - A)\tilde{x}, x - \tilde{x}\rangle \leq 0, \quad \forall\, x \in C, \tag{1.5}$$

or equivalently $\tilde{x} = P_{Fix(T)}(I - A + \gamma f)\tilde{x}$, where C is the fixed point set of a nonexpansive mapping T.

In 2009, Maingè [4] considered the viscosity approximation method (1.1), and expanded the strong convergence to quasi-nonexpansive mappings in Hilbert space.

In 2010, Tian [5] considered the following general iterative method under the frame of nonexpansive mappings:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \mu\alpha_n F)Tx_n, \quad \forall\, n \geq 0, \tag{1.6}$$

and gave some strong convergent theorems.

Very recently, Tian [6] extended (1.6) to a more general scheme, that is: the mapping f: $H \to H$ is no longer a contraction but a L-Lipschitzian continuous operator with coefficient $L > 0$, and proved that if the sequence $\{\alpha_n\}$ satisfies appropriate conditions, the sequence $\{x_n\}$ generated by $x_{n+1} = \alpha_n\gamma f(x_n) + (I - \mu\alpha_n F)Tx_n$ converges strongly to the unique solution $\tilde{x} \in Fix(T)$ of the variational inequality where T is still nonexpansive:

$$\langle(\gamma f - \mu F)\tilde{x}, x - \tilde{x}\rangle \leq 0, \quad \forall\, x \in Fix(T). \tag{1.7}$$

Motivated by Maingè [4] and Tian [6], we consider the following iterative process:

$$\begin{cases} x_0 = x \in H \quad \text{arbitrarily chosen}, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n \mu F)T_\omega x_n, \quad \forall\, n \geq 0, \end{cases} \tag{1.8}$$

where f is L-Lipschitzian, $T_\omega = (1 - \omega)I + \omega T$, and T is a quasi-nonexpansive mapping. Under some appropriate conditions on ω and $\{\alpha_n\}$, we obtain strong convergence over the class of quasi-nonexpansive mappings in Hilbert spaces. Our result is more general than Maingè's [4] conclusion.

2. Preliminaries

Throughout this article, we write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. $x_n \to x$ implies that the sequence $\{x_n\}$ converges strongly to x. The following lemmas are useful for our article.

The following statements are valid in a Hilbert space H: for each $x,y \in H$, $t \in [0,1]$

(i) $||x + y|| \leq ||x||^2 + 2\langle y, x + y\rangle$;
(ii) $||(1 - t)x + ty||^2 = (1 - t)||x||^2 + t||y||^2 - (1 - t)t||x - y||^2$;
(iii) $\langle x, y\rangle = -\frac{1}{2}\|x-y\|^2 + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|y\|^2$.

Lemma 2.1. *Let f: $H \to H$ be a L-Lipschitzian continuous operator with coefficient $L > 0$. F: $H \to H$ is a κ-Lipschitzian continuous and η-strongly monotone operator with $\kappa > 0$ and $\eta > 0$. Then, for $0 < \gamma \leq \mu\eta/L$,*

$$\langle x-y, (\mu F-\gamma f)x-(\mu F-\gamma f)y\rangle \geq (\mu\eta-\gamma L)\|x-y\|^2. \tag{2.1}$$

That is, $\mu F - \gamma f$ is strongly monotone with coefficient $\mu\eta - \gamma L$.

Lemma 2.2. [4]*Let $T_\omega := (1 - \omega)I + \omega T$, with T quasi-nonexpansive on H, $Fix(T) \neq \emptyset$, and $\omega \in (0,1]$. Then, the following statements are reached:*

(a1) $Fix(T) = Fix(T_\omega)$;
(a2) T_ω *is quasi-nonexpansive*;
(a3) $||T_\omega x - q||^2 \leq ||x - q||^2 - \omega(1 - \omega)||Tx - x||^2$ *for all* $x \in H$ *and* $q \in Fix(T)$;
(a4) $\langle x - T_\omega x, x - q\rangle \geq \frac{\omega}{2}\|x - Tx\|^2$ *for all* $x \in H$ *and* $q \in Fix(T)$.

Proposition 2.3. *From the equality (iii) and the fact that T is quasi-nonexpansive, we have*

$$\langle x-Tx, x-q\rangle = -\frac{1}{2}\|Tx-q\|^2 + \frac{1}{2}\|x-Tx\|^2 + \frac{1}{2}\|x-q\|^2 \geq \frac{1}{2}\|x-Tx\|^2.$$

(a4) is easily deduced by $I-T_\omega = \omega(I-T)$ and the previous inequality.

Lemma 2.4. [7]*Let $\{\Gamma_n\}$ be a sequence of real numbers that does not decrease at infinity, in the sense that there exist a subsequence $\{\Gamma_{n_j}\}_{j\geq 0}$ of $\{\Gamma_n\}$ which satisfies $\Gamma_{n_j} < \Gamma_{n_j+1}$ for all $j \geq 0$. Also, consider the sequence of integers $\{\tau(n)\}_{n\geq n_0}$ defined by*

$$\tau(n) = \max\{k \leq n \mid \Gamma_k < \Gamma_{k+1}\}.$$

Then, $\{\tau(n)\}_{n\geq n_0}$ is a nondecreasing sequence verifying $\lim_{n\to\infty} \tau(n) = \infty$ and for all $n \geq n_0$, it holds that $\Gamma_{\tau(n)} < \Gamma_{\tau(n)+1}$ and we have

$$\Gamma_n \leq \Gamma_{\tau(n)+1}.$$

Recall the metric projection P_K from a Hilbert space H to a closed convex subset K of H is defined: for each $x \in H$ the unique element $P_K x \in K$ such that

$$\|x - P_K x\| := \inf\{\|x-y\| : y \in K\}.$$

Lemma 2.5. *Let K be a closed convex subset of H. Given $x \in H$, and $z \in K$, $z = P_K x$, if and only if there holds the inequality:*

$$\langle x-z, y-z\rangle \leq 0, \quad \forall\, y \in K.$$

Lemma 2.6. *If x^* is the solution of the variational inequality (1.7) with T: $H \to H$ demi-closed and $\{y_n\} \in H$ is a bounded sequence such that $||Ty_n - y_n|| \to 0$, then*

$$\liminf_{n\to\infty}\langle(\mu F-\gamma f)x^*, y_n - x^*\rangle \geq 0. \tag{2.2}$$

Proof. We assume that there exists a subsequence $\{y_{n_j}\}$ of $\{y_n\}$ such that $y_{n_j} \rightharpoonup \tilde{y}$. From the given conditions $\|Ty_n - y_n\| \to 0$ and T: $H \to H$ demi-closed, we have that any weak cluster point of $\{y_n\}$ belongs to the fixed point set $Fix(T)$. Hence, we conclude that $\tilde{y} \in Fix(T)$, and also have that

$$\liminf_{n\to\infty}\langle(\mu F - \gamma f)x^*, y_n - x^*\rangle = \lim_{j\to\infty}\langle(\mu F - \gamma f)x^*, y_{n_j} - x^*\rangle.$$

Recalling (1.7), we immediately obtain

$$\liminf_{n\to\infty}\langle(\mu F - \gamma f)x^*, y_n - x^*\rangle = \langle(\mu F - \gamma f)x^*, \tilde{y} - x^*\rangle \geq 0.$$

This completes the proof. □

3. Main results

Let H be a real Hilbert space, let F be a κ-Lipschitzian and η-strongly monotone operator on H with $k > 0$, $\eta > 0$, and let T be a quasi-nonexpansive mapping on H, and f is a L-Lipschitzian mapping with coefficient $L > 0$ for all $x,y \in H$. Assume the set $Fix(T)$ of fixed points of T is nonempty and we note that $Fix(T)$ is closed and convex.

Theorem 3.1. *Let* $0 < \mu < 2\eta/\kappa^2, 0 < \gamma < \mu(\eta - \frac{\mu\kappa^2}{2})/L = \tau/L$, *and start with an arbitrary chosen* $x_0 \in H$, *let the sequence* $\{x_n\}$ *be generated by*

$$x_{n+1} = \alpha_n\gamma f(x_n) + (I - \alpha_n\mu F)T_\omega x_n, \tag{3.1}$$

where the sequence $\{\alpha_n\} \subset (0,1)$ *satisfies* $\lim_{n\to\infty} \alpha_n = 0$, *and* $\sum_{n=0}^{\infty}\alpha_n = \infty$. *Also* $\omega \in (0, \frac{1}{2})$, $T_\omega := (1 - \omega)I + \omega I$ *with two conditions on* T:

(C1) $||Tx - q|| \leq ||x - q||$ *for any* $x \in H$, *and* $q \in Fix(T)$; *this means that* T *is a quasi-nonexpansive mapping*;
(C2) T *is demi-closed on H; that is: if* $\{y_k\} \in H$, $y_k \rightharpoonup z$, *and* $(I - T)y_k \to 0$, *then* $z \in Fix(T)$.

Then, $\{x_n\}$ *converges strongly to the* $x^* \in Fix(T)$ *which is the unique solution of the VIP*:

$$\langle(\mu F - \gamma f)x^*, x - x^*\rangle \geq 0, \quad \forall\, x \in Fix(T). \tag{3.2}$$

Proof. First, we show that $\{x_n\}$ is bounded.
Take any $p \in Fix(T)$, by Lemma 2.2 (a3), we have

$$\begin{aligned}
&\|x_{n+1} - p\| \\
&\quad = \|\alpha_n\gamma f(x_n) + (I - \alpha_n\mu F)T_\omega x_n - p\| \\
&\quad = \|\alpha_n\gamma(f(x_n) - f(p)) + \alpha_n(\gamma f(p) - \mu Fp) + (I - \alpha_n\mu F)T_\omega x_n - (I - \alpha_n\mu F)p\| \\
&\quad \leq \alpha_n\gamma L\|x_n - p\| + \alpha_n\|\gamma f(p) - \mu Fp\| + (1 - \alpha_n\tau)\|x_n - p\| \\
&\quad \leq (1 - \alpha_n(\tau - \gamma L))\|x_n - p\| + \alpha_n\|\gamma f(p) - \mu Fp\|.
\end{aligned} \tag{3.3}$$

By induction, we have

$$\|x_n - p\| \leq \max\left\{\|x_0 - p\|, \frac{\|\gamma f(p) - \mu Fp\|}{\tau - \gamma L}\right\}, \quad \forall\, n \geq 0.$$

Hence, $\{x_n\}$ is bounded, so are the $\{f(x_n)\}$ and $\{F(x_n)\}$.

From (3.1), we have

$$x_{n+1} - x_n + \alpha_n(\mu F x_n - \gamma f(x_n)) = (I - \alpha_n \mu F)T_\omega x_n - (I - \alpha_n \mu F)x_n. \tag{3.4}$$

Since $x^* \in Fix(T)$, from Lemma 2.2 (a4), and together with (3.4), we obtain

$$\begin{aligned}
&\langle x_{n+1} - x_n + \alpha_n(\mu F(x_n) - \gamma f(x_n)), x_n - x^*\rangle \\
&\quad = \langle (I - \alpha_n\mu F)T_\omega x_n - (I - \alpha_n \mu F)x_n, x_n - x^*\rangle \\
&\quad = (1 - \alpha_n)\langle T_\omega x_n - x_n, x_n - x^*\rangle + \alpha_n\langle (I - \mu F)T_\omega x_n - (I - \mu F)x_n, x_n - x^*\rangle \\
&\quad \le -\frac{\omega}{2}(1 - \alpha_n)\|x_n - Tx_n\|^2 + \alpha_n\|(I - \mu F)T_\omega x_n - (I - \mu F)x_n\|\|x_n - x^*\| \\
&\quad \le -\frac{\omega}{2}(1 - \alpha_n)\|x_n - Tx_n\|^2 + \alpha_n(1 - \tau)\|T_\omega x_n - x_n\|\|x_n - x^*\| \\
&\quad = -\frac{\omega}{2}(1 - \alpha_n)\|x_n - Tx_n\|^2 + \omega\alpha_n(1 - \tau)\|Tx_n - x_n\|\|x_n - x^*\|,
\end{aligned}$$

it follows from the previous inequality that

$$\begin{aligned}
-\langle x_n - x_{n+1}, x_n - x^*\rangle \le -\alpha_n\langle(\mu F - \gamma f)x_n, x_n - x^*\rangle - \frac{\omega}{2}(1 - \alpha_n)\|x_n - Tx_n\|^2 \\
+ \omega\alpha_n(1 - \tau)\|Tx_n - x_n\|\|x_n - x^*\|.
\end{aligned} \tag{3.5}$$

From (iii), we obviously have

$$\langle x_n - x_{n+1}, x_n - x^*\rangle = -\frac{1}{2}\|x_{n+1} - x^*\|^2 + \frac{1}{2}\|x_n - x^*\|^2 + \frac{1}{2}\|x_{n+1} - x_n\|^2. \tag{3.6}$$

Set $\Gamma_n := \frac{1}{2}\|x_n - x^*\|^2$, and combine (3.5) with (3.6), it follows that

$$\begin{aligned}
\Gamma_{n+1} - \Gamma_n - \frac{1}{2}\|x_{n+1} - x_n\|^2 \le -\alpha_n\langle(\mu F - \gamma f)x_n, x_n - x^*\rangle - \frac{\omega}{2}(1 - \alpha_n)\|x_n - Tx_n\|^2 \\
+ \omega\alpha_n(1 - \tau)\|Tx_n - x_n\|\|x_n - x^*\|.
\end{aligned} \tag{3.7}$$

Now, we calculate $||x_n+1 - x_n||$.

From the given condition: $T_\omega := (1 - \omega)I + \omega T$, it is easy to deduce that $||T_\omega x_n - x_n|| = \omega||x_n - Tx_n||$. Thus, it follows from (3.4) that

$$\begin{aligned}
\|x_{n+1} - x_n\|^2 &= \|\alpha_n(\gamma f(x_n) - \mu F x_n) + (I - \alpha_n\mu F)T_\omega x_n - (I - \alpha_n \mu F)x_n\|^2 \\
&\le 2\alpha_n^2\|\gamma f(x_n) - \mu F x_n\|^2 + 2(1 - \alpha_n\tau)^2\|T_\omega x_n - x_n\|^2 \\
&= 2\alpha_n^2\|\gamma f(x_n) - \mu F x_n\|^2 + 2\omega^2(1 - \alpha_n\tau)^2\|Tx_n - x_n\|^2.
\end{aligned} \tag{3.8}$$

Then, from (3.7) and (3.8), we have

$$\begin{aligned}
&\Gamma_{n+1} - \Gamma_n + \left[\frac{\omega}{2}(1 - \alpha_n) - \omega^2(1 - \alpha_n\tau)^2\right]\|x_n - Tx_n\|^2 \\
&\quad \le \alpha_n[\alpha_n\|\gamma f(x_n) - \mu F x_n\|^2 - \langle(\mu F - \gamma f)x_n, x_n - x^*\rangle \\
&\qquad + \omega(1 - \tau)\|Tx_n - x_n\|\|x_n - x^*\|].
\end{aligned} \tag{3.9}$$

Finally, we prove $x_n \to x^*$. To this end, we consider two cases.

Case 1: Suppose that there exists n_0 such that $\{\Gamma_n\}_{n \ge n_0}$ is nonincreasing, it is equal to $\Gamma_{n+1} \le \Gamma_n$ for all $n \ge n_0$. It follows that $\lim_{n\to\infty} \Gamma_n$ exists, so we conclude that

$$\lim_{n\to\infty}(\Gamma_{n+1} - \Gamma_n) = 0. \tag{3.10}$$

It follows from (3.9),(3.10) and combine with the fact that $\lim_{n\to\infty} \alpha_n = 0$, we have $\lim_{n\to\infty}||x_n - Tx_n|| = 0$. Considering (3.9) again, from (3.10), we have

$$\begin{aligned} &-\alpha_n[\alpha_n\|\gamma f(x_n) - \mu F x_n\|^2 - \langle(\mu F - \gamma f)x_n, x_n - x^*\rangle \\ &\quad + \omega(1-\tau)\|Tx_n - x_n\|\|x_n - x^*\|] \\ &\quad\le \Gamma_n - \Gamma_{n+1}. \end{aligned} \tag{3.11}$$

Then, by $\sum_{n=0}^{\infty} \alpha_n = \infty$, we conclude that

$$\begin{aligned} &\liminf_{n\to\infty} -[\alpha_n\|\gamma f(x_n) - \mu F x_n\|^2 - \langle(\mu F - \gamma f)x_n, x_n - x^*\rangle \\ &\quad + \omega(1-\tau)\|Tx_n - x_n\|\|x_n - x^*\|] \\ &\quad\le 0. \end{aligned} \tag{3.12}$$

Since $\{f(x_n)\}$ and $\{x_n\}$ are both bounded, as well as $\alpha_n \to 0$, and $\lim_{n\to\infty}||x_n - Tx_n|| = 0$, it follows from (3.12) that

$$\liminf_{n\to\infty}\langle(\mu F - \gamma f)x_n, x_n - x^*\rangle \le 0. \tag{3.13}$$

From Lemma 2.1, it is obvious that

$$\langle(\mu F - \gamma f)x_n, x_n - x^*\rangle \ge \langle(\mu F - \gamma f)x^*, x_n - x^*\rangle + 2(\mu\eta - \gamma L)\Gamma_n. \tag{3.14}$$

Thus, from (3.14), and the fact that $\lim_{n\to\infty}\Gamma_n$ exists, we immediately obtain
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$$\begin{aligned} &\liminf_{n\to\infty}\langle(\mu F - \gamma f)x^*, x_n - x^*\rangle + 2(\mu\eta - \gamma L)\Gamma_n \\ &\quad = 2(\mu\eta - \gamma L)\lim_{n\to\infty}\Gamma_n + \liminf_{n\to\infty}\langle(\mu F - \gamma f)x^*, x_n - x^*\rangle \le 0, \end{aligned} \tag{3.15}$$

or equivalently

$$2(\mu\eta - \gamma L)\lim_{n\to\infty}\Gamma_n \le -\liminf_{n\to\infty}\langle(\mu F - \gamma f)x^*, x_n - x^*\rangle. \tag{3.16}$$

Finally, by Lemma 2.6, we have

$$2(\mu\eta - \gamma L)\lim_{n\to\infty}\Gamma_n \le 0, \tag{3.17}$$

so we conclude that $\lim_{n\to\infty}\Gamma_n = 0$, which equivalently means that $\{x_n\}$ converges strongly to x^*.

Case 2: Assume that there exists a subsequence $\{\Gamma_{n_j}\}_{j\ge 0}$ of $\{\Gamma_n\}_{n\ \ge\ 0}$ such that $\Gamma_{n_j} < \Gamma_{n_j+1}$ for all $j \in \mathbb{N}$. In this case, it follows from Lemma 2.4 that there exists a subsequence $\{\Gamma_{\tau(n)}\}$ of $\{\Gamma_n\}$ such that $\Gamma_{\tau(n)+1} > \Gamma_{\tau(n)}$, and $\{\tau(n)\}$ is defined as in Lemma 2.4.

Invoking (3.9) again, it follows that

$$\begin{aligned} &\Gamma_{\tau(n)+1} - \Gamma_{\tau(n)} + \left[\frac{\omega}{2}(1-\alpha_{\tau(n)}) - \omega^2(1-\alpha_{\tau(n)}\tau)^2\right]\|x_{\tau(n)} - Tx_{\tau(n)}\|^2 \\ &\quad\le \alpha_{\tau(n)}[\alpha_{\tau(n)}\|\gamma f(x_{\tau(n)}) - \mu F x_{\tau(n)}\|^2 - \langle(\mu F - \gamma f)x_{\tau(n)}, x_{\tau(n)} - x^*\rangle \\ &\qquad + \omega(1-\tau)\|Tx_{\tau(n)} - x_{\tau(n)}\|\|x_{\tau(n)} - x^*\|]. \end{aligned}$$

Recalling the fact that $\Gamma_{\tau(n)+1} > \Gamma_{\tau(n)}$, we have

$$\begin{aligned}&\left[\frac{\omega}{2}(1-\alpha_{\tau(n)}) - \omega^2(1-\alpha_{\tau(n)}\tau)^2\right]\|x_{\tau(n)} - Tx_{\tau(n)}\|^2 \\ &\quad \le \alpha_{\tau(n)}[\alpha_{\tau(n)}\|\gamma f(x_{\tau(n)}) - \mu F x_{\tau(n)}\|^2 - \langle(\mu F - \gamma f)x_{\tau(n)}, x_{\tau(n)} - x^*\rangle \\ &\qquad + \omega(1-\tau)\|Tx_{\tau(n)} - x_{\tau(n)}\|\|x_{\tau(n)} - x^*\|].\end{aligned} \tag{3.18}$$

From the preceding results, we get the boundedness of $\{x_n\}$ and $\alpha_n \to 0$ which obviously lead to

$$\lim_{n\to\infty}\|x_{\tau(n)} - Tx_{\tau(n)}\| = 0. \tag{3.19}$$

Hence, combining (3.18) with (3.19), we immediately deduce that

$$\begin{aligned}\langle(\mu F - \gamma f)x_{\tau(n)}, x_{\tau(n)} - x^*\rangle &\le \alpha_{\tau(n)}\|\gamma f(x_{\tau(n)}) - \mu F x_{\tau(n)}\|^2 \\ &\quad + \omega(1-\tau)\|Tx_{\tau(n)} - x_{\tau(n)}\|\|x_{\tau(n)} - x^*\|.\end{aligned} \tag{3.20}$$

Again, (3.14) and (3.20) yield

$$\begin{aligned}\langle(\mu F - \gamma f)x^*, x_{\tau(n)} - x^*\rangle + 2(\mu\eta - \gamma L)\Gamma_{\tau(n)} &\le \alpha_{\tau(n)}\|\gamma f(x_{\tau(n)}) - \mu F x_{\tau(n)}\|^2 \\ &\quad + \omega(1-\tau)\|Tx_{\tau(n)} - x_{\tau(n)}\|\|x_{\tau(n)} - x^*\|.\end{aligned} \tag{3.21}$$

Recall that $\lim_{n\to\infty}\alpha_{\tau(n)} = 0$, from (3.19) and (3.21), we immediately have

$$2(\mu\eta - \gamma L)\limsup_{n\to\infty}\Gamma_{\tau(n)} \le -\liminf_{n\to\infty}\langle(\mu F - \gamma f)x^*, x_{\tau(n)} - x^*\rangle. \tag{3.22}$$

By Lemma 2.6, we have

$$\liminf_{n\to\infty}\langle(\mu F - \gamma f)x^*, x_{\tau(n)} - x^*\rangle \ge 0. \tag{3.23}$$

Consider (3.22) again, we conclude that

$$\limsup_{n\to\infty}\Gamma_{\tau(n)} = 0, \tag{3.24}$$

which means that $\lim_{n\to\infty}\Gamma_{\tau(n)} = 0$. By Lemma 2.4, it follows that $\Gamma_n \le \Gamma_{\tau(n)}$, thus, we get $\lim_{n\to\infty}\Gamma_n = 0$, which is equivalent to $x_n \to x^*$. □

Remark 3.2. Corollary 3.3 is only valid for $\omega \in (0, \frac{1}{2})$. This is revised by Wongchan and Saejung [8].

corollary 3.3. [4]*Let the sequence* $\{x_n\}$ *be generated by*

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T_\omega x_n, \tag{3.25}$$

where the sequence $\{\alpha_n\} \subset (0,1)$ *satisfies* $\lim_{n\to\infty}\alpha_n = 0$, *and* $\sum_{n=0}^{\infty}\alpha_n = \infty$. *Also* $\omega \in (0, \frac{1}{2})$, *and* $T_\omega := (1 - \omega)I + \omega T$ *with two conditions on T:*

(C1) $||Tx - q|| \le ||x - q||$ *for any* $x \in H$, *and* $q \in Fix(T)$; *this means that T is a quasi-nonexpansive mapping;*
(C2) *T is demi-closed on H; that is: if* $\{y_k\} \in H$, $y_k \rightharpoonup z$, *and* $(I - T)y_k \to 0$, $z \in Fix(T)$.

Then, $\{x_n\}$ *converges strongly to the* $x^* \in Fix(T)$ *which is the unique solution of the*

VIP(3.26):

$$\langle (I-f)x^*, x-x^* \rangle \geq 0, \quad \forall\, x \in Fix(T). \tag{3.26}$$

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Authors' contributions
All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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