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Research Article

Systems of Quasilinear Parabolic Equations with Discontinuous Coefficients and Continuous Delays

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This paper is concerned with a weakly coupled system of quasilinear parabolic equations where the coefficients are allowed to be discontinuous and the reaction functions may depend on continuous delays. By the method of upper and lower solutions and the associated monotone iterations and by difference ratios method and various estimates, we obtained the existence and uniqueness of the global piecewise classical solutions under certain conditions including mixed quasimonotone property of reaction functions. Applications are given to three 2-species Volterra-Lotka models with discontinuous coefficients and continuous delays.

1. Introduction

Reaction-diffusion equations with time delays have been studied by many researchers (see [1–8] and references therein). However, all of the discussions in the literature are devoted to the equations with continuous coefficients. In this paper, we consider a weakly coupled system of quasilinear parabolic equations where the coefficients are allowed to be discontinuous and the reaction functions may depend on continuous (infinite or finite) delays.

To describe the problem, we first introduce some notations. Let Ω be a bounded domain with the boundary $\partial\Omega$ in $\mathbb{R}^n$ $(n \geq 1)$. Suppose that Ω consists of a finite number of domains Ω_k $(k = 1, \ldots, K)$ separated by $\Gamma_{k'}$, where $\Gamma_{k'}$, $k' = 1, \ldots, K'$, are surfaces which do not intersect with each other and with $\partial\Omega$. $\Gamma := \cup\Gamma_{k'}$ and $\overrightarrow{\mathbf{n}}$ is the normal to Γ. The symbol $[v]_{\Gamma\times[0,+\infty)}$ denotes the jump in the function v as it crosses $\Gamma\times[0,+\infty)$. For any vector function $\mathbf{u} = (u^1, \ldots, u^N)$, we write $u^l_t := \partial u^l/\partial t$, $u^l_{x_i} := \partial u^l/\partial x_i$, $u^l_x := (u^l_{x_1}, \ldots, u^l_{x_n})$, $l = 1, \ldots, N$, $i = 1, \ldots, n$.

In this paper, we consider the following reaction-diffusion system:

$$\begin{gathered}
u_t^l - \mathcal{L}^l\left(u^l\right) = g^l(x,t,\mathbf{u},J*\mathbf{u}) \quad (x \in \Omega,\ t>0), \\
\left[u^l\right]_{\Gamma\times[0,+\infty)} = 0, \quad \left[\sum_{i,j=1}^{n} a_{ij}^l\left(x,t,u^l\right)u_{x_j}^l \cos(\vec{\mathbf{n}},x_i)\right]_{\Gamma\times[0,+\infty)} = 0, \\
u^l = h^l(x,t) \quad (x \in \partial\Omega,\ t \ge 0), \\
u^l(x,t) = \psi^l(x,t) \quad \left(x\in\Omega, t \in I^l\right),\ l=1,\dots,N,
\end{gathered} \tag{1.1}$$

where

$$J*\mathbf{u} := \left(J^1 * u^1,\dots,J^N*u^N\right), \quad J^l * u^l := \int_{I^l\cup[0,t]} J^l(x,t-s)u^l(x,s)\mathrm{d}s, \tag{1.2}$$

$$\begin{gathered}
\mathcal{L}^l\left(u^l\right) := \sum_{i=1}^{n}\frac{\mathrm{d}}{\mathrm{d}x_i}\left(\sum_{j=1}^{n} a_{ij}^l\left(x,t,u^l\right)u_{x_j}^l\right) + \sum_{j=1}^{n} b_j^l\left(x,t,u^l\right)u_{x_j}^l, \\
I^l := \begin{cases} (-\infty,0] & \text{for } l = 1,\dots,N_0, \\ \left[-r^l,0\right] & \text{for } l = N_0+1,\dots,N, \end{cases}
\end{gathered} \tag{1.3}$$

the expressions $(\mathrm{d}/\mathrm{d}x_i)(a_{ij}^l(x,t,u^l)u_{x_j}^l)$ mean that

$$\frac{\mathrm{d}}{\mathrm{d}x_i}\left(a_{ij}^l\left(x,t,u^l\right)u_{x_j}^l\right) = \left[\frac{\partial a_{ij}^l(x,t,u^l)}{\partial x_i} + \frac{\partial a_{ij}^l(x,t,u^l)}{\partial u^l}u_{x_i}^l\right]u_{x_j}^l + a_{ij}^l\left(x,t,u^l\right)u_{x_j x_i}^l, \tag{1.4}$$

N_0 is a nonnegative integer, and r^l, $l = N_0+1,\dots,N$, are positive constants.

The equations with discontinuous coefficients have been investigated extensively in the literature (see [9–16] and references therein]). However, the discussions in these literature are devoted either to scalar equations without time delays or to coupled system of equations without time delays and with the restrictive conditions that the principal parts are the same and the convection functions $b^l(x,t,\mathbf{u},u_x^l)$ satisfy (see [16])

$$u^l b^l(x,t,\mathbf{u},0) \ge -C_1|\mathbf{u}|^2 - C_2 \quad \left(x\in\overline{\Omega}_k,\ t\in[0,T],\ \mathbf{u}\in\mathbb{R}^N\right),\ k=1,\dots,K,\ l=1,\dots,N. \tag{1.5}$$

In this paper we will extend the method of upper and lower solutions and the monotone iteration scheme to reaction-diffusion system with discontinuous coefficients and continuous delays and use these methods and the results of [15, 16] to prove the existence and uniqueness of the piecewise classical solutions for (1.1) under hypothesis (H) in Section 2.

This paper is organized as follows. In the next section we will prove a weak comparison principle and construct two monotone sequences. Section 3 is devoted to investigate the uniform estimates of the sequences. In Section 4 we prove the existence and uniqueness of the piecewise classical solutions for (1.1). Applications of these results are given in Section 5 to three 2-species Volterra-Lotka models with discontinuous coefficients and continuous delays.

2. Two Monotone Sequences

The aim of this section is to prove a weak comparison principle and construct two monotone sequences. In Section 4 we will show that these sequences converge to the unique solution of (1.1).

2.1. The Definitions, Hypotheses, and Weak Comparison Principle

In all that follows, pairs of indices i or j imply a summation from 1 to n. The symbol $\Omega' \subset\subset \Omega$ means that $\Omega' \subset \Omega$ and $\operatorname{dist}(\Omega', \partial\Omega) > 0$. For any $T > 0$, we set

$$\begin{gathered}
\overline{\Omega} := \Omega \cup \partial\Omega, \qquad \overline{\Omega}_k := \Omega_k \cup \partial\Omega_k, \qquad S_T := \partial\Omega \times [0,T], \qquad \Gamma_T := \Gamma \times [0,T], \\
D_T := \Omega \times (0,T], \qquad D_{k,T} := \Omega_k \times (0,T], \qquad \overline{D}_T := \overline{\Omega} \times [0,T], \qquad \overline{D}_{k,T} := \overline{\Omega}_k \times [0,T], \\
\mathfrak{D}_T := \underbrace{D_T \times \cdots \times D_T}_{N}, \qquad \mathfrak{D}_{k,T} := \underbrace{D_{k,T} \times \cdots \times D_{k,T}}_{N}, \qquad \overline{\mathfrak{D}}_T := \underbrace{\overline{D}_T \times \cdots \times \overline{D}_T}_{N}, \\
Q_0^l := \Omega \times I^l, \qquad Q_{k,0}^l := \Omega_k \times I^l, \qquad \overline{Q}_0^l := \overline{\Omega} \times I^l, \qquad \overline{Q}_{k,0}^l = \overline{\Omega}_k \times I^l, \\
Q_T^l := \Omega \times \left(I^l \cup [0,T]\right), \qquad Q_{k,T}^l := \Omega_k \times \left(I^l \cup [0,T]\right), \qquad \overline{Q}_T^l := \overline{\Omega} \times \left(I^l \cup [0,T]\right), \\
Q_T := Q_T^1 \times \cdots \times Q_T^N, \quad Q_{k,T} := Q_{k,T}^1 \times \cdots \times Q_{k,T}^N, \quad k = 1,\ldots,K, \ l = 1,\ldots,N.
\end{gathered} \tag{2.1}$$

Let $|\mathbf{u}| := (\sum_{l=1}^{N}(u^l)^2)^{1/2}$, $|u_x^l| := (\sum_{i=1}^{n}(u_{x_i}^l)^2)^{1/2}$, $|u_{xx}^l| := (\sum_{i,j=1}^{n}(u_{x_ix_j}^l)^2)^{1/2}$.

$W_2^{1,0}(D_T)$ and $W_2^{1,1}(D_T)$ are the Hilbert spaces with scalar products $(v,w)_{W_2^{1,0}(D_T)} = \iint_{D_T}(vw + v_{x_i}w_{x_i})\mathrm{d}x\,\mathrm{d}t$ and $(v,w)_{W_2^{1,1}(D_T)} = \iint_{D_T}(vw + v_t w_t + v_{x_i}w_{x_i})\mathrm{d}x\,\mathrm{d}t$, respectively. $\overset{\circ}{W}{}_2^{1,1}(D_T)$ and $\overset{\circ}{W}{}_2^{1,0}(D_T)$ are the sets of all functions in $W_2^{1,1}(D_T)$ and $W_2^{1,0}(D_T)$ that vanish on S_T in the sense of trace, respectively. For vector functions with N-components, we use the notations

$$\begin{gathered}
C^\alpha\left(\overline{\mathfrak{D}}_T\right) := \underbrace{C^\alpha\left(\overline{D}_T\right) \times \cdots \times C^\alpha\left(\overline{D}_T\right)}_{N}, \qquad \mathcal{W}_2^{1,1}(\mathfrak{D}T) := \underbrace{\mathcal{W}_2^{1,1}(DT) \times \cdots \times \mathcal{W}_2^{1,1}(DT)}_{N}, \\
C^\alpha\left(\overline{Q}_T\right) := C^\alpha\left(\overline{Q}_T^1\right) \times \cdots \times C^\alpha\left(\overline{Q}_T^N\right).
\end{gathered} \tag{2.2}$$

In Section 3 the same notations are also used to denote the spaces of the vector functions with $2N$-components. Similar notations are used for other function spaces and other domains.

Definition 2.1 (see [3, 5]). Write $\mathbf{u}$, $\mathbf{v}$ in the split form

$$\mathbf{u} = \left(u^l, [\mathbf{u}]_{a^l}, [\mathbf{u}]_{b^l}\right), \qquad \mathbf{v} = ([\mathbf{v}]_{c^l}, [\mathbf{v}]_{d^l}). \tag{2.3}$$

The vector function $\mathbf{g}(\cdot, \mathbf{u}, \mathbf{v}) := (g^1(\cdot, \mathbf{u}, \mathbf{v}), \ldots, g^N(\cdot, \mathbf{u}, \mathbf{v}))$ is said to be mixed quasimonotone in $\mathfrak{A} \subset \mathbb{R}^N \times \mathbb{R}^N$ if, for each $l = 1, \ldots, N$, there exist nonnegative integers a^l, b^l, c^l, and d^l satisfying

$$a^l + b^l = N - 1, \qquad c^l + d^l = N, \tag{2.4}$$

such that $g^l(\cdot, u^l, [\mathbf{u}]_{a^l}, [\mathbf{u}]_{b^l}, [\mathbf{v}]_{c^l}, [\mathbf{v}]_{d^l})$ is nondecreasing in $[\mathbf{u}]_{a^l}$ and $[\mathbf{v}]_{c^l}$, and is nonincreasing in $[\mathbf{u}]_{b^l}$ and $[\mathbf{v}]_{d^l}$ for all $(\mathbf{u}, \mathbf{v}) \in \mathfrak{A}$.

Let

$$\mathscr{L}^l(\tau; \mathbf{v}, \boldsymbol{\eta}) := \iint_{D_\tau} \left\{ v_t^l \eta^l + a_{ij}^l\left(x, t, v^l\right) v_{x_j}^l \eta_{x_i}^l + b_j^l\left(x, t, v^l\right) v_{x_j}^l \eta^l \right\} \mathrm{d}x \, \mathrm{d}t, \tag{2.5}$$

where $D_\tau := \Omega \times (0, \tau]$.

Definition 2.2. A pair of functions $\tilde{\mathbf{u}} = (\tilde{u}_1, \ldots, \tilde{u}_N)$, $\hat{\mathbf{u}} = (\hat{u}_1, \ldots, \hat{u}_N)$ are called coupled weak upper and lower solutions of (1.1) if (i) $\tilde{\mathbf{u}}$ and $\hat{\mathbf{u}}$ are in $C^{\alpha_0}(\overline{Q}_T) \cap C^{1+\alpha_0}(\overline{\mathfrak{D}}_{k,T})$ $(k = 1, \ldots, K)$ for some $\alpha_0 \in (0, 1)$, (ii) $\tilde{\mathbf{u}} \geq \hat{\mathbf{u}}$ and (iii) for any nonnegative vector function $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_N) \in \overset{\circ}{\mathcal{W}}{}_2^{1,1}(\mathfrak{D}_T)$ and any $\tau \in (0, T]$

$$\begin{gathered}
\mathscr{L}^l(\tau; \tilde{\mathbf{u}}, \boldsymbol{\eta}) \geq \iint_{D_\tau} g^l\left(x, \mathrm{t}, \tilde{u}^l, [\tilde{\mathbf{u}}]_{a^l}, [\hat{\mathbf{u}}]_{b^l}, [J * \tilde{\mathbf{u}}]_{c^l}, [J * \hat{\mathbf{u}}]_{d^l}\right) \eta^l \, \mathrm{d}x \, \mathrm{d}t, \\
\mathscr{L}^l(\tau; \hat{\mathbf{u}}, \boldsymbol{\eta}) \leq \iint_{D_\tau} g^l\left(x, t, \hat{u}^l, [\hat{\mathbf{u}}]_{a^l}, [\tilde{\mathbf{u}}]_{b^l}, [J * \hat{\mathbf{u}}]_{c^l}, [J * \tilde{\mathbf{u}}]_{d^l}\right) \eta^l \, \mathrm{d}x \, \mathrm{d}t, \\
\hat{u}^l \leq g^l(x, t) \leq \tilde{u}^l \quad ((x, t) \in S_T), \\
\hat{u}^l(x, t) \leq \psi^l(x, t) \leq \tilde{u}^l(x, t) \quad \left((x, t) \in Q_0^l\right), \; l = 1, \ldots, N.
\end{gathered} \tag{2.6}$$

Throughout this paper the following hypotheses will be used.

(H) (i) $\partial\Omega$ and Γ_k, $k = 1, \ldots, K'$, are of $C^{2+\alpha_0}$ for some exponent $\alpha_0 \in (0, 1)$, and there exist positive numbers a_0 and θ_0 such that

$$\operatorname{mes}(K_\rho \cap \Omega) \leq (1 - \theta_0) \operatorname{mes} K_\rho \tag{2.7}$$

holds for any open ball K_ρ with center on $\partial\Omega$ of radius $\rho \leq a_0$.

(ii) There exist a pair of bounded and coupled weak upper and lower solutions $\tilde{\mathbf{u}}$, $\hat{\mathbf{u}}$. We set

$$\begin{gathered}\mathcal{S} := \left\{\mathbf{u} \in \mathcal{C}\left(\overline{Q}_T\right) : \hat{\mathbf{u}} \leq \mathbf{u} \leq \tilde{\mathbf{u}}\right\}, \qquad \mathcal{S}^* := \left\{\mathbf{w} \in \mathcal{C}\left(\overline{Q}_T\right) : J * \hat{\mathbf{u}} \leq \mathbf{w} \leq J * \tilde{\mathbf{u}}\right\}, \\ \mathcal{S}^{*l} := \left\{w^l \in C\left(\overline{Q}_T^l\right) : J^l * \hat{u} \leq w^l \leq J^l * \tilde{u}\right\}, \quad l = 1, \ldots, N.\end{gathered} \tag{2.8}$$

(iii) For each $k = 1, \ldots, K$, $l = 1, \ldots, N$, $a_{ij}^l(x,t,u^l)$, $b_j^l(x,t,u^l) \in C^{1+\alpha_0}(\overline{D}_{k,T} \times \mathbb{R})$ $(i,j = 1, \ldots, n)$, $g^l(x,t,\mathbf{u},\mathbf{v}) \in C^{1+\alpha_0}(\overline{D}_{k,T} \times \mathcal{S} \times \mathcal{S}^*)$, $h^l(x,t) \in C^{2+\alpha_0}(S_T)$, $\psi^l(x,0) \in C^{\alpha_0}(\overline{\Omega}) \cap C^{2+\alpha_0}(\overline{\Omega}_k)$. There exist a positive nonincreasing function $\nu(\theta)$, a positive nondecreasing function $\mu(\theta)$ for $\theta \in [0,+\infty)$, and a positive constant μ_1 such that

$$\nu\left(\left|u^l\right|\right)\sum_{i'=1}^{n}\xi_{i'}^2 \leq \sum_{i',j'=1}^{n} a_{i'j'}^l\left(x,t,u^l\right)\xi_{i'}\xi_{j'} \leq \mu\left(\left|u^l\right|\right)\sum_{i'=1}^{n}\xi_{i'}^2, \tag{2.9}$$

$$a_{ij}^l = a_{ji}^l, \quad \left|a_{ij}^l\left(x,t,u^l\right); b_j^l\left(x,t,u^l\right)\right| \leq \mu\left(\left|u^l\right|\right), \quad i,j = 1, \ldots, N, \tag{2.10}$$

$$\left\|g^l(x,t,\mathbf{u},\mathbf{v})\right\|_{C^1(\overline{D}_{k,T}\times\mathcal{S}\times\mathcal{S}^*)} \leq \mu_1, \tag{2.11}$$

$$\left\|h^l\right\|_{C^{2+\alpha_0}(S_T)} + \left\|\psi^l(x,0)\right\|_{C^{\alpha_0}(\overline{\Omega})} + \left\|\psi^l(x,0)\right\|_{C^{2+\alpha_0}(\overline{\Omega}_k)} \leq \mu_1. \tag{2.12}$$

(iv) For each $l = 1, \ldots, N$, $J^l(x,t) \in C^{\alpha_0}(\overline{\Omega} \times I_*^l) \cap C^{1+\alpha_0}(\overline{D}_{k,T})$,

$$J^l(x,t) \geq 0 \quad \left((x,t) \in \Omega \times I_*^l\right), \quad \int_{I_*^l} J^l(x,t)\mathrm{d}t = 1 \quad (x \in \Omega), \tag{2.13}$$

where $I_*^l := [0,+\infty)$ for $l = 1, \ldots, N_0$ and $I_*^l := [0,r^l]$ for $l = N_0+1, \ldots, N$, $\psi^l(x,t) \in C^{\alpha_0}(\overline{Q}_0^l)$, and $\int_{I^l} J^l(x,t-s)\psi^l(x,s)\mathrm{d}s$, $J^l * \tilde{u}^l, J^l * \hat{u}^l \in C^{1+\alpha_0}(\overline{D}_{k,T})$. There exists a constant μ_2 such that

$$\left\|\psi^l(x,t)\right\|_{C(\overline{Q}_0^l)} \leq \mu_2, \qquad \left\|\int_{I^l} J^l(x,t-s)\psi^l(x,s)\mathrm{d}s; J^l * \tilde{u}^l; J^l * \hat{u}^l\right\|_{C^{1+\alpha_0}(\overline{D}_{k,T})} \leq \mu_2. \tag{2.14}$$

(v) The vector function $\mathbf{g}(\cdot,\mathbf{u},\mathbf{v}) = (g^1(\cdot,\mathbf{u},\mathbf{v}), \ldots, g^N(\cdot,\mathbf{u},\mathbf{v}))$ is mixed quasimonotone in $\mathcal{S} \times \mathcal{S}^*$.

(vi) The following compatibility conditions hold:

$$\begin{gathered}h^l(x,0) = \psi^l(x,0) \quad (x \in \partial\Omega), \\ \left[a_{ij}^l\left(x,0,\psi^l(x,0)\right)\frac{\partial\psi^l(x,0)}{\partial x_j}\cos\left(\vec{\mathbf{n}},x_i\right)\right]_\Gamma = 0, \quad l = 1, \ldots, N.\end{gathered} \tag{2.15}$$

The weak upper and lower solutions $\tilde{\mathbf{u}}$, $\hat{\mathbf{u}}$ in hypothesis (H)-(ii) will be used as the initial iterations to construct two monotone convergent sequences.

Definition 2.3. A function $\mathbf{u}$ is called a piecewise classical solution of (1.1) if (i) $\mathbf{u} \in \mathcal{C}^{\alpha}(\overline{Q}_T)$, $\mathbf{u}_t \in \mathcal{C}^{\alpha,\alpha/2}(\overline{\mathfrak{D}}_T)$, $\mathbf{u}_{x_j} \in \mathcal{C}^{\alpha,\alpha/2}(\overline{\mathfrak{D}}_{k,T})$ for some $\alpha \in (0,1)$, $\mathbf{u}_{x_j t} \in \mathcal{L}^2(\mathfrak{D}_T)$, $j = 1,\dots,n$; and for any given $k, k = 1,\dots,K$, and any given $\Omega'' \subset\subset \Omega_k$ and $t'' \in (0,T)$, there exists $\alpha'' \in (0,1)$ such that $u^l_{x_i x_j} \in \mathcal{C}^{\alpha'',\alpha''/2}(\overline{\Omega}'' \times [t'',T])$, $i,j = 1,\dots,n$, $l = 1,\dots,N$, and if (ii) $\mathbf{u}$ satisfies pointwise the equations in (1.1) for $(x,t) \in D_{k,T}$, $k = 1,\dots,K$, and satisfies pointwise the inner boundary conditions in (1.1) on Γ_T, the parabolic conditions on S_T, and the initial conditions $u^l(x,t) = \varphi^l(x,t)$ in Q^l_0.

To construct the monotone sequences, we next prove the weak comparison principle.

Lemma 2.4. *Let functions $a^l_{ij}(x,t,u^l)$, $b^l_j(x,t,u^l)$, $l = 1,\dots,N$, satisfy the conditions in hypothesis (H).*

(i) *Assume that $q^l(x,t,\mathbf{Y},\mathbf{Z}) \in C^{1+\alpha_0}(\overline{D}_{k,T} \times \mathcal{S} \times \mathcal{S}^*)$, $l = 1,\dots,N$, and the vector function $\mathbf{q}(\cdot,\mathbf{Y},\mathbf{Z}) = (q^1(\cdot,\mathbf{Y},\mathbf{Z}),\dots,q^N(\cdot,\mathbf{Y},\mathbf{Z}))$ is mixed quasimonotone in $\mathcal{S} \times \mathcal{S}^*$. If $\mathbf{v},\mathbf{u} \in \mathcal{C}(\overline{\mathfrak{D}}_T) \cap \mathcal{W}^{1,1}_\infty(\mathfrak{D}_T) \cap \mathcal{S}$ and if*

$$\begin{gathered}
\mathcal{H}^l(\tau;\mathbf{v},\boldsymbol{\eta}) - \iint_{D_\tau} q^l\Big(x,t,v^l,[\mathbf{v}]_{a^l},[\mathbf{u}]_{b^l},[J*\mathbf{v}]_{c^l},[J*\mathbf{u}]_{d^l}\Big)\eta^l \,\mathrm{d}x\,\mathrm{d}t \\
\le \mathcal{H}^l(\tau;\mathbf{u},\boldsymbol{\eta}) - \iint_{D_\tau} q^l\Big(x,t,u^l,[\mathbf{u}]_{a^l},[\mathbf{v}]_{b^l},[J*\mathbf{u}]_{c^l},[J*\mathbf{v}]_{d^l}\Big)\eta^l \,\mathrm{d}x\,\mathrm{d}t, \\
v^l(x,t) \le u^l(x,t) \quad ((x,t) \in S_T), \\
v^l(x,t) = u^l(x,t) = \varphi^l(x,t) \quad \Big((x,t) \in Q^l_0\Big),\ l = 1,\dots,N,
\end{gathered} \tag{2.16}$$

for any nonnegative bounded vector function $\boldsymbol{\eta} = (\eta^1,\dots,\eta^N) \in \overset{\circ}{\mathcal{W}}{}^{1,1}_2(\mathfrak{D}_T)$ and any $\tau \in (0,T]$, then $\mathbf{v} \le \mathbf{u}$ for $(x,t) \in \overline{D}_T$.

(ii) *If $\mathbf{v},\mathbf{u} \in \mathcal{C}(\overline{\mathfrak{D}}_T) \cap \mathcal{W}^{1,1}_\infty(\mathfrak{D}_T)$ and if*

$$\begin{gathered}
\mathcal{H}^l(\tau;\mathbf{v},\boldsymbol{\eta}) + \iint_{D_\tau} e^l(x,t)v^l\eta^l \,\mathrm{d}x\,\mathrm{d}t \le \mathcal{H}^l(\tau;\mathbf{u},\boldsymbol{\eta}) + \iint_{D_\tau} e^l(x,t)u^l\eta^l \,\mathrm{d}x\,\mathrm{d}t, \\
v^l(x,t) \le u^l(x,t) \quad ((x,t) \in S_T), \quad v^l(x,0) \le u^l(x,0) \quad (x \in \Omega),\ l = 1,\dots,N,
\end{gathered} \tag{2.17}$$

for any nonnegative bounded vector function $\boldsymbol{\eta} \in \overset{\circ}{\mathcal{W}}{}^{1,1}_2(\mathfrak{D}_T)$, where $e^l(x,t)$, $l = 1,\dots,N$, are functions in $C(\overline{D}_{k,T})$ $(k = 1,\dots,K)$, then $\mathbf{v} \le \mathbf{u}$ for $(x,t) \in \overline{D}_T$.

Proof. We first prove part (i) of the lemma. Let $\mathbf{w} = \mathbf{v} - \mathbf{u}$, $\mathbf{w}^+ = ((w^1)^+, \dots, (w^N)^+) := (\max(w^1, 0), \dots, \max(w^N, 0))$. Then $(w^l)^+ = 0$ for $(x,t) \in S_T \cup Q_0^l$, $l = 1, \dots, N$. Choosing $\boldsymbol{\eta} = \mathbf{w}^+$ in (2.16), we obtain

$$\begin{aligned}
&\sum_{l=1}^{N}\Big[\mathscr{H}^l(\tau;\mathbf{v},\mathbf{w}^+) - \mathscr{H}^l(\tau;\mathbf{u},\mathbf{w}^+)\Big] \\
&\quad\le \sum_{l=1}^{N}\iint_{D_\tau}\Big[q^l\Big(x,t,v^l,[\mathbf{v}]_{a^l},[\mathbf{u}]_{b^l},[J*\mathbf{v}]_{c^l},[J*\mathbf{u}]_{d^l}\Big) \\
&\qquad - q^l\Big(x,t,u^l,[\mathbf{u}]_{a^l},[\mathbf{v}]_{b^l},[J*\mathbf{u}]_{c^l},[J*\mathbf{v}]_{d^l}\Big)\Big]\Big(w^l\Big)^+ \mathrm{d}x\,\mathrm{d}t \\
&\quad= \sum_{l=1}^{N}\iint_{D_\tau}\Bigg[E^l_{1l}w^l + \sum_{w^{l'}\in[\mathbf{w}]_{a^l}} E^l_{1l'}w^{l'} + \sum_{w^{l'}\in[\mathbf{w}]_{b^l}} E^l_{1l'}\Big(-w^{l'}\Big) \\
&\qquad + \sum_{J^{l'}*w^{l'}\in[J*\mathbf{w}]_{c^l}} E^l_{2l'}\Big(J^{l'}*w^{l'}\Big) + \sum_{J^{l'}*w^{l'}\in[J*\mathbf{w}]_{d^l}} E^l_{2l'}\Big(-J^{l'}*w^{l'}\Big)\Bigg]\Big(w^l\Big)^+ \mathrm{d}x\,\mathrm{d}t,
\end{aligned} \tag{2.18}$$

where

$$\begin{aligned}
E^l_{1l'} &= \int_0^1 \frac{\partial q^l(x,t,\mathbf{Y}_\theta,\mathbf{Z}_\theta)}{\partial y^{l'}_\theta}\mathrm{d}\theta, \qquad E^l_{2l'} = \int_0^1 \frac{\partial q^l(x,t,\mathbf{Y}_\theta,\mathbf{Z}_\theta)}{\partial z^{l'}_\theta}\mathrm{d}\theta, \\
(\mathbf{Y}_\theta,\mathbf{Z}_\theta) &= \Big(y^l_\theta,[\mathbf{Y}_\theta]_{a^l},[\mathbf{Y}_\theta]_{b^l},[\mathbf{Z}_\theta]_{c^l},[\mathbf{Z}_\theta]_{d^l}\Big) \\
&:= \theta\Big(v^l,[\mathbf{v}]_{a^l},[\mathbf{u}]_{b^l},[J*\mathbf{v}]_{c^l},[J*\mathbf{u}]_{d^l}\Big) + (1-\theta)\Big(u^l,[\mathbf{u}]_{a^l},[\mathbf{v}]_{b^l},[J*\mathbf{u}]_{c^l},[J*\mathbf{v}]_{d^l}\Big).
\end{aligned} \tag{2.19}$$

Let us estimate the terms in (2.18). It follows from the mixed quasimonotone property of $\mathbf{q}(\cdot,\mathbf{Y},\mathbf{Z})$, (2.13) and (2.14) that, for each $l = 1, \dots, N$,

$$\begin{aligned}
&E^l_{1l'}w^{l'} \le E^l_{1l'}\Big(w^{l'}\Big)^+ \quad \text{for } w^{l'} \in [\mathbf{w}]_{a^l}, \quad -E^l_{1l'}w^{l'} \le -E^l_{1l'}\Big(w^{l'}\Big)^+ \quad \text{for } w^{l'} \in [\mathbf{w}]_{b^l}, \\
&E^l_{2l'}\Big(J^{l'}*w^{l'}\Big) \le E^l_{2l'}\Big(J^{l'}*w^{l'}\Big)^+ \le E^l_{2l'}\Big[J^{l'}*\Big(w^{l'}\Big)^+\Big] \quad \text{for } J^{l'}*w^{l'} \in [J*\mathbf{w}]_{c^l}, \\
&-E^l_{2l'}J^{l'}*w^{l'} \le -E^l_{2l'}\Big(J^{l'}*w^{l'}\Big)^+ \le -E^l_{2l'}\Big[J^{l'}*\Big(w^{l'}\Big)^+\Big] \quad \text{for } J^{l'}*w^{l'} \in [J*\mathbf{w}]_{d^l},
\end{aligned} \tag{2.20}$$

$$\Big|E^l_{1l'}\Big| + \Big|E^l_{2l'}\Big| \le C(O), \quad l' = 1, \dots, N, \tag{2.21}$$

where $O = |||\mathbf{u}|||_{C(\overline{D}_T)} + |||\mathbf{v}|||_{C(\overline{D}_T)} + |||J*\mathbf{u}|||_{C(\overline{D}_T)} + |||J*\mathbf{v}|||_{C(\overline{D}_T)}$. Here and below in this section, $C(\cdots)$ denotes the constant depending only on μ_1, μ_2, and the quantities appearing in parentheses. Constant C in different expressions may be different. By hypothesis (H)-(iv) and Hölder's inequality, we have that

$$\begin{aligned}
\iint_{D_\tau}\left[J^{l'}*\left(w^{l'}\right)^{+}\right]^{2}\mathrm{d}x\mathrm{d}t &= \iint_{D_\tau}\left[\int_0^t J^{l'}(x,t-s)\left(w^{l'}(x,s)\right)^{+}\mathrm{d}s\right]^{2}\mathrm{d}x\,\mathrm{d}t \\
&\le \iint_{D_\tau}\left[\int_0^t\left(J^{l'}(x,t-s)\right)^{2}\mathrm{d}s\int_0^t\left(\left(w^{l'}(x,s)\right)^{+}\right)^{2}\mathrm{d}s\right]\mathrm{d}x\,\mathrm{d}t \\
&\le C\iint_{D_\tau}\left\{\int_0^\tau\left[\left(w^{l'}(x,s)\right)^{+}\right]^{2}\mathrm{d}s\right\}\mathrm{d}x\,\mathrm{d}t \\
&\le C\tau\iint_{D_\tau}\left[\left(w^{l'}(x,t)\right)^{+}\right]^{2}\mathrm{d}x\,\mathrm{d}t,\quad l'=1,\dots,N,
\end{aligned}\tag{2.22}$$

and by (2.5), (2.9), (2.10), and Cauchy's inequality, we have that

$$\begin{aligned}
&\sum_{l=1}^{N}\left[\mathscr{H}^{l}(\tau;\mathbf{v},\mathbf{w}^{+})-\mathscr{H}^{l}(\tau;\mathbf{u},\mathbf{w}^{+})\right] \\
&\quad=\frac{1}{2}\int_{\Omega}\left[(\mathbf{w}(x,\tau))^{+}\right]^{2}\mathrm{d}x \\
&\qquad+\sum_{l=1}^{N}\iint_{D_\tau}\left\{\left[a_{ij}^{l}\left(x,t,v^{l}\right)\left(\left(w^{l}\right)^{+}\right)_{x_j}\left(a_{ij}^{l}\left(x,t,v^{l}\right)-a_{ij}^{l}\left(x,t,u^{l}\right)\right)u_{x_j}^{l}\right]\left(\left(w^{l}\right)^{+}\right)_{x_i}\right. \\
&\qquad\qquad\left.+\left[b_{j}^{l}\left(x,t,v^{l}\right)v_{x_j}^{l}-b_{j}^{l}\left(x,t,u^{l}\right)u_{x_j}^{l}\right]\left(w^{l}\right)^{+}\right\}\mathrm{d}x\,\mathrm{d}t \\
&\quad\ge\frac{1}{2}\int_{\Omega}\left[(\mathbf{w}(x,\tau))^{+}\right]^{2}\mathrm{d}x+(\nu(O)-\varepsilon)\sum_{l=1}^{N}\iint_{D_\tau}\left|\left(\left(w^{l}\right)^{+}\right)_{x}\right|^{2}\mathrm{d}x\,\mathrm{d}t-C(O_1)\iint_{D_\tau}|\mathbf{w}^{+}|^{2}\mathrm{d}x\,\mathrm{d}t,
\end{aligned}\tag{2.23}$$

where $O_1=|||\mathbf{u}|||_{C(\overline{D}_T)}+|||\mathbf{v}|||_{C(\overline{D}_T)}+\sum_{l=1}^{N}(|||u_x^l|||_{L^\infty(D_T)}+|||v_x^l|||_{L^\infty(D_T)})$.

Setting $\varepsilon=\nu(O)/2$ and substituting relations (2.20)–(2.23) into (2.18), we see that

$$\begin{aligned}
\int_{\Omega}\left[(\mathbf{w}(x,\tau))^{+}\right]^{2}\mathrm{d}x+\sum_{l=1}^{N}\iint_{D_\tau}\left|\left(\left(w^{l}\right)^{+}\right)_{x}\right|^{2}\mathrm{d}x\,\mathrm{d}t &\le C(O,O_1)\iint_{D_\tau}\left[|\mathbf{w}^{+}|^{2}+|J*\mathbf{w}^{+}|^{2}\right]\mathrm{d}x\,\mathrm{d}t \\
&\le C(O,O_1)\iint_{D_\tau}|\mathbf{w}^{+}|^{2}\mathrm{d}x\,\mathrm{d}t.
\end{aligned}\tag{2.24}$$

Hence, we deduce the relation $\mathbf{w}^{+}\equiv 0$ from this inequality with the use of Gronwall inequality. Then, $\mathbf{v}\le\mathbf{u}$ in $\overline{D}_T$, and the proof of part (i) of the lemma is completed. The similar argument gives the proof of part (ii) of the lemma. □

2.2. Construction of Monotone Sequences

In this subsection, we construct the monotone sequences. By hypothesis (H)-(iii), for each $l = 1,\dots,N$, there exists $\varrho^l = \varrho^l(x,t) \in C^2(\overline{D}_k)$ $(k = 1,\dots,K)$ satisfying

$$\varrho^l(x,t) \ge \max\left\{-\frac{\partial g^l(x,t,\mathbf{u},\mathbf{v})}{\partial u^l} : (\mathbf{u},\mathbf{v}) \in \mathcal{S}\times\mathcal{S}^*\right\}. \tag{2.25}$$

Define

$$G^l(x,t,\mathbf{u},\mathbf{v}) = G^l\left(x,t,u^l,[\mathbf{u}]_{a^l},[\mathbf{u}]_{b^l},[\mathbf{v}]_{c^l},[\mathbf{v}]_{d^l}\right) := \varrho^l u^l + g^l\left(x,t,u^l,[\mathbf{u}]_{a^l},[\mathbf{u}]_{b^l},[\mathbf{v}]_{c^l},[\mathbf{v}]_{d^l}\right). \tag{2.26}$$

Since $\mathbf{g}(\cdot,\mathbf{u}) = (g^1(\cdot,\mathbf{u}),\dots,g^N(\cdot,\mathbf{u}))$ is mixed quasimonotone in $\mathcal{S}\times\mathcal{S}^*$, then, for any $(\mathbf{u},\mathbf{v}),(\mathbf{u}^*,\mathbf{v}^*) \in \mathcal{S}\times\mathcal{S}^*$, $(\mathbf{u},\mathbf{v}) \le (\mathbf{u}^*,\mathbf{v}^*)$,

$$G^l\left(\cdot,u^l,[\mathbf{u}]_{a^l},[\mathbf{u}^*]_{b^l},[\mathbf{v}]_{c^l},[\mathbf{v}^*]_{d^l}\right) \le G^l\left(\cdot,u^{*l},[\mathbf{u}^*]_{a^l},[\mathbf{u}]_{b^l},[\mathbf{v}^*]_{c^l},[\mathbf{v}]_{d^l}\right). \tag{2.27}$$

It is obvious that the following problem is equivalent to (1.1):

$$\begin{gathered}
\mathbb{L}^l\left(u^l\right) := u^l_t - \mathcal{L}^l\left(u^l\right) + \varrho^l u^l = G^l\left(x,t,u^l,[\mathbf{u}]_{a^l},[\mathbf{u}]_{b^l},[J*\mathbf{u}]_{c^l},[J*\mathbf{u}]_{d^l}\right) \quad ((x,t)\in D_T),\\
\left[u^l\right]_{\Gamma_T} = 0, \qquad \left[a^l_{ij}\left(x,t,u^l\right)u^l_{x_j}\cos(\vec{\mathbf{n}},x_i)\right]_{\Gamma_T} = 0,\\
u^l = h^l(x,t) \quad ((x,t)\in S_T), \quad u^l(x,t) = \psi^l(x,t) \quad \left((x,t)\in Q^l_0\right),\ l = 1,\dots,N.
\end{gathered} \tag{2.28}$$

We construct two sequences $\{\overline{\mathbf{u}}_m\}$, $\{\underline{\mathbf{u}}_m\}$ from the iteration process

$$\begin{gathered}
\mathbb{L}^l\left(\overline{u}^l_m\right) = G^l\left(x,t,\overline{u}^l_{m-1},[\overline{\mathbf{u}}_{m-1}]_{a^l},\left[\underline{\mathbf{u}}_{m-1}\right]_{b^l},[J*\overline{\mathbf{u}}_{m-1}]_{c^l},\left[J*\underline{\mathbf{u}}_{m-1}\right]_{d^l}\right) \quad ((x,t)\in D_T),\\
\mathbb{L}^l\left(\underline{u}^l_m\right) = G^l\left(x,t,\underline{u}^l_{m-1},\left[\underline{\mathbf{u}}_{m-1}\right]_{a^l},[\overline{\mathbf{u}}_{m-1}]_{b^l},\left[J*\underline{\mathbf{u}}_{m-1}\right]_{c^l},[J*\overline{\mathbf{u}}_{m-1}]_{d^l}\right) \quad ((x,t)\in D_T),\\
\left[\overline{u}^l_m\right]_{\Gamma_T} = 0, \qquad \left[\underline{u}^l_m\right]_{\Gamma_T} = 0,\\
\left[a^l_{ij}\left(x,t,\overline{u}^l_m\right)\overline{u}^l_{mx_j}\cos(\vec{\mathbf{n}},x_i)\right]_{\Gamma_T} = 0, \qquad \left[a^l_{ij}\left(x,t,\underline{u}^l_m\right)\underline{u}^l_{mx_j}\cos(\vec{\mathbf{n}},x_i)\right]_{\Gamma_T} = 0,\\
\overline{u}^l_m = h^l(x,t), \quad \underline{u}^l_m = h^l(x,t) \quad ((x,t)\in S_T),\\
\overline{u}^l_m(x,t) = \psi^l(x,t), \quad \underline{u}^l_m(x,t) = \psi^l(x,t) \quad \left((x,t)\in Q^l_0\right),\ l = 1,\dots,N,\ m = 1,2,\dots,
\end{gathered} \tag{2.29}$$

where $\overline{\mathbf{u}}_0 = \tilde{\mathbf{u}}$, $\underline{\mathbf{u}}_0 = \hat{\mathbf{u}}$, $\overline{\mathbf{u}}_m = (\overline{u}^1_m,\dots,\overline{u}^N_m)$, and $\underline{\mathbf{u}}_m = (\underline{u}^1_m,\dots,\underline{u}^N_m)$.

Lemma 2.5. *The sequences* $\{\overline{\mathbf{u}}_m\}$, $\{\underline{\mathbf{u}}_m\}$ *given by* (2.29) *are well defined and possess the regularity*

$$\mathbf{u}_m \in \mathcal{C}^{\beta_m}\left(\overline{Q}_T\right), \qquad \mathbf{u}_{mt} \in \mathcal{C}^{\beta_m,\beta_m/2}\left(\overline{\mathfrak{D}}_T\right), \qquad \mathbf{u}_{mx_j} \in \mathcal{C}^{\beta_m,\beta_m/2}\left(\overline{\mathfrak{D}}_{k,T}\right),$$
$$\mathbf{u}_{mx_ix_j} \in \mathcal{C}^{\beta_m,\beta_m/2}(\mathfrak{D}_{k,T}), \quad \mathbf{u}_{mx_jt} \in \mathcal{L}^2(\mathfrak{D}_T) \quad \textit{for some } \beta_m \in (0,\alpha_0), \tag{2.30}$$

and the monotone property

$$\hat{\mathbf{u}} \le \underline{\mathbf{u}}_{m-1} \le \underline{\mathbf{u}}_m \le \overline{\mathbf{u}}_m \le \overline{\mathbf{u}}_{m-1} \le \tilde{\mathbf{u}} \quad \left((x,t) \in \overline{Q}_T^l\right), \; m = 1,2,\ldots. \tag{2.31}$$

Proof. Let

$$\overline{f}^l_{m-1} = \overline{f}^l_{m-1}(x,t) := G^l\left(x,t,\overline{u}^l_{m-1}, [\overline{\mathbf{u}}_{m-1}]_{a^l}, [\underline{\mathbf{u}}_{m-1}]_{b^l}, [J * \overline{\mathbf{u}}_{m-1}]_{c^l}, [J * \underline{\mathbf{u}}_{m-1}]_{d^l}\right),$$
$$\underline{f}^l_{m-1} = \underline{f}^l_{m-1}(x,t) := G^l\left(x,t,\underline{u}^l_{m-1}, [\underline{\mathbf{u}}_{m-1}]_{a^l}, [\overline{\mathbf{u}}_{m-1}]_{b^l}, [J * \underline{\mathbf{u}}_{m-1}]_{c^l}, [J * \overline{\mathbf{u}}_{m-1}]_{d^l}\right). \tag{2.32}$$

Then, for any fixed $l,m, l \in \{1,\ldots,N\}$, $m \in \{1,2,\ldots\}$, and for given $\overline{\mathbf{u}}_{m-1}$ and $\underline{\mathbf{u}}_{m-1}$, problem (2.29) is equivalent to require that $\underline{u}^l_m = \overline{u}^l_m = \varphi^l$ for $(x,t) \in Q^l_0$, $\overline{u}^l_m$ is governed by the problem for one equation with discontinuous coefficients

$$\mathbb{L}^l\left(\overline{u}^l_m\right) = \overline{f}^l_{m-1}(x,t) \quad ((x,t) \in D_T),$$
$$\left[\overline{u}^l_m\right]_{\Gamma_T} = 0, \qquad \left[a^l_{ij}(x,t,\overline{u}_m)\overline{u}_{mx_j}\cos(\vec{\mathbf{n}},x_i)\right]_{\Gamma_T} = 0, \tag{2.33}$$
$$\overline{u}^l_m = h^l(x,t) \quad ((x,t) \in S_T), \quad \overline{u}^l_m(x,0) = \varphi^l(x,0) \quad (x \in \Omega),$$

and $\underline{u}^l_m$ is governed by the problem

$$\mathbb{L}^l\left(\underline{u}^l_m\right) = \underline{f}^l_{m-1}(x,t) \quad ((x,t) \in D_T),$$
$$\left[\underline{u}^l_m\right]_{\Gamma_T} = 0, \qquad \left[a^l_{ij}\left(x,t,\underline{u}^l_m\right)\underline{u}^l_{mx_j}\cos(\vec{\mathbf{n}},x_i)\right]_{\Gamma_T} = 0, \tag{2.34}$$
$$\underline{u}^l_m = h^l(x,t) \quad ((x,t) \in S_T), \quad \underline{u}^l_m(x,0) = \varphi^l(x,0) \quad (x \in \Omega).$$

Problems (2.33) and (2.34) are the special case of [16, problem (1), (2), (5)] for one equation. Reference [16, Theorem 5] shows that problems (2.33) and (2.34) have a unique piecewise classical solution $\overline{u}^l_m$ and $\underline{u}^l_m$ satisfying (2.30), respectively, whenever $\overline{f}^l_{m-1}(x,t)$, $\underline{f}^l_{m-1}(x,t) \in \mathcal{C}^{1+\beta'_{m-1}}(\overline{D}_{k,T})$ $(k = 1,\ldots,K)$ for some $\beta'_{m-1} \in (0,\alpha_0]$. Furthermore, by the formula of integration by parts we get from (2.33) and (2.34) that for, any nonnegative bounded vector function $\boldsymbol{\eta} = (\eta^1, \dots, \eta^N) \in \overset{\circ}{\mathcal{W}}{}_2^{1,1}(\mathfrak{D}_T)$ and any $\tau \in [0, T]$,

$$\begin{aligned}
&\mathscr{H}^l(\tau; \overline{\mathbf{u}}_m, \boldsymbol{\eta}) + \iint_{D_\tau} \varrho^l \overline{u}_m^l \eta^l \mathrm{d}x\, \mathrm{d}t = \iint_{D_\tau} \overline{f}_{m-1}^l(x,t) \eta^l \mathrm{d}x\, \mathrm{d}t, \\
&\mathscr{H}^l(\tau; \underline{\mathbf{u}}_m, \boldsymbol{\eta}) + \iint_{D_\tau} \varrho^l \underline{u}_m^l \eta^l \mathrm{d}x\, \mathrm{d}t = \iint_{D_\tau} \underline{f}_{m-1}^l(x,t) \eta^l \mathrm{d}x\, \mathrm{d}t.
\end{aligned} \tag{2.35}$$

We next prove the lemma by the principle of induction. When $m = 1$, Definition 2.2 and hypotheses (H)-(iii) and (iv) show that $\tilde{\mathbf{u}}, \hat{\mathbf{u}} \in C^{\alpha_0}(\overline{Q}_T) \cap C^{1+\alpha_0}(\overline{\mathfrak{D}}_{k,T})$, $J * \tilde{\mathbf{u}}, J * \hat{\mathbf{u}} \in C^{1+\alpha_0}(\overline{\mathfrak{D}}_{k,T})$ and $g^l(x,t,\mathbf{u},\mathbf{v}) \in C^{1+\alpha_0}(\overline{D}_{k,T} \times \mathcal{S} \times \mathcal{S}^*)$. Thus, for each $l = 1, \dots, N$, $\overline{f}_0^l(x,t)$ and $\underline{f}_0^l(x,t)$ are in $C^{1+\beta_0'}(\overline{D}_{k,T})$ for some $\beta_0' \in (0, \alpha_0)$ and problems (2.33) and (2.34) for $m = 1$ have a unique piecewise classical solution $\overline{u}_1^l$ and $\underline{u}_1^l$, respectively. Since the relation $\hat{\mathbf{u}} \le \tilde{\mathbf{u}}$ implies that $J * \hat{\mathbf{u}} \le J * \tilde{\mathbf{u}}$, then (2.27) and (2.32) yield that $\underline{f}_0^l - \overline{f}_0^l \le 0$. By using (2.6) and (2.35) for $m = 1$, we have that

$$\begin{aligned}
&\mathscr{H}^l(\tau; \overline{\mathbf{u}}_1, \boldsymbol{\eta}) - \mathscr{H}^l(\tau; \tilde{\mathbf{u}}, \boldsymbol{\eta}) + \iint_{D_\tau} \left(\varrho^l \overline{u}_1^l - \varrho^l \tilde{u}^l \right) \eta^l \mathrm{d}x\, \mathrm{d}t \\
&\quad \le \iint_{D_\tau} \left(\overline{f}_0^l - \overline{f}_0^l \right) \eta^l \mathrm{d}x \mathrm{d}t = 0, \quad l = 1, \dots, N, \\
&\mathscr{H}^l(\tau; \underline{\mathbf{u}}_1, \boldsymbol{\eta}) - \mathscr{H}^l(\tau; \overline{\mathbf{u}}_1, \boldsymbol{\eta}) + \iint_{D_\tau} \left(\varrho^l \underline{u}_1^l - \varrho^l \overline{u}_1^l \right) \eta^l \mathrm{d}x\, \mathrm{d}t \\
&\quad = \iint_{D_\tau} \left(\underline{f}_0^l - \overline{f}_0^l \right) \eta^l \mathrm{d}x \mathrm{d}t \le 0, \quad l = 1, \dots, N.
\end{aligned} \tag{2.36}$$

Note that $\underline{\mathbf{u}}_1(x,t) = \overline{\mathbf{u}}_1(x,t) \le \tilde{\mathbf{u}}(x,t)$ for $(x,t) \in S_T \cup \{(x,t) : x \in \Omega, t = 0\}$. It follows from part (ii) of Lemma 2.4 that $\underline{\mathbf{u}}_1 \le \overline{\mathbf{u}}_1 \le \tilde{\mathbf{u}}$ for $(x,t) \in \overline{D}_T$. Similar argument gives the relation $\hat{\mathbf{u}} \le \underline{\mathbf{u}}_1$ for $(x,t) \in \overline{D}_T$. Since $\hat{u}^l \le \underline{u}_1^l = \overline{u}_1^l = \psi^l \le \tilde{u}^l$ for $(x,t) \in \overline{Q}_0^l$, the above conclusions show that $\underline{\mathbf{u}}_1$ and $\overline{\mathbf{u}}_1$ are well defined and possess the properties (2.30) and (2.31) for $m = 1$.

Assume, by induction, that $\overline{\mathbf{u}}_m$ and $\underline{\mathbf{u}}_m$ given by (2.29) are well defined and possess the properties (2.30) and (2.31). Thus, $\overline{u}_m^l(x,t) = \underline{u}_m^l(x,t) = \psi^l(x,t)$ for $(x,t) \in Q_0^l$. By (1.2) and hypothesis (H)-(iv),

$$\begin{aligned}
&J^l * \underline{u}_m^l = \int_0^t J^l(x, t-s) \underline{u}_m^l(x,s) \mathrm{d}s + \int_{I^l} J^l(x, t-s) \psi^l(x,s) \mathrm{d}s \in C^{1+\beta_m^*}\left(\overline{D}_{k,T}\right) \cap \mathcal{S}^{*l}, \\
&J^l * \overline{u}_m^l = \int_0^t J^l(x, t-s) \overline{u}_m^l(x,s) \mathrm{d}s + \int_{I^l} J^l(x, t-s) \psi^l(x,s) \mathrm{d}s \in C^{1+\beta_m^*}\left(\overline{D}_{k,T}\right) \cap \mathcal{S}^{*l}, \\
&J^l * \hat{u}^l \le J^l * \underline{u}_{m-1}^l \le J^l * \underline{u}_m^l \le J^l * \overline{u}_m^l \le J^l * \overline{u}_{m-1}^l \le J^l * \tilde{u}^l, \quad l = 1, \dots, N,
\end{aligned} \tag{2.37}$$

where $\beta_m^* \in (0, \alpha_0]$. Hypothesis (H)-(iii) and (2.37)imply that $\overline{f}_m^l(x,t)$ and $\underline{f}_m^l(x,t)$ are in $C^{1+\beta_m'}(\overline{D}_{k,T})$ $(k = 1, \dots, K)$ for some $\beta_m' \in (0, \alpha_0]$. Again by using [16, Theorem 5], we obtain that for each $l = 1, \dots, N$, problems (2.33) and (2.34) for the case $m+1$ have a unique piecewise classical solution $\overline{u}_{m+1}^l$ and $\underline{u}_{m+1}^l$, respectively. It follows from (2.27), (2.32), (2.37), and (2.35) for the cases m and $m+1$ that

$$\begin{aligned}
&\mathscr{L}^l(\tau; \overline{\mathbf{u}}_{m+1}, \boldsymbol{\eta}) - \mathscr{L}^l(\tau; \overline{\mathbf{u}}_m, \boldsymbol{\eta}) + \iint_{D_\tau} \left(\varrho^l \overline{u}_{m+1}^l - \varrho^l \overline{u}_m^l\right) \eta^l \mathrm{d}x\, \mathrm{d}t \\
&\quad = \iint_{D_\tau} \Big\{ G^l\Big(x, t, \overline{u}_m^l, [\overline{\mathbf{u}}_m]_{a^l}, [\underline{\mathbf{u}}_m]_{b^l}, [J * \overline{\mathbf{u}}_m]_{c^l}, [J * \underline{\mathbf{u}}_m]_{d^l}\Big) \\
&\qquad - G^l\Big(x, t, \overline{u}_{m-1}^l, [\overline{\mathbf{u}}_{m-1}]_{a^l}, [\underline{\mathbf{u}}_{m-1}]_{b^l}, [J * \overline{\mathbf{u}}_{m-1}]_{c^l}, [J * \underline{\mathbf{u}}_{m-1}]_{d^l}\Big) \Big\} \eta^l \mathrm{d}x\, \mathrm{d}t \\
&\quad \leq 0, \quad l = 1, \dots, N, \\
&\mathscr{L}^l(\tau; \underline{\mathbf{u}}_{m+1}, \boldsymbol{\eta}) - \mathscr{L}^l(\tau; \overline{\mathbf{u}}_{m+1}, \boldsymbol{\eta}) + \iint_{D_\tau} \left(\varrho^l \underline{u}_{m+1}^l - \varrho^l \overline{u}_{m+1}^l\right) \eta^l \mathrm{d}x\, \mathrm{d}t \\
&\quad = \iint_{D_\tau} \Big\{ G^l\Big(x, t, \underline{u}_m^l, [\underline{\mathbf{u}}_m]_{a^l}, [\overline{\mathbf{u}}_m]_{b^l}, [J * \underline{\mathbf{u}}_m]_{c^l}, [J * \overline{\mathbf{u}}_m]_{d^l}\Big) \\
&\qquad - G^l\Big(x, t, \overline{u}_m^l, [\overline{\mathbf{u}}_m]_{a^l}, [\underline{\mathbf{u}}_m]_{b^l}, [J * \overline{\mathbf{u}}_m]_{c^l}, [J * \underline{\mathbf{u}}_m]_{d^l}\Big) \Big\} \eta^l \mathrm{d}x\, \mathrm{d}t \\
&\quad \leq 0, \quad l = 1, \dots, N.
\end{aligned} \tag{2.38}$$

Since $\underline{\mathbf{u}}_{m+1} = \overline{\mathbf{u}}_{m+1} = \overline{\mathbf{u}}_m$ for $(x,t) \in S_T \cup \{(x,t) : x \in \Omega, t = 0\}$, using again part (ii) of Lemma 2.4, we obtain that $\underline{\mathbf{u}}_{m+1} \leq \overline{\mathbf{u}}_{m+1} \leq \overline{\mathbf{u}}_m$ in $\overline{D}_T$. The similar proof gives that $\underline{\mathbf{u}}_m \leq \underline{\mathbf{u}}_{m+1}$ in $\overline{D}_T$. Notice that $\underline{u}_m^l = \underline{u}_{m+1}^l = \overline{u}_{m+1}^l = \overline{u}_m^l = \psi^l$ for $(x,t) \in Q_0^l$, $l = 1, \dots, N$. We get that $\underline{\mathbf{u}}_{m+1}$ and $\overline{\mathbf{u}}_{m+1}$ are well defined and possess the properties (2.30) and (2.31) for the case $m+1$. By the principle of induction, we complete the proof of the lemma. □

3. Uniform Estimates of $\{\overline{u}_m\}$, $\{\underline{u}_m\}$

To prove the existence of solutions to (1.1), in this section, we show the uniform estimates of $\{\overline{\mathbf{u}}_m\}$, $\{\underline{\mathbf{u}}_m\}$.

3.1. Preliminaries

In this section we introduce more notations. Let

$$\begin{gathered}
a_{ij}^{N+l}(x,t,v) := a_{ij}^l(x,t,v), \qquad b_j^{N+l}(x,t,v) := b_j^l(x,t,v), \qquad h^{N+l}(x,t) := h^l(x,t), \\
J^{N+l}(x,t) := J^l(x,t), \qquad \psi^{N+l}(x,t) := \psi^l(x,t), \qquad Q_0^{N+l} := Q_0^l, \\
\check{G}^l(x,t,\mathbf{U}_{m-1}, J * \mathbf{U}_{m-1}) := G^l\Big(x,t,\overline{u}_{m-1}^l, [\overline{\mathbf{u}}_{m-1}]_{a^l}, [\underline{\mathbf{u}}_{m-1}]_{b^l}, [J * \overline{\mathbf{u}}_{m-1}]_{c^l}, [J * \underline{\mathbf{u}}_{m-1}]_{d^l}\Big), \\
\check{G}^{N+l}(x,t,\mathbf{U}_{m-1}, J * \mathbf{U}_{m-1}) := G^l\Big(x,t,\underline{u}_{m-1}^l, [\underline{\mathbf{u}}_{m-1}]_{a^l}, [\overline{\mathbf{u}}_{m-1}]_{b^l}, [J * \underline{\mathbf{u}}_{m-1}]_{c^l}, [J * \overline{\mathbf{u}}_{m-1}]_{d^l}\Big), \\
\mathbf{U}_m = \Big(U_m^1, \dots, U_m^{2N}\Big) := (\overline{\mathbf{u}}_m, \underline{\mathbf{u}}_m), \quad J * \mathbf{U}_{m-1} := \Big(J^1 * U_{m-1}^1, \dots, J^{2N} * U_{m-1}^{2N}\Big), \quad l = 1, \dots, N.
\end{gathered} \tag{3.1}$$

K_ρ is an arbitrary open ball of radius ρ with center at x^0, and Q_ρ is an arbitrary cylinder of the form $K_\rho \times [t^0 - \rho^2, t^0]$. $K_{2\rho}$ is concentric with K_ρ. $\Omega_\rho := K_\rho \cap \Omega$.

In this section, $C(\cdots)$ denotes the constant depending only on the parameters M, a_0, θ_0, α_0, μ_1, μ_2, $\nu(M)$, $\mu(M)$, and ϱ_0 from hypothesis (H) and (2.25) and on the quantities appearing in parentheses, independent of m, where $M := \max_{l=1,\dots,N}\{\|\widetilde{u}^l\|_{C(\overline{Q}_T^l)} + \|\widehat{u}^l\|_{C(\overline{Q}_T^l)}\}$ and $\varrho_0 := \max_{1\le l\le N}\max_{k=1,\dots,K}\|\varrho^l(x,t)\|_{C^1(\overline{D}_{k,T})}$.

Write (2.29) in the form

$$\begin{gathered}
\mathbb{L}^l\left(U_m^l\right) = \check{G}^l(x,t,\mathbf{U}_{m-1}, J*\mathbf{U}_{m-1}) \quad ((x,t)\in D_T),\\
\left[U_m^l\right]_{\Gamma_T} = 0, \qquad \left[a_{ij}^l\left(x,t,U_m^l\right)U_{mx_j}^l\cos(\overrightarrow{\mathbf{n}},x_i)\right]_{\Gamma_T} = 0,\\
U_m^l = h^l(x,t) \quad ((x,t)\in S_T),\\
U_m^l(x,t) = \psi^l(x,t) \quad \left((x,t)\in Q_0^l\right),\ l=1,\dots,2N,\ m=1,2,\dots.
\end{gathered} \tag{3.2}$$

Consider the equalities $\sum_{k=1}^K\int_0^T\int_{\Omega_k}\mathbb{L}^l(U_m^l)\eta^l\mathrm{d}x\,\mathrm{d}t = \iint_{D_T}\check{G}^l(x,t,\mathbf{U}_{m-1},J*\mathbf{U}_{m-1})\eta^l\mathrm{d}x\,\mathrm{d}t$ for any $\boldsymbol{\eta} = \boldsymbol{\eta}(x,t) = (\eta^1,\dots,\eta^{2N}) \in \overset{\circ}{\mathcal{W}}{}_2^{1,0}(\mathfrak{D}_T)$. From the formula of integration by parts, we see that

$$\begin{aligned}
&\iint_{D_T} a_{ij}^l\left(x,t,U^l\right)U_{mx_j}^l\eta_{x_i}^l\mathrm{d}x\mathrm{d}t\\
&\quad= \iint_{D_T}\left[-U_{mt} - b_j^l\left(x,t,U_m^l\right)U_{mx_j}^l - \varrho^lU^l + \check{G}^l(x,t,\mathbf{U}_{m-1},J*\mathbf{U}_{m-1})\right]\eta^l\mathrm{d}x\,\mathrm{d}t,\\
&\qquad\qquad l=1,\dots,2N,\ m=1,2,\dots.
\end{aligned} \tag{3.3}$$

Similarly, for any $\phi = \phi(x) = (\phi^1,\dots,\phi^{2N}) \in \overset{\circ}{\mathcal{W}}{}_2^1(\Omega)$ and for every $t\in[0,T]$, we get

$$\begin{aligned}
&\int_\Omega a_{ij}^l\left(x,t,U^l\right)U_{mx_j}^l\phi_{x_i}^l\,\mathrm{d}x\\
&\quad= \int_\Omega\left[-U_{mt} - b_j^l\left(x,t,U_m^l\right)U_{mx_j}^l - \varrho^lU^l + \check{G}^l(x,t,\mathbf{U}_{m-1},J*\mathbf{U}_{m-1})\right]\phi^l\mathrm{d}x,\\
&\qquad\qquad l=1,\dots,2N,\ m=1,2,\dots.
\end{aligned} \tag{3.4}$$

3.2. Uniform Estimates of $\|U_m^l\|_{C^{\alpha_1,\alpha_1/2}(\overline{D}_T)}$, $\|U_{mx}^l\|_{L^2(D_T)}$

Lemma 3.1. *There exist constants α_1 and C depending only on M, a_0, θ_0, α_0, μ_1, μ_2, $\nu(M)$, $\mu(M)$, and ϱ_0, independent of m, such that*

$$\left\|U_m^l\right\|_{C^{\alpha_1,\alpha_1/2}(\overline{D}_T)} \le C, \quad 0<\alpha_1<1, \tag{3.5}$$

$$\left\|U_{mx}^l\right\|_{L^2(D_T)} \le C, \quad l=1,\dots,2N,\ m=1,2,\dots. \tag{3.6}$$

Proof. Fix $l, m, l \in \{1, \dots, 2N\}$, $m \in \{1, 2, \dots\}$. Let $w = U_m^l$. Then w is the bounded generalized solution of the following single equation:

$$\mathbb{L}^l(w) = \check{G}^l(x, t, \mathbf{U}_{m-1}, J * \mathbf{U}_{m-1}) \quad ((x,t) \in D_T) \tag{3.7}$$

in the sense of [10, Section 1, Chapter V]. Equation (3.7) is the special case of [10, Chapter V, (0.1)] with $a_i(x, t, w, w_x) = a_{ij}^l(x, t, w) w_{x_j}$ and $a(x, t, w, w_x) = b_j^l(x, t, w) w_{x_j} + \varrho^l(x, t) w - \check{G}^l(x, t, \mathbf{U}_{m-1}, J * \mathbf{U}_{m-1})$. From (2.31) and hypotheses (H)-(iii)–(v), we see that

$$\begin{gathered}
a_i^l\left(x, t, u^l, p\right) p_i = a_{ij}^l(x, t, w) p_j p_i \geq \nu(M) |p|^2, \\
\left|a_i(x, t, w, p)\right| + \left|\frac{\partial a_i(x, t, w, p)}{\partial x_j}\right| + \left|\frac{\partial a_i(x, t, w, p)}{\partial w}\right| \leq C|p|, \\
\left|a(x, t, w, p)\right| = \left|b_j^l(x, t, w) p_j + \varrho^l(x, t) w - \check{G}^l(x, t, \mathbf{U}_{m-1}, J * \mathbf{U}_{m-1})\right| \leq C(|p| + 1),
\end{gathered} \tag{3.8}$$

where $p = (p_1, \dots, p_n)$. Then (3.8) and [10, Chapter V, Theorem 1.1] give (3.5), and the proof similar to that of [10, Chapter V, formula 4.1] gives (3.6). □

Lemma 3.2. *There exists a positive constant ρ_1 depending only on M,a_0, θ_0, α_0, μ_1, μ_2, $\nu(M)$, $\mu(M)$, and ϱ_0, such that when $\rho \leq \rho_1$, for any cylinder Q_ρ with $(x^0, t^0) \in \overline{D}_T$ and for any bounded function $\zeta = \zeta(x, t) \in \overset{\circ}{W}{}_2^{1,0}(Q_\rho)$,*

$$\begin{aligned}
&\iint_{Q_\rho \cap D_T} \left|U_{mx}^l\right|^2 \zeta^2 \mathrm{d}x\, \mathrm{d}t \\
&\quad \leq C \rho^{\alpha_1} \iint_{Q_\rho \cap D_T} \left[|\zeta_x|^2 + \left(1 + \left|U_{mt}^l\right|\right) \zeta^2\right] \mathrm{d}x\, \mathrm{d}t, \quad l = 1, \dots, 2N, \ m = 1, 2, \dots .
\end{aligned} \tag{3.9}$$

and, for any bounded function $\lambda = \lambda(x) \in \overset{\circ}{W}{}_2^1(K_\rho)$ and for every $t \in [0, T]$,

$$\int_{\Omega_\rho} \left|U_{mx}^l\right|^2 \lambda^2 \mathrm{d}x \leq C \rho^{\alpha_1} \int_{\Omega_\rho} \left[|\lambda_x|^2 + \left(1 + \left|U_{mt}^l\right|\right) \lambda^2\right] \mathrm{d}x, \quad l = 1, \dots, 2N, \ m = 1, 2, \dots . \tag{3.10}$$

Proof. When $K_\rho \subset \Omega$, set $\eta^l = (U_m^l(x, t) - U_m^l(x_1, t_1))\zeta^2$ in (3.3) and $\phi^l = (U_m^l(x, t) - U_m^l(x_1, t))\lambda^2$ in (3.4), where (x_1, t_1) is an arbitrary point in Q_ρ. When $K_\rho \cap \partial\Omega \neq \emptyset$, set $\eta^l = (U_m^l(x, t) - h^l(x, t))\zeta^2$ in (3.3) and $\phi^l = (U_m^l(x, t) - h^l(x, t))\lambda^2$ in (3.4). Thus (3.9) and (3.10) follow from (3.8) and the proofs similar to those of [15, formulas (2.7) and (4.2)]. □

3.3. Uniform Estimates on $\overline{\Omega}' \times [0, T]$

The bounds in this subsection will be of a local nature. By hypothesis (H)-(i) for any given point $x^0 \in \Gamma$ there exists a ball K_ρ with center at x^0 such that we can straighten $\Gamma \cap K_\rho$ out introducing new nondegenerate coordinates $y = y(x)$ possessing bounded first and second derivatives with respect to x. It is possible to divide Γ into a finite number of pieces and introduce for each of them coordinates y (see [11, Chapter 3, Section 16]). Therefore, without loss of generality we assume that the interface Γ lies in the plane $x_n = 0$.

In [15], Tan and Leng investigate the Hölder estimates for the first derivatives of the generalized solution u for one parabolic equation with discontinuous coefficients and without time delays. The estimates $\|u_{x_j}\|_{C^\alpha((\overline{\Omega'\cap\Omega_k})\times[t',T])}$, $\|u_t\|_{C^\alpha(\overline{\Omega}'\times[t',T])}$ in [15] depend on $\max_{[t',T]}(\|u_t\|_{L^{q/2}(\Omega)} + \|u_t\|_{L^2(\Omega)})$ for some $q > n$, where $\Omega' \subset\subset \Omega$, $0 < t' < T$. The results of [15] can not be used directly in this paper, but, by a slight modification, the methods and the framework of [15] can be used to obtain the uniform estimates of $\|U^l_{mx_j}\|_{C^\alpha((\overline{\Omega'\cap\Omega_k})\times[0,T])}$, $\|U^l_{mt}\|_{C^\alpha(\overline{\Omega}'\times[0,T])}$ in this subsection. We omit most of the detailed proofs and only sketch the main steps. The main changes in the derivations are the following: (i) [15, formulas (2.7) and (4.2)] are replaced by (3.9) and (3.10), respectively; (ii) the estimates in this subsection are on $\Omega' \times [0,T]$, while the estimates in [15] are on $\Omega' \times [t',T]$; (iii) the behavior of the reaction functions with continuous delays requires special considerations.

Lemma 3.3. *Let $K_\rho, K_{2\rho} \subset \Omega$. Then there exists a positive constant ρ_2 depending only on M, a_0, θ_0, α_0, μ_1, μ_2, $\nu(M)$, $\mu(M)$, and ϱ_0, such that, when $\rho \le \rho_2$,*

$$\int_0^T \int_{K_\rho} \left[\left(U^l_{mt}\right)^2 + \left|U^l_{mx}\right|^4 + \left|U^l_{mxx}\right|^2 \right] \mathrm{d}x\,\mathrm{d}t \le C\left(\frac{1}{\rho}\right), \quad l = 1,\dots,2N,\ m = 1,2,\dots, \tag{3.11}$$

where $\int_0^T \int_{K_\rho} |U^l_{mxx}|^2 \mathrm{d}x\,\mathrm{d}t := \sum_{k=1}^K \int_0^T \int_{K_\rho\cap\Omega_k} |U^l_{mxx}|^2 \mathrm{d}x\,\mathrm{d}t$.

Proof. Let $\lambda = \lambda(x,t)$ be an arbitrary smooth function taking values in $[0,1]$ such that $\lambda = 0$ for $x \notin K_{2\rho}$ or $t \le t^0 - 4\rho^2$, and $|\lambda_x|^2 + |\lambda_t| \le C/\rho^2$ for $(x,t) \in Q_{2\rho}$. Hypothesis (H)-(iv) shows that for $m = 1$, $\int_0^T \int_{\Omega_{2\rho}} |(J^l * U^l_{(m-1)})_x|^2 \mathrm{d}x\,\mathrm{d}t (l = 1,\dots,2N)$ are estimated by a constant C, and, for $m > 1$,

$$\begin{aligned}
&\int_0^T \int_{K_{2\rho}} \left| \left(J^l * U^l_{(m-1)}\right)_x \right|^2 \mathrm{d}x\,\mathrm{d}t \\
&\quad= \int_0^T \int_{K_{2\rho}} \left| \left(\int_0^t J^l(x,t-s) U^l_{(m-1)}(x,s)\mathrm{d}s + \int_{I^l} J^l(x,t-s)\psi^l_{(m-1)}(x,s)\mathrm{d}s \right)_x \right|^2 \mathrm{d}x\,\mathrm{d}t \\
&\quad\le C + C\int_0^T \int_{K_{2\rho}} \left| U^l_{(m-1)x} \right|^2 \mathrm{d}x\,\mathrm{d}t, \quad l = 1,\dots,2N.
\end{aligned} \tag{3.12}$$

These inequalities, together with (2.11), (3.6), and (2.37), imply that

$$\begin{aligned}
&\int_0^T \int_{K_{2\rho}} \left| \frac{\mathrm{d}\check{G}^l(x,t,\mathbf{U}_{m-1}, J * \mathbf{U}_{m-1})}{\mathrm{d}x_s} \right| \left| U^l_{mx_s} \right| \lambda^2 \mathrm{d}x\,\mathrm{d}t \\
&\quad\le C + \int_0^T \int_{K_{2\rho}} \left[|\mathbf{U}_{mx}|^2 + \left| \left(J * \mathbf{U}_{(m-1)}\right)_x \right|^2 \right] \mathrm{d}x\,\mathrm{d}t \\
&\quad\le C, \quad s = 1,\dots,n-1,\ l = 1,\dots,2N,\ m = 1,2,\dots.
\end{aligned} \tag{3.13}$$

Based on these inequalities, we can get

$$\sum_{s=1}^{n-1}\int_0^T\int_{K_{2\rho}}\left|U^l_{mx_sx}\right|^2\lambda^2\,\mathrm{d}x\,\mathrm{d}t\le C\left(\frac{1}{\rho}\right)+\int_0^T\int_{K_{2\rho}}\left|U^l_{mx}\right|^4\lambda^2\,\mathrm{d}x\,\mathrm{d}t,\quad l=1,\ldots,2N,\ m=1,2,\ldots. \tag{3.14}$$

For this purpose, similar to [15, Lemma 3.1], we consider not the estimate of the second derivatives of U^l but the estimate of the difference ratios $\Delta/\Delta x_s$ of the first derivatives $U^l_{x_i}$ by setting $\eta^l=(\Delta/\Delta x_s)(((\Delta U^l(x-\Delta x_s,t))/\Delta x_s)\lambda^2(x-\Delta x_s,t))$ in (3.3), where $\Delta v(x,t)/\Delta x_s$, $s=1,\ldots,n-1$, denote the difference ratios $v(x+\Delta x_s,t)-(v(x,t))/\Delta x_s$ with respect to x_s, and then we obtain (3.14) by letting $\Delta x_s\to 0$.

We next show that

$$\int_0^T\int_{K_{2\rho}}\left|U^l_{mt}\right|^2\lambda^2\,\mathrm{d}x\,\mathrm{d}t\le C\left(\frac{1}{\rho}\right)+C\int_0^T\int_{K_{2\rho}}\left|U^l_{mx}\right|^4\lambda^2\,\mathrm{d}x\,\mathrm{d}t,\quad l=1,\ldots,2N,\ m=1,2,\ldots. \tag{3.15}$$

To do this, consider $\sum_{k=1}^K\int_0^T\int_{\Omega_k}\mathbb{L}^l(U^l_m)U^l_{mt}\lambda^l\,\mathrm{d}x\,\mathrm{d}t=\iint_{D_T}\check{G}^l(x,t,\mathbf{U}_{m-1},J*\mathbf{U}_{m-1})U^l_{mt}\lambda^l\,\mathrm{d}x\,\mathrm{d}t$. From an integration by parts we get

$$\begin{aligned}
&\frac{1}{2}\int_{K_{2\rho}}a^l_{ij}\left(x,t,U^l_m\right)U^l_{mx_j}U^l_{mx_i}\lambda^2\,\mathrm{d}x\Big|_{t=0}^{t=t^0}\\
&\quad+\int_0^T\int_{K_{2\rho}}\Bigg\{\left(U^l_{mt}\right)^2\lambda^2-\frac{1}{2}\frac{\partial a^l_{ij}}{\partial U^l_m}U^l_{mx_j}U^l_{mx_i}U_{mt}\lambda^2\\
&\qquad-\frac{1}{2}\frac{\partial a^l_{ij}}{\partial t}U^l_{mx_j}U^l_{mx_i}\lambda^2-a^l_{ij}U^l_{mx_j}U^l_{mx_i}\lambda\lambda_t+2a^l_{ij}U^l_{mx_j}U_{mt}\lambda\lambda_{x_i}\\
&\qquad+\left[b^l_jU^l_{mx_j}+\varrho^lU^l_m-\check{G}^l(x,t,\mathbf{U}_{m-1},J*\mathbf{U}_{m-1})\right]U^l_{mt}\lambda^2\Bigg\}\mathrm{d}x\,\mathrm{d}t=0,\\
&\qquad\qquad l=1,\ldots,2N,\ m=1,2,\ldots.
\end{aligned} \tag{3.16}$$

By hypothesis (H)-(iii) and Cauchy's inequality with ε we conclude from the above equalities that

$$\begin{aligned}
&\int_0^T\int_{K_{2\rho}}\left(U^l_{mt}\right)^2\lambda^2\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\le\varepsilon\int_0^T\int_{K_{2\rho}}\left(U^l_{mt}\right)^2\lambda^2\,\mathrm{d}x\,\mathrm{d}t+C\\
&\qquad+C(\varepsilon)\int_0^T\int_{K_{2\rho}}\left[\left(1+\left|U^l_{mx}\right|^2\right)\left(\lambda^2+\lambda_t^2+|\lambda_x|^2\right)+\left|U^l_{mx}\right|^4\lambda^2\right]\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{3.17}$$

In view of (3.6), setting $\varepsilon=1/2$, we have (3.15).

Next, the proof similar to the first inequality of (3.5) of [15] gives that there exists a positive constant ρ_2 depending only on M, a_0, θ_0, α_0, μ_1, μ_2, $\nu(M)$, $\mu(M)$, and ϱ_0, such that, when $\rho \le \rho_2$,

$$\int_0^T \int_{K_{2\rho}} \left|U^l_{mx}\right|^4 \lambda^2 \,\mathrm{d}x\,\mathrm{d}t \le C\left(\frac{1}{\rho}\right), \quad l = 1,\dots,2N,\ m = 1,2,\dots. \tag{3.18}$$

Furthermore, since the equations in (3.2) and Hypothesis (H)-(iii) show that

$$\left|U^l_{mx_nx_n}\right| \le C\left(\left|U^l_{mt}\right| + \sum_{s=1}^{n-1}\left|U^l_{mx_sx}\right| + \left|U^l_x\right|^2 + 1\right) \quad ((x,t) \in D_{k,T}),\ k = 1,\dots,K, \tag{3.19}$$

then (3.11) follows from (3.14)–(3.19). □

Lemma 3.4. *Let $K_\rho, K_{2\rho} \subset \Omega$. Then there exists a positive constant ρ_3 depending only on M, a_0, θ_0,α_0, μ_1, μ_2, $\nu(M)$, $\mu(M)$, and ϱ_0, such that, when $\rho \le \rho_3$,*

$$\begin{aligned}\max_{[0,T]} \int_{K_{2\rho}} |\mathbf{U}_{mt}|^{r+1}\mathrm{d}x + \int_0^T \int_{K_{2\rho}} \left(|\mathbf{U}_{mt}|^{r-1}|\mathbf{U}_{mtx}|^2 + |\mathbf{U}_{mt}|^{r+2}\right)\mathrm{d}x\,\mathrm{d}t \\ \le C\left(q,\frac{1}{\rho}\right). \quad r = 1,\dots,q,\ m = 1,2,\dots,\end{aligned} \tag{3.20}$$

where $|\mathbf{U}_{mt}| := [\sum_{l=1}^{2N} |U^l_{mt}|^2]^{1/2}$, $|\mathbf{U}_{mtx}| := [\sum_{l=1}^{2N} |U^l_{mtx}|^2]^{1/2}$.

Proof. Let $\lambda = \lambda(x,\mathrm{t})$ be an arbitrary smooth function taking values in $[0,1]$ such that $\lambda = 0$ for $x \notin K_{2\rho}$ or $t \le t^0 - 4\rho^2$, and $|\lambda_x|^2 + |\lambda_t| \le C/\rho^2$ for $(x,t) \in Q_{2\rho}$. Similar to (3.12), from hypotheses (H)-(iii)-(iv), (2.37) for the case $m-1$, and Hölder's inequality we see that

$$\begin{aligned}&\int_0^\tau \int_{K_{2\rho}} \left|\frac{\mathrm{d}\check{G}^l(x,t,\mathbf{U}_{m-1},J*\mathbf{U}_{m-1})}{\mathrm{d}t}\right|\left|U^l_{mt}\right|^r \lambda^2\,\mathrm{d}x\,\mathrm{d}t \\ &\quad\le C + \int_0^\tau \int_{K_{2\rho}} \left(|\mathbf{U}_{mt}|^{r+1} + \left|\mathbf{U}_{(m-1)t}\right|^{r+1}\right)\mathrm{d}x\,\mathrm{d}t, \quad l = 1,\dots,2N,\ m = 1,2,\dots.\end{aligned} \tag{3.21}$$

Next, let us examine the difference ratio with respect to t on both sides of $\mathbb{L}^l(U^l_m) = \check{G}^l(x,t,\mathbf{U}_{m-1},J*\mathbf{U}_{m-1})$. Multiplying the equations obtained by $|U^l_{m(t)}|^{r-1}U^l_{m(t)}\lambda^2$, where $U^l_{m(t)} = (U^l_m(x,t+\Delta t) - U^l_m(x,t))/(\Delta t)$, integrating by parts, and then letting $\Delta t \to 0$, from (3.9) and the proof similar to that of [15, formula (3.26)], we find that there exists a positive constant $\rho_{3,1}$ depending only on M, a_0, θ_0, α_0, μ_1, μ_2, $\nu(M)$, $\mu(M)$, and ϱ_0, such that, when $\rho \le \rho_{3,1}$,

$$\int_{K_{2\rho}} |\mathbf{U}_{mt}|^{r+1}\lambda^2\,\mathrm{d}x\Big|_{t=0}^{t=\tau}$$
$$+\int_0^\tau\int_{K_{2\rho}} |\mathbf{U}_{mt}|^{r-1}|\mathbf{U}_{mtx}|^2\lambda^2\,\mathrm{d}x\,\mathrm{d}t \le C\rho^{\alpha_1}\int_0^\tau\int_{K_{2\rho}} |\mathbf{U}_{mt}|^{r+2}\lambda^2\,\mathrm{d}x\,\mathrm{d}t \tag{3.22}$$
$$+C\int_0^\tau\int_{K_{2\rho}}\Big[\Big(1+|\mathbf{U}_{mt}|^{r+1}\Big)\Big(\lambda^2+\lambda|\lambda_t|+|\lambda_x|^2\Big)+\Big(1+|(\mathbf{U}_{m-1})_t|^{r+1}\Big)\lambda^2\Big]\mathrm{d}x\,\mathrm{d}t.$$

To estimate $\int_0^\tau\int_{K_{2\rho}}|\mathbf{U}_{mt}|^{r+2}\lambda^2\mathrm{d}x\mathrm{d}t$, we take $\eta = |U^l_{mt}|^r U^l_{mt}\lambda^2$ in (3.3). Hence, by hypotheses (H)-(iii)-(iv) and Cauchy's inequality we get

$$\int_0^\tau\int_{K_{2\rho}} |\mathbf{U}_{mt}|^{r+2}\lambda^2\,\mathrm{d}x\,\mathrm{d}t$$
$$\le C(q)\int_0^\tau\int_{K_{2\rho}}\Big[|\mathbf{U}_{mt}|^{r-1}|\mathbf{U}_{mtx}|^2\lambda^2+|\mathbf{U}_{mx}|^2|\mathbf{U}_{mt}|^{r+1}\xi^2+|\mathbf{U}_{mt}|^{r+1}\Big(\lambda^2+|\lambda_x|^2\Big)\Big]\mathrm{d}x\,\mathrm{d}t, \tag{3.23}$$

and by (3.9) with $\zeta = (|\mathbf{U}_{mt}|^2)^{(r+1)/4}\lambda$ we get

$$\int_0^\tau\int_{K_{2\rho}} |\mathbf{U}_{mx}|^2|\mathbf{U}_{mt}|^{r+1}\,\lambda^2\,\mathrm{d}x\,\mathrm{d}t$$
$$\le C(q)\rho^{\alpha_1}\int_0^\tau\int_{K_{2\rho}}\Big\{|\mathbf{U}_{mt}|^{r-1}|\mathbf{U}_{mtx}|^2\lambda^2+\Big(1+|\mathbf{U}_{mt}|^{r+1}\Big)|\lambda_x|^2+|\mathbf{U}_{mt}|^{r+2}\lambda^2\Big\}\mathrm{d}x\,\mathrm{d}t. \tag{3.24}$$

Furthermore, (3.22)–(3.24) show that

$$\int_0^\tau\int_{K_{2\rho}} |\mathbf{U}_{mt}|^{r+2}\lambda^2\,\mathrm{d}x\,\mathrm{d}t \le C_{3,1}(q)\rho^{\alpha_1}\int_0^\tau\int_{K_{2\rho}} |\mathbf{U}_{mt}|^{r+2}\lambda^2\,\mathrm{d}x\,\mathrm{d}t$$
$$+C\int_0^\tau\int_{K_{2\rho}}\Big[\Big(1+|\mathbf{U}_{mt}|^{r+1}\Big)\Big(\lambda^2+\lambda|\lambda_t|+|\lambda_x|^2\Big)+\Big(1+|(\mathbf{U}_{m-1})_t|^{r+1}\Big)\lambda^2\Big]\mathrm{d}x\,\mathrm{d}t. \tag{3.25}$$

Set $\rho_{3,2} := \min\{\rho_{3,1}, (2C_{3,1}(q))^{-1/\alpha_1}\}$. Thus, when $0<\rho\le\rho_{3,2}$,

$$\int_0^\tau\int_{K_{2\rho}} |\mathbf{U}_{mt}|^{r+2}\lambda^2\mathrm{d}x\,\mathrm{d}t$$
$$\le C\int_0^\tau\int_{K_{2\rho}}\Big[\Big(1+|\mathbf{U}_{mt}|^{r+1}\Big)\Big(\lambda^2+\lambda|\lambda_t|+|\lambda_x|^2\Big)+\Big(1+|(\mathbf{U}_{m-1})_t|^{r+1}\Big)\lambda^2\Big]\mathrm{d}x\,\mathrm{d}t. \tag{3.26}$$

Note that by property (2.30), hypothesis (H)-(iii), and the equations in (3.2),

$$U^l_{mt}(x,0) = \frac{\mathrm{d}}{\mathrm{d}x_i}\Big(a^l_{ij}\Big(x,0,\varphi^l\Big)\varphi^l_{x_j}\Big)+b^l_j\Big(x,t,\varphi^l\Big)\varphi^l_{x_j}+\check{G}^l(x,0,\boldsymbol{\varphi},J*\boldsymbol{\varphi}) \quad \Big(x\in\overline{\Omega}\Big), \tag{3.27}$$

where $\boldsymbol{\psi} := (\psi^1, \ldots, \psi^{2N})$. Therefore, $\int_\Omega |\mathbf{U}_{mt}(x,0)|^r \mathrm{d}x$ can be estimated from above by $C(q)$. Thus, using the same arguments given in the derivation of [15, formula (3.29)], we get (3.20) from (3.26), (3.22), and (3.27). □

Lemma 3.5. *Let $K_\rho, K_{2\rho} \subset \Omega$. For any given positive integer q, one has that*

$$\int_{K_\rho} \left[\left| U^l_{mxx} \right|^2 \left(1 + \left| U^l_{mx} \right|^{2r} \right) + \left| U^l_{mx} \right|^{2r+4} \right] \mathrm{d}x \\ \leq C\left(q, \frac{1}{\rho} \right), \quad r = 0, 1, \ldots, q, \; l = 1, \ldots, 2N, \; m = 1, 2, \ldots \tag{3.28}$$

for every $t \in [0, T]$.

Proof. By using (3.20) and [10, Chapter II, Lemmas 5.2 and 5.3′], from the same argument as that in the proof of [15, formula (2.2)], we find that, for every $t \in [0, T]$,

$$\int_{K_\rho} \left(U^l_{mt} \right)^2 \zeta^2 \, \mathrm{d}x \leq C\rho^{\alpha_1} \int_{K_\rho} |\zeta_x|^2 \mathrm{d}x, \quad l = 1, \ldots, 2N, \; m = 1, 2, \ldots, \tag{3.29}$$

where $\zeta = \zeta(x)$ is an arbitrary bounded function from $\overset{\circ}{W}{}^1_2(K_\rho)$. Then by (3.4), (3.8), (3.10), and (3.29), the proof similar to [15, formula (4.6)] implies (3.6). □

Based on the above uniform estimates, we can get the following local Hölder estimates of the first derivatives.

Lemma 3.6. *Let $K_\rho \subset \Omega' \subset\subset \Omega$. There exist positive constants α_2, α_3, and $C(d')$, $0 < \alpha_2$, $\alpha_3 < 1$, such that*

$$\max_{Q_\rho \cap \overline{D}_{k,T}} \left| U^l_{x_j} \right| + \rho^{-\alpha_2} \operatorname{osc}\left\{ U^l_{x_j}, Q_\rho \cap \overline{D}_{k,T} \right\} \leq C(d'), \quad j = 1, \ldots, n, \; k = 1, \ldots, K, \; l = 1, \ldots, 2N, \tag{3.30}$$

$$\max_{Q_\rho \cap \overline{D}_T} \left| U^l_t \right| + \rho^{-\alpha_3} \operatorname{osc}\left\{ U^l_t, Q_\rho \cap \overline{D}_T \right\} \leq C(d'), \quad l = 1, \ldots, 2N, \; m = 1, 2, \ldots, \tag{3.31}$$

where α_2 and α_3 depend only on d' and the parameters M, a_0, θ_0, α_0, μ_1, μ_2, $\nu(M)$, $\mu(M)$, and ϱ_0, independent of m.

Proof. By Hypothesis (H), (3.20), and (3.27), the proof similar to that of [15, Lemma 4.4] gives (3.31), and, by (3.20) and (3.28), the proof similar to that of [15, Lemma 4.3] gives

$$\max_{K_\rho} \left| U^l_{x_s} \right| + \rho^{-\beta_1^*} \operatorname{osc}\left\{ U^l_{x_s}, K_\rho \right\} \leq C(d'), \quad s = 1, \ldots, n-1, \; l = 1, \ldots, 2N, \tag{3.32}$$

$$\max_{K_\rho \cap \overline{\Omega}_k} \left| U^l_{x_n} \right| + \rho^{-\beta_2^*} \operatorname{osc}\left\{ U^l_{x_n}, K_\rho \cap \Omega_k \right\} \leq C(d'), \quad l = 1, \ldots, 2N, \; m = 1, 2, \ldots, \; k = 1, \ldots, K, \tag{3.33}$$

where β_1^* and β_2^* depend only on d' and the parameters M, a_0, $\theta_0, \alpha_0, \mu_1$, μ_2, $\nu(M)$, $\mu(M)$, and ϱ_0. By using (3.5), (3.32), (3.33), and [10, Chapter II, Lemma 3.1] we see that for any given k $(k = 1, \ldots, K)$,

$$\begin{aligned}&\left|U^l_{mx_j}(x, t_1) - U^l_{mx_j}(x, t_2)\right| \\ &\quad\le C(d')|t_1 - t_2|^{\beta_3^*} \quad \left((x, t_1), (x, t_2) \in \left(\overline{\Omega' \cap \Omega_k}\right) \times [0, T]\right),\ j = 1, \ldots, n,\end{aligned} \tag{3.34}$$

where $\beta_3^* = (\alpha_1/2)\min(\beta_1^*, \beta_2^*)/(1 + \min(\beta_1^*, \beta_2^*))$. Then (3.30) follows from (3.32)–(3.34). □

3.4. *Uniform Estimates on* $\overline{D}_T$

Theorem 3.7. *Let hypothesis (H) holds, and let the sequence* $\{\mathbf{U}_m\}$ *be given by* (3.2). *Then*

$$\left\|U^l_{mx_j}\right\|_{C^{\alpha_4,\alpha_4/2}(\overline{D}_{k,T})} + \left\|U^l_{mt}\right\|_{C^{\alpha_4,\alpha_4/2}(\overline{D}_T)} \le C, \quad 0 < \alpha_4 < 1, \tag{3.35}$$

$$\left\|U^l_{mx_ix_j}\right\|_{L^2(D_{k,T})} + \left\|U^l_{mx_jt}\right\|_{L^2(D_T)} \le C, \quad k = 1, \ldots, K,\ l = 1, \ldots, 2N, \tag{3.36}$$

where α_4 *depends only on* M, a_0, θ_0, α_0, μ_1, μ_2, $\nu(M)$, $\mu(M)$, *and* ϱ_0, *independent of* m. *For any given* k, $k \in \{1, \ldots, K\}$, *letting* $\Omega'' \subset\subset \Omega_k$ *and* $t'' < T$, *there exists a positive constant* $\alpha_5 \in (0, 1)$ *depending only on* $d'' := \operatorname{dist}(\Omega'', \partial\Omega_k)$, t'' *and the parameters* M,a_0, θ_0,α_0, μ_1, μ_2, $\nu(M)$, $\mu(M)$, *and* ϱ_0, *such that*

$$\left\|U^l_m\right\|_{C^{2+\alpha_5,1+\alpha_5/2}(\overline{\Omega}''\times[t'',T])} \le C(d'', t''), \quad l = 1, \ldots, 2N,\ m = 1, 2, \ldots. \tag{3.37}$$

Proof. Since $\Gamma \cap \partial\Omega = \emptyset$, then there exists a subdomain of Ω, denoted by Ω_K, such that $\partial\Omega \subset \overline{\Omega}_K$. Then the coefficients of the equations $\mathbb{L}^l(U^l_m) = \check{G}^l(x, t, \mathbf{U}_{m-1}, J * \mathbf{U}_{m-1})$ are continuous in Ω_K. In [10], the estimates near $\partial\Omega$ for the equations with continuous coefficients and without time delays are well known. By the methods of Section 3.3 and [10] we can get the estimates near $\partial\Omega$. The details are omitted. Then the estimates near $\partial\Omega$ and the results of the above subsections give (3.35) and (3.36).

We next prove (3.37). For any fixed $l, m, k, l \in \{1, \ldots, 2N\}$, $m \in \{1, 2, \ldots\}$, $k \in \{1, \ldots, K\}$, U^l_m satisfies the linear equation with continuous coefficients

$$U^l_{mt} - \widehat{a}_{ij}(x, t)U^l_{mx_ix_j} + \widehat{b}_j(x, t)U^l_{mx_j} = \widehat{f}(x, t) \quad ((x, t) \in \Omega_k \times (0, T]), \tag{3.38}$$

where

$$\begin{gathered}\widehat{a}_{ij}(x, t) = a^l_{ij}\left(x, t, U^l_m\right), \qquad \widehat{b}_j(x, t) = -\frac{\partial a^l_{ij}(x, t, U^l_m)}{\partial U^l_m}U^l_{mx_i} - \frac{\partial a^l_{ij}(x, t, U^l_m)}{\partial x_i} + b^l_j\left(x, t, U^l_m\right), \\ \widehat{f}(x, t) = -\varrho^l(x, t)U^l_m + \check{G}^l(x, t, \mathbf{U}_{m-1}, J * \mathbf{U}_{m-1}).\end{gathered} \tag{3.39}$$

It follows from (3.35), (3.36), and hypotheses (H)-(iii)-(iv) that

$$\left\| \widehat{a}_{ij}(x,t); \widehat{b}_j(x,t); \widehat{f}(x,t) \right\|_{C^{\beta_4^*}(\overline{D}_{k,T})} \le C, \quad i,j = 1,\ldots,n, \tag{3.40}$$

where $\beta_4^* \in (0,1)$ depends only on α_4 and the parameters M, a_0,θ_0,α_0,μ_1, μ_2, $\nu(M)$, $\mu(M)$, and ϱ_0. Therefore, (3.40) and the Schauder estimate for linear parabolic equation yield (3.37). □

4. Existence and Uniqueness of Solutions for (1.1)

In this section we show that the sequences $\{\overline{\mathbf{u}}_m\}$, $\{\underline{\mathbf{u}}_m\}$ converge to the unique solution of (1.1) and prove the main theorem of this paper.

Theorem 4.1. *Let hypothesis (H) hold. Then, problem* (1.1) *has a unique piecewise classical solution* $\mathbf{u}^*$ *in* $\mathcal{S}$*, and the sequences* $\{\overline{\mathbf{u}}_m\}$*,* $\{\underline{\mathbf{u}}_m\}$ *given by* (2.29) *converge monotonically to* $\mathbf{u}^*$*. The relation*

$$\widehat{\mathbf{u}} \le \underline{\mathbf{u}}_{m-1} \le \underline{\mathbf{u}}_m \le \mathbf{u}^* \le \overline{\mathbf{u}}_m \le \overline{\mathbf{u}}_{m-1} \le \widetilde{\mathbf{u}} \quad \left((x,t) \in \overline{Q}_T^l \right), \; m = 1,2,\ldots \tag{4.1}$$

holds.

Proof. It follows from Lemma 2.5 that the pointwise limits

$$\lim_{m\to\infty} \overline{\mathbf{u}}_m = \overline{\mathbf{u}}, \qquad \lim_{m\to\infty} \underline{\mathbf{u}}_m = \underline{\mathbf{u}} \tag{4.2}$$

exist and satisfy the relation

$$\widehat{\mathbf{u}} \le \underline{\mathbf{u}}_{m-1} \le \underline{\mathbf{u}}_m \le \underline{\mathbf{u}} \le \overline{\mathbf{u}} \le \overline{\mathbf{u}}_m \le \overline{\mathbf{u}}_{m-1} \le \widetilde{\mathbf{u}}. \tag{4.3}$$

Let $\{\mathbf{u}_m\}$ denote either the sequence $\{\overline{\mathbf{u}}_m\}$ or the sequence $\{\underline{\mathbf{u}}_m\}$, and let $\mathbf{u}$ be the corresponding limit.

Estimates (3.5), (3.6), (3.35), and (3.36) imply that there exists a subsequence $\{\mathbf{u}_{m'}\}$ (denoted by $\{\mathbf{u}_m\}$ still) such that $\{\mathbf{u}_m\}$ and $\{\mathbf{u}_{mt}\}$ converge in $C(\overline{\mathfrak{D}}_T)$ to $\mathbf{u}$ and $\mathbf{u}_t$, respectively, for each $i,j = 1,\ldots,n$, $\{\mathbf{u}_{mx_j}\}$ converges in $C(\overline{\mathfrak{D}}_{k,T})$ to $\mathbf{u}_{x_j}$, $\{\mathbf{u}_{mx_ix_j}\}$ converges weakly in $\mathcal{L}^2(\mathfrak{D}_{k,T})$ to $\mathbf{u}_{x_ix_j}$, and $\{\mathbf{u}_{mx_jt}\}$ converges weakly in $\mathcal{L}^2(\mathfrak{D}_T)$ to $\mathbf{u}_{x_jt}$ $(k = 1,\ldots,K)$. Thus, $\mathbf{u} \in C^{\alpha_1,\alpha_1/2}(\overline{\mathfrak{D}}_T)$, $\mathbf{u}_{x_j} \in C^{\alpha_4,\alpha_4/2}(\overline{\mathfrak{D}}_{k,T})$, $\mathbf{u}_t \in C^{\alpha_4,\alpha_4/2}(\overline{\mathfrak{D}}_T)$, and $\mathbf{u}_{x_jt} \in \mathcal{L}^2(\mathfrak{D}_T)$. Since $u^l = \psi^l$ in Q_0^l, $l = 1,\ldots,N$, then $\mathbf{u} \in C^{\alpha_1,\alpha_1/2}(\overline{Q}_T)$. For any given k, $k = 1,\ldots,K$, and any given $\Omega'' \subset\subset \Omega_k$ and $t'' \in (0,T)$, (3.37) in Theorem 3.7 implies that there exists a subsequence $\{\mathbf{u}_{m'}\}$ (denoted by $\{\mathbf{u}_m\}$ still) such that $\{\mathbf{u}_m\}$ converges in $C^{2,1}(\overline{\Omega}'' \times [t',T])$ to $\mathbf{u}$. Then $\mathbf{u} \in C^{2+\alpha_5,1+\alpha_5/2}(\overline{\Omega}'' \times [t'',T])$.

Let $m \to \infty$. The above conclusions and (2.29) yield that $\overline{\mathbf{u}}, \underline{\mathbf{u}}$ satisfy

$$\begin{gathered}
\overline{u}^l_t - \mathcal{L}^l\left(\overline{u}^l\right) = g^l\left(x,t,\overline{u}^l, [\overline{\mathbf{u}}]_{a^l}, [\underline{\mathbf{u}}]_{b^l}, [J * \overline{\mathbf{u}}]_{c^l}, [J * \underline{\mathbf{u}}]_{d^l}\right) \quad ((x,t) \in D_T), \\
\underline{u}^l_t - \mathcal{L}^l\left(\underline{u}^l\right) = g^l\left(x,t,\underline{u}^l, [\underline{\mathbf{u}}]_{a^l}, [\overline{\mathbf{u}}]_{b^l}, [J * \underline{\mathbf{u}}]_{c^l}, [J * \overline{\mathbf{u}}]_{d^l}\right) \quad ((x,t) \in D_T), \\
\left[\overline{u}^l\right]_{\Gamma_T} = 0, \qquad \left[\underline{u}^l\right]_{\Gamma_T} = 0, \\
\left[a^l_{ij}\left(x,t,\overline{u}^l\right)\overline{u}^l_{x_j} \cos(\vec{\mathbf{n}}, x_i)\right]_{\Gamma_T} = 0, \qquad \left[a^l_{ij}\left(x,t,\underline{u}^l\right)\underline{u}^l_{x_j} \cos(\vec{\mathbf{n}}, x_i)\right]_{\Gamma_T} = 0, \\
\overline{u}^l = h^l(x,t), \quad \underline{u}^l = h^l(x,t) \quad ((x,t) \in S_T), \\
\overline{u}^l(x,t) = \psi^l(x,t), \quad \underline{u}^l(x,t) = \psi^l(x,t) \quad \left((x,t) \in Q^l_0\right), \; l = 1,\dots,N, \; m = 1,2,\dots.
\end{gathered} \tag{4.4}$$

Furthermore, (2.35) shows that $\overline{\mathbf{u}}, \underline{\mathbf{u}}$ satisfy (2.16) with $\mathbf{u}, \ \mathbf{v}, \ q^l$ and the symbol "$\leq$" replaced by $\underline{\mathbf{u}}, \overline{\mathbf{u}}, \ g^l$, and the symbol "$=$", respectively. By Lemma 2.4 we get that $\overline{\mathbf{u}} = \underline{\mathbf{u}}$ for $(x,t) \in \overline{D}_T$. In view of $\overline{u}^l(x,t) = \underline{u}^l(x,t) = \psi^l(x,t)$ for $(x,t) \in Q^l_0$, then $\overline{u}^l = \underline{u}^l$ for $(x,t) \in \overline{Q}^l_T, \ l = 1,\dots,N$. Consequently, by (4.4) and Definition 2.3, $\mathbf{u}^* := \overline{\mathbf{u}} = \underline{\mathbf{u}}$ is a piecewise classical solution of (1.1) in $\mathcal{S}$ and satisfies the relation (4.1). If $\mathbf{u}^{**}$ is also a piecewise classical solution of (1.1) in $\mathcal{S}$, then by Lemma 2.4 the same argument shows that $\mathbf{u}^* \equiv \mathbf{u}^{**}$. Therefore, the piecewise classical solution of (1.1) in $\mathcal{S}$ is unique. □

Since T is an arbitrary positive number, the piecewise classical solution $\mathbf{u}^*$ given by Theorem 4.1 is global.

5. Applications in Ecology

Consider 2-species Volterra-Lotka models with diffusion and continuous delays (see [2, 3]). Suppose that the natural conditions for the subdomains $\Omega_k, \ k = 1,\dots,K$, are different. Then the diffusion coefficients are allowed to be discontinuous on the interface Γ. Assume that near Γ, the density and the flux are continuous. Then

$$\left[u^l\right]_{\Gamma_T} = 0, \quad \left[a^l_{ij}\left(x,t,u^l\right)u^l_{x_j} \cos(\vec{\mathbf{n}}, x_i)\right]_{\Gamma_T} = 0, \quad l = 1,2, \tag{5.1}$$

where $u^1, \ u^2$ are the densities of the populations of the two species. Therefore, $u^1, \ u^2$ are governed by the system (1.1), where the reaction functions are explicitly given as follows.

(1) For the Volterra-Lotka cooperation model with continuous delays,

$$\begin{aligned}
g^1(x,t,\mathbf{u}, J * \mathbf{u}) &= u^1\left(r^1_k - \delta^1_k u^1 + \sigma^1_k J^2 * u^2\right) \quad ((x,t) \in D_{k,T}), \ k = 1,\dots,K, \\
g^2(x,t,\mathbf{u}, J * \mathbf{u}) &= u^2\left(r^2_k + \delta^2_k J^1 * u^1 - \sigma^2_k u^2\right) \quad ((x,t) \in D_{k,T}), \ k = 1,\dots,K.
\end{aligned} \tag{5.2}$$

(2) For the Volterra-Lotka competition model with continuous delays,

$$\begin{aligned} g^1(x,t,\mathbf{u},J*\mathbf{u}) &= u^1\left(r_k^1 - \delta_k^1 u^1 - \sigma_k^1 J^2 * u^2\right) \quad ((x,t)\in D_{k,T}),\ k=1,\dots,K, \\ g^2(x,t,\mathbf{u},J*\mathbf{u}) &= u^2\left(r_k^2 - \delta_k^2 J^1 * u^1 - \sigma_k^2 u^2\right) \quad ((x,t)\in D_{k,T}),\ k=1,\dots,K. \end{aligned} \tag{5.3}$$

(3) For the Volterra-Lotka prey-predator model with continuous delays,

$$\begin{aligned} g^1(x,t,\mathbf{u},J*\mathbf{u}) &= u^1\left(r_k^1 - \delta_k^1 u^1 - \sigma_k^1 J^2 * u^2\right) \quad ((x,t)\in D_{k,T}),\ k=1,\dots,K, \\ g^2(x,t,\mathbf{u},J*\mathbf{u}) &= u^2\left(r_k^2 + \delta_k^2 J^1 * u^1 - \sigma_k^2 u^2\right) \quad ((x,t)\in D_{k,T}),\ k=1,\dots,K. \end{aligned} \tag{5.4}$$

Here r_k^l, δ_k^l, and σ_k^l are all positive constants for $k=1,\dots,K$, $l=1,2$.

Theorem 5.1. *Let the functions $a_{ij}^l(x,t,u^l)$, $b_j^l(x,t,u^l)$, $h^l(x,t)$, and $\psi^l(x,t)$, $l=1,2$, satisfy the hypotheses in (H). If $h^l(x,t)$ and $\psi^l(x,t)$, $l=1,2$, are nonnegative functions and the condition $\overline{b}^2/\overline{b}^1 < \overline{c}^2/\overline{c}^1$ holds for the cooperation model, where $\overline{b}^1 = \min_{k=1,\dots,K}(\delta_k^1/r_k^1)$, $\overline{c}^1 = \max_{k=1,\dots,K}(\sigma_k^1/r_k^1)$, $\overline{b}^2 = \max_{k=1,\dots,K}(\delta_k^2/r_k^2)$, and $\overline{c}^2 = \min_{k=1,\dots,K}(\sigma_k^2/r_k^2)$, and if $N=2$ and $g^l(x,t,\mathbf{u},J*\mathbf{u})$, $l=1,2$, are given by one of* (5.2)–(5.4), *then problem* (1.1) *has a unique nonnegative piecewise classical solution.*

Proof. By Theorem 4.1, the proof of this theorem is completed if there exist a pair of coupled weak upper and lower solutions $\tilde{\mathbf{u}} = (M^1, M^2)$, $\hat{\mathbf{u}} = (0,0)$ for each case of (5.2)–(5.4), where M^1 and M^2 are positive constants. We next prove the existence of M^1 and M^2 for each case. Note that (1.2) and (2.13) imply that $J*(M^1,M^2) = (M^1,M^2)$.

Case 1. $g^l(x,t,\mathbf{u},J*\mathbf{u})$, $l=1,2$, are given by (5.2). Then $\mathbf{g}(\cdot,\mathbf{u},\mathbf{v}) = (g^1(\cdot,\mathbf{u},\mathbf{v}), g^2(\cdot,\mathbf{u},\mathbf{v}))$ is quasimonotone nondecreasing. The requirement of $\tilde{\mathbf{u}} = (M^1,M^2)$, $\hat{\mathbf{u}} = (0,0)$ in Definition 2.2 becomes

$$M^1\left[1 - \frac{\delta_k^1}{r_k^1}M^1 + \frac{\sigma_k^1}{r_k^1}M^2\right] \le 0, \quad M^2\left[1 + \frac{\delta_k^2}{r_k^2}M^1 - \frac{\sigma_k^2}{r_k^2}M^2\right] \le 0, \quad k=1,\dots,K. \tag{5.5}$$

Since $\overline{b}^2/\overline{b}^1 < \overline{c}^2/\overline{c}^1$, by the argument in [1, Page 676] we conclude that there exist positive constants η^1 and η^2 such that, for any $R \ge 1$,

$$1 - \overline{b}^1 R\eta^1 + \overline{c}^1 R\eta^2 \le 0, \qquad 1 + \overline{b}^2 R\eta^1 - \overline{c}^2 R\eta^2 \le 0. \tag{5.6}$$

There exists R_0 such that $R_0\eta^l \ge h^l(x,t)$ for $(x,t)\in S_T$ and $R_0\eta^l \ge \psi^l(x,t)$ for $(x,t)\in \overline{Q}_0^l$, $l=1,2$. If $(M^1,M^2) \ge (R_0\eta^1, R_0\eta^2)$, then (M^1,M^2) satisfies (5.5), and $\tilde{\mathbf{u}} = (M^1,M^2)$, $\hat{\mathbf{u}} = (0,0)$ are a pair of coupled weak upper and lower solutions of (1.1).

Case 2. $g^l(x,t,\mathbf{u},J*\mathbf{u})$, $l=1,2$, are given by (5.3). $\mathbf{g}(\cdot,\mathbf{u},\mathbf{v})$ is mixed quasimonotone. The requirement of $\tilde{\mathbf{u}}=(M^1,M^2)$, $\hat{\mathbf{u}}=(0,0)$ in Definition 2.2 becomes

$$M^1\left(r_k^1-\delta_k^1M^1\right)\le 0,\quad M^2\left(r_k^2-\sigma_k^2M^2\right)\le 0,\quad k=1,\ldots,K. \tag{5.7}$$

If $(M^1,M^2)\ge(\max_{k=1,\ldots,K}(r_k^1/\delta_k^1),\max_{k=1,\ldots,K}(r_k^2/\sigma_k^2))$, $M^l\ge h^l(x,t)$ for $(x,t)\in S_T$, and $M^l\ge\psi^l(x,t)$ for $(x,t)\in\overline{Q}_0^l$, $l=1,2$, then $\tilde{\mathbf{u}}=(M^1,M^2)$, $\hat{\mathbf{u}}=(0,0)$ are a pair of coupled weak upper and lower solutions of (1.1).

Case 3. $g^l(x,t,\mathbf{u},J*\mathbf{u})$, $l=1,2$, are given by (5.4). $\mathbf{g}(\cdot,\mathbf{u},\mathbf{v})$ is mixed quasimonotone. The requirement of $\tilde{\mathbf{u}}=(M^1,M^2)$, $\hat{\mathbf{u}}=(0,0)$ in Definition 2.2 becomes

$$M^1\left(r_k^1-\delta_k^1M^1\right)\le 0,\quad M^2\left(r_k^2+\delta_k^2M^1-\sigma_k^2M^2\right)\le 0,\quad k=1,\ldots,K. \tag{5.8}$$

We first choose M^1 satisfying $M^1\ge\max_{k=1,\ldots,K}(r_k^1,/\delta_k^1)$, $M^1\ge h^1(x,t)$ for $(x,t)\in S_T$ and $M^1\ge\psi^1(x,t)$ for $(x,t)\in\overline{Q}_0^1$, and then we choose M^2 satisfying $M^2\ge\max_{k=1,\ldots,K}(r_k^2/\sigma_k^2+\delta_k^2M^1/\sigma_k^2)$, $M^2\ge h^2(x,t)$ for $(x,t)\in S_T$, and $M^2\ge\psi^2(x,t)$ for $(x,t)\in\overline{Q}_0^2$. Thus, $\tilde{\mathbf{u}}=(M^1,M^2)$, $\hat{\mathbf{u}}=(0,0)$ are a pair of coupled weak upper and lower solutions of (1.1). □

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