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Global and blow-up solutions for nonlinear parabolic problems with a gradient term under Robin boundary conditions

Juntang Ding*

*Correspondence:
djuntang@sxu.edu.cn
School of Mathematical Sciences,
Shanxi University, Taiyuan, 030006,
P.R. China

Abstract
In this paper, we study the global and blow-up solutions of the following nonlinear parabolic problems with a gradient term under Robin boundary conditions:

$$\begin{cases} (b(u))_t = \nabla\cdot(g(u)\nabla u) + f(x,u,|\nabla u|^2,t) & \text{in } D\times(0,T), \\ \frac{\partial u}{\partial n} + \gamma u = 0 & \text{on } \partial D\times(0,T), \\ u(x,0) = u_0(x) > 0 & \text{in } \overline{D}, \end{cases}$$

where $D\subset\mathbb{R}^N$ ($N\geq 2$) is a bounded domain with smooth boundary ∂D. By constructing auxiliary functions and using maximum principles, the sufficient conditions for the existence of a global solution, an upper estimate of the global solution, the sufficient conditions for the existence of a blow-up solution, an upper bound for 'blow-up time', and an upper estimate of 'blow-up rate' are specified under some appropriate assumptions on the functions f, g, b and initial value u_0.
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Keywords: global solution; blow-up solution; parabolic problem; Robin boundary condition; gradient term

1 Introduction

In this paper, we study the global and blow-up solutions of the following nonlinear parabolic problems with a gradient term under Robin boundary conditions:

$$\begin{cases} (b(u))_t = \nabla\cdot(g(u)\nabla u) + f(x,u,q,t) & \text{in } D\times(0,T), \\ \frac{\partial u}{\partial n} + \gamma u = 0 & \text{on } \partial D\times(0,T), \\ u(x,0) = u_0(x) > 0 & \text{in } \overline{D}, \end{cases} \tag{1.1}$$

where $q := |\nabla u|^2$, $D\subset\mathbb{R}^N$ ($N\geq 2$) is a bounded domain with smooth boundary ∂D, $\partial/\partial n$ represents the outward normal derivative on ∂D, γ is a positive constant, u_0 is the initial value, T is the maximal existence time of u, and $\overline{D}$ is the closure of D. Set $\mathbb{R}^+ := (0,+\infty)$. We assume, throughout the paper, that $b(s)$ is a $C^3(\mathbb{R}^+)$ function, $b'(s)>0$ for any $s\in\mathbb{R}^+$, $g(s)$ is a positive $C^2(\mathbb{R}^+)$ function, $f(x,s,d,t)$ is a nonnegative $C^1(\overline{D}\times\mathbb{R}^+\times\overline{\mathbb{R}^+}\times\mathbb{R}^+)$ function, and $u_0(x)$ is a positive $C^2(\overline{D})$ function. Under the above assumptions, the classical theory [1] of parabolic equation assures that there exists a unique classical solution $u(x,t)$ with some $T > 0$ for problem (1.1) and the solution is positive over $\overline{D} \times [0, T)$. Moreover, the regularity theorem [2] implies $u(x,t) \in C^3(D \times (0,T)) \cap C^2(\overline{D} \times [0,T))$.

Many papers have studied the global and blow-up solutions of parabolic problems with a gradient term (see, for instance, [3–13]). Some authors have discussed the global and blow-up solutions of parabolic problems under Robin boundary conditions and have got a lot of meaningful results (see [14–20] and the references cited therein). Some special cases of problem (1.1) have been treated already. Zhang [21] dealt with the following problem:

$$\begin{cases} u_t = \nabla \cdot (g(u)\nabla u) + f(u) & \text{in } D \times (0,T), \\ \frac{\partial u}{\partial n} + \gamma u = 0 & \text{on } \partial D \times (0,T), \\ u(x,0) = u_0(x) > 0 & \text{in } \overline{D}, \end{cases}$$

where $D \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with smooth boundary ∂D. By constructing auxiliary functions and using maximum principles, the sufficient conditions characterized by functions f, g and u_0 were given for the existence of a blow-up solution. Zhang [22] investigated the following problem:

$$\begin{cases} (b(u))_t = \Delta u + f(u) & \text{in } D \times (0,T), \\ \frac{\partial u}{\partial n} + \gamma u = 0 & \text{on } \partial D \times (0,T), \\ u(x,0) = u_0(x) > 0 & \text{in } \overline{D}, \end{cases}$$

where $D \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with smooth boundary ∂D. By constructing some auxiliary functions and using maximum principles, the sufficient conditions were obtained there for the existence of global and blow-up solutions. Meanwhile, the upper estimate of a global solution, the upper bound of 'blow-up time' and the upper estimate of 'blow-up rate' were also given. Ding [21] considered the following problem:

$$\begin{cases} (b(u))_t = \nabla \cdot (g(u)\nabla u) + f(u) & \text{in } D \times (0,T), \\ \frac{\partial u}{\partial n} + \gamma u = 0 & \text{on } \partial D \times (0,T), \\ u(x,0) = u_0(x) > 0 & \text{in } \overline{D}, \end{cases}$$

where $D \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with smooth boundary ∂D. By constructing some appropriate auxiliary functions and using a first-order differential inequality technique, the sufficient conditions were obtained for the existence of global and blow-up solutions. For the blow-up solution, an upper and a lower bound on blow-up time were also given.

In this paper, we study problem (1.1). Since the function $f(x,u,q,t)$ contains a gradient term $q = |\nabla u|^2$, it seems that the methods of [21–23] are not applicable for problem (1.1). In this paper, by constructing completely different auxiliary functions with those in [21–23] and technically using maximum principles, we obtain some existence theorems of a global solution, an upper estimate of the global solution, the existence theorems of a blow-up solution, an upper bound of 'blow-up time', and an upper estimates of 'blow-up rate'. Our results extend and supplement those obtained [21–23].

We proceed as follows. In Section 2 we study the global solution of (1.1). Section 3 is devoted to the blow-up solution of (1.1). A few examples are presented in Section 4 to illustrate the applications of the abstract results.

2 Global solution

The main result for the global solution is the following theorem.

Theorem 2.1 *Let u be a solution of problem* (1.1). *Assume that the following conditions* (i)-(iv) *are satisfied*:

(i) *for any* $s \in \mathbb{R}^+$,

$$\begin{aligned}&\bigl(sb'(s)\bigr)' \geq 0, \qquad sb'(s) - \bigl(sb'(s)\bigr)' \leq 0, \qquad \left(\frac{g(s)}{b'(s)}\right)' \leq 0,\\&\left[\frac{1}{g(s)}\left(\frac{g(s)}{b'(s)}\right)' + \frac{1}{b'(s)}\right]' + \frac{1}{g}\left(\frac{g(s)}{b'(s)}\right)' + \frac{1}{b'(s)} \leq 0;\end{aligned} \tag{2.1}$$

(ii) *for any* $(x, s, d, t) \in D \times \mathbb{R}^+ \times \overline{\mathbb{R}^+} \times \mathbb{R}^+$,

$$\begin{aligned}&f_t(x, s, d, t) \leq 0, \qquad f_d(x, s, d, t)\left[\left(\frac{1}{b'(s)}\right)' + \frac{1}{b'(s)}\right] \leq 0,\\&\left(\frac{f(x, s, d, t)b'(s)}{g(s)}\right)_s - \frac{f(x, s, d, t)b'(s)}{g(s)} \leq 0;\end{aligned} \tag{2.2}$$

(iii)

$$\int_{m_0}^{+\infty} \frac{b'(s)}{\mathrm{e}^s}\,\mathrm{d}s = +\infty, \qquad m_0 := \min_{\overline{D}} u_0(x); \tag{2.3}$$

(iv)

$$\alpha := \max_{\overline{D}} \frac{\nabla \cdot (g(u_0)\nabla u_0) + f(x, u_0, q_0, 0)}{\mathrm{e}^{u_0}} > 0, \qquad q_0 := |\nabla u_0|^2. \tag{2.4}$$

Then the solution u to problem (1.1) *must be a global solution and*

$$u(x, t) \leq H^{-1}\bigl(\alpha t + H\bigl(u_0(x, t)\bigr)\bigr), \qquad (x, t) \in \overline{D} \times \overline{\mathbb{R}^+}, \tag{2.5}$$

where

$$H(z) := \int_{m_0}^{z} \frac{b'(s)}{\mathrm{e}^s}\,\mathrm{d}s, \qquad z \geq m_0, \tag{2.6}$$

and H^{-1} *is the inverse function of* H.

Proof Consider the auxiliary function

$$P(x, t) := b'(u)u_t - \alpha \mathrm{e}^u. \tag{2.7}$$

Now we have

$$\nabla P = b'' u_t \nabla u + b' \nabla u_t - \alpha \mathrm{e}^u \nabla u, \tag{2.8}$$

$$\Delta P = b''' u_t |\nabla u|^2 + 2b'' \nabla u \cdot \nabla u_t + b'' u_t \Delta u + b' \Delta u_t - \alpha \mathrm{e}^u |\nabla u|^2 - \alpha \mathrm{e}^u \Delta u, \tag{2.9}$$

and

$$\begin{aligned}
P_t &= b''(u_t)^2 + b'(u_t)_t - \alpha \mathrm{e}^u u_t \\
&= b''(u_t)^2 + b'\left(\frac{g}{b'}\Delta u + \frac{g'}{b'}|\nabla u|^2 + \frac{f}{b'}\right)_t - \alpha \mathrm{e}^u u_t \\
&= b''(u_t)^2 + \left(g' - \frac{b''g}{b'}\right)u_t \Delta u + g\Delta u_t + \left(g'' - \frac{b''g'}{b'}\right)u_t|\nabla u|^2 \\
&\quad + \left(2g' + 2f_q\right)\nabla u \cdot \nabla u_t + \left(f_u - \frac{b''f}{b'} - \alpha \mathrm{e}^u\right)u_t + f_t.
\end{aligned} \tag{2.10}$$

It follows from (2.9) and (2.10) that

$$\begin{aligned}
\frac{g}{b'}\Delta P - P_t = {}& \left(\frac{b'''g}{b'} + \frac{b''g'}{b'} - g''\right)u_t|\nabla u|^2 + \left(2\frac{b''g}{b'} - 2g' - 2f_q\right)\nabla u \cdot \nabla u_t \\
& + \left(2\frac{b''g}{b'} - g'\right)u_t \Delta u - \alpha\frac{g}{b'}\mathrm{e}^u|\nabla u|^2 - \alpha\frac{g}{b'}\mathrm{e}^u\Delta u - b''(u_t)^2 \\
& + \left(\frac{b''f}{b'} - f_u + \alpha \mathrm{e}^u\right)u_t - f_t.
\end{aligned} \tag{2.11}$$

By (1.1), we have

$$\Delta u = \frac{b'}{g}u_t - \frac{g'}{g}|\nabla u|^2 - \frac{f}{g}. \tag{2.12}$$

Substitute (2.12) into (2.11), to get

$$\begin{aligned}
\frac{g}{b'}\Delta P - P_t = {}& \left(\frac{b'''g}{b'} - \frac{b''g'}{b'} - g'' + \frac{(g')^2}{g}\right)u_t|\nabla u|^2 + \left(2\frac{b''g}{b'} - 2g' - 2f_q\right)\nabla u \cdot \nabla u_t \\
& - \frac{(b')^2}{g}\left(\frac{g}{b'}\right)'(u_t)^2 + \left(\frac{fg'}{g} - \frac{b''f}{b'} - f_u\right)u_t + \left(\alpha\frac{g'}{b'}\mathrm{e}^u - \alpha\frac{g}{b'}\mathrm{e}^u\right)|\nabla u|^2 \\
& + \alpha\frac{f}{b'}\mathrm{e}^u - f_t.
\end{aligned} \tag{2.13}$$

With (2.8), we have

$$\nabla u_t = \frac{1}{b'}\nabla P - \frac{b''}{b'}u_t\nabla u + \alpha\frac{\mathrm{e}^u}{b'}\nabla u. \tag{2.14}$$

Next, we substitute (2.14) into (2.13) to obtain

$$\begin{aligned}
& \frac{g}{b'}\Delta P + \left[2\left(\frac{g}{b'}\right)' + 2\frac{f_q}{b'}\right]\nabla u \cdot \nabla P - P_t \\
&\quad = \left(\frac{b'''g}{b'} + \frac{b''g'}{b'} - g'' + \frac{(g')^2}{g} - 2\frac{(b'')^2 g}{(b')^2} + 2\frac{b''f_q}{b'}\right)u_t|\nabla u|^2 \\
&\qquad + \left(2\alpha\frac{b''g}{(b')^2}\mathrm{e}^u - \alpha\frac{g'}{b'}\mathrm{e}^u - \alpha\frac{g}{b'}\mathrm{e}^u - 2\alpha\frac{f_q}{b'}\mathrm{e}^u\right)|\nabla u|^2 \\
&\qquad - \frac{(b')^2}{g}\left(\frac{g}{b'}\right)'(u_t)^2 + \left(\frac{fg'}{g} - \frac{b''f}{b} - f_u\right)u_t + \alpha\frac{f}{b'}\mathrm{e}^u - f_t.
\end{aligned} \tag{2.15}$$

In view of (2.7), we have

$$u_t = \frac{1}{b'}P + \alpha\frac{e^u}{b'}. \tag{2.16}$$

Substituting (2.16) into (2.15), we get

$$\begin{aligned}
&\frac{g}{b'}\Delta P + \left[2\left(\frac{g}{b'}\right)' + 2\frac{f_q}{b'}\right]\nabla u\cdot\nabla P \\
&\quad + \left\{\left[g\left(\frac{1}{g}\left(\frac{g}{b'}\right)'\right)' + 2f_q\left(\frac{1}{b'}\right)'\right]|\nabla u|^2 + \frac{g}{(b')^2}\left(\frac{fb'}{g}\right)_u\right\}P - P_t \\
&= -\alpha e^u\left\{g\left[\left(\frac{1}{g}\left(\frac{g}{b'}\right)' + \frac{1}{b'}\right)' + \frac{1}{g}\left(\frac{g}{b'}\right)' + \frac{1}{b'}\right] + 2f_q\left[\left(\frac{1}{b'}\right)' + \frac{1}{b'}\right]\right\}|\nabla u|^2 \\
&\quad - \frac{(b')^2}{g}\left(\frac{g}{b'}\right)'(u_t)^2 - \alpha\frac{ge^u}{(b')^2}\left[\left(\frac{fb'}{g}\right)_u - \frac{fb'}{g}\right] - f_t.
\end{aligned} \tag{2.17}$$

The assumptions (2.1) and (2.2) guarantee that the right-hand side of (2.17) is nonnegative, *i.e.*,

$$\begin{aligned}
&\frac{g}{b'}\Delta P + \left[2\left(\frac{g}{b'}\right)' + 2\frac{f_q}{b'}\right]\nabla u\cdot\nabla P \\
&\quad + \left\{\left[g\left(\frac{1}{g}\left(\frac{g}{b'}\right)'\right)' + 2f_q\left(\frac{1}{b'}\right)'\right]|\nabla u|^2 + \frac{g}{(b')^2}\left(\frac{fb'}{g}\right)_u\right\}P - P_t \\
&\geq 0 \quad \text{in } D\times(0,T).
\end{aligned} \tag{2.18}$$

By applying the maximum principle [24], it follows from (2.18) that P can attain its nonnegative maximum only for $\overline{D}\times\{0\}$ or $\partial D\times(0,T)$. For $\overline{D}\times\{0\}$, by (2.4), we have

$$\begin{aligned}
\max_{\overline{D}} P(x,0) &= \max_{\overline{D}}\left\{b'(u_0)(u_0)_t - \alpha e^{u_0}\right\} = \max_{\overline{D}}\left\{\nabla\cdot\left(g(u_0)\nabla u_0\right) + f(x,u_0,q_0,0) - \alpha e^{u_0}\right\} \\
&= \max_{\overline{D}}\left\{e^{u_0}\left[\frac{\nabla\cdot(g(u_0)\nabla u_0) + f(x,u_0,q_0,0)}{e^{u_0}} - \alpha\right]\right\} = 0.
\end{aligned}$$

We claim that P cannot take a positive maximum at any point $(x,t)\in\partial D\times(0,T)$. In fact, suppose that P takes a positive maximum at a point $(x_0,t_0)\in\partial D\times(0,T)$, then

$$P(x_0,t_0) > 0 \quad \text{and} \quad \left.\frac{\partial P}{\partial n}\right|_{(x_0,t_0)} > 0. \tag{2.19}$$

With (1.1) and (2.16), we have

$$\begin{aligned}
\frac{\partial P}{\partial n} &= b''u_t\frac{\partial u}{\partial n} + b'\frac{\partial u_t}{\partial n} - \alpha e^u\frac{\partial u}{\partial n} = -\gamma b''uu_t + b'\left(\frac{\partial u}{\partial n}\right)_t + \gamma\alpha ue^u \\
&= -\gamma b''uu_t + b'(-\gamma u)_t + \gamma\alpha ue^u = -\gamma\left(ub'\right)'u_t + \gamma\alpha ue^u \\
&= -\gamma\left(ub'\right)'\left(\frac{1}{b'}P + \alpha\frac{1}{b'}e^u\right) + \gamma\alpha ue^u \\
&= -\gamma\frac{(ub')'}{b'}P + \gamma\alpha e^u\frac{ub' - (ub')'}{b'} \quad \text{on } \partial D\times(0,T).
\end{aligned} \tag{2.20}$$

Next, by using the fact that $(sb'(s))' \geq 0$, $sb'(s) - (sb'(s))' \leq 0$ for any $s \in \mathbb{R}^+$, it follows from (2.20) that

$$\left.\frac{\partial P}{\partial n}\right|_{(x_0,t_0)} \leq 0,$$

which contradicts with inequality (2.19). Thus we know that the maximum of P in $\overline{D} \times [0,T)$ is zero, *i.e.*,

$$P \leq 0 \quad \text{in } \overline{D} \times [0,T),$$

and

$$\frac{b'(u)}{\mathrm{e}^u} u_t \leq \alpha. \tag{2.21}$$

For each fixed $x \in \overline{D}$, integration of (2.21) from 0 to t yields

$$\int_0^t \frac{b'(u)}{\mathrm{e}^u} u_t \,\mathrm{d}t = \int_{u_0(x)}^{u(x,t)} \frac{b'(s)}{\mathrm{e}^s} \,\mathrm{d}s \leq \alpha t, \tag{2.22}$$

which implies that u must be a global solution. Actually, if that u blows up at finite time T, then

$$\lim_{t \to T^-} u(x,t) = +\infty.$$

Passing to the limit as $t \to T^-$ in (2.22) yields

$$\int_{u_0(x)}^{+\infty} \frac{b'(s)}{\mathrm{e}^s} \,\mathrm{d}s \leq \alpha T$$

and

$$\int_{m_0}^{+\infty} \frac{b'(s)}{\mathrm{e}^s} \,\mathrm{d}s = \int_{m_0}^{u_0(x)} \frac{b'(s)}{\mathrm{e}^s} \,\mathrm{d}s + \int_{u_0(x)}^{+\infty} \frac{b'(s)}{\mathrm{e}^s} \,\mathrm{d}s \leq \int_{m_0}^{u_0(x)} \frac{b'(s)}{\mathrm{e}^s} \,\mathrm{d}s + \alpha T < +\infty,$$

which contradicts with assumption (2.3). This shows that u is global. Moreover, it follows from (2.22) that

$$\int_{u_0(x)}^{u(x,t)} \frac{b'(s)}{\mathrm{e}^s} \,\mathrm{d}s = \int_{m_0}^{u(x,t)} \frac{b'(s)}{\mathrm{e}^s} \,\mathrm{d}s - \int_{m_0}^{u_0(x)} \frac{b'(s)}{\mathrm{e}^s} \,\mathrm{d}s = H\big(u(x,t)\big) - H\big(u_0(x)\big) \leq \alpha t.$$

Since H is an increasing function, we have

$$u(x,t) \leq H^{-1}\big(\alpha t + H\big(u_0(x)\big)\big).$$

The proof is complete. □

3 Blow-up solution

The following theorem is the main result for the blow-up solution.

Theorem 3.1 *Let u be a solution of problem* (1.1). *Assume that the following conditions* (i)-(iv) *are fulfilled*:

(i) *for any* $s \in \mathbb{R}^+$,

$$\begin{aligned} &\big(sb'(s)\big)' \geq 0, \qquad sb'(s) - \big(sb'(s)\big)' \geq 0, \qquad \left(\frac{g(s)}{b'(s)}\right)' \geq 0, \\ &\left[\frac{1}{g(s)}\left(\frac{g(s)}{b'(s)}\right)' + \frac{1}{b'(s)}\right]' + \frac{1}{g}\left(\frac{g(s)}{b'(s)}\right)' + \frac{1}{b'(s)} \geq 0; \end{aligned} \tag{3.1}$$

(ii) *for any* $(x,s,d,t) \in D \times \mathbb{R}^+ \times \overline{\mathbb{R}^+} \times \mathbb{R}^+$,

$$\begin{aligned} &f_t(x,s,d,t) \geq 0, \qquad f_d(x,s,d,t)\left[\left(\frac{1}{b'(s)}\right)' + \frac{1}{b'(s)}\right] \geq 0, \\ &\left(\frac{f(x,s,d,t)b'(s)}{g(s)}\right)_s - \frac{f(x,s,d,t)b'(s)}{g(s)} \geq 0; \end{aligned} \tag{3.2}$$

(iii)

$$\int_{M_0}^{+\infty} \frac{b'(s)}{\mathrm{e}^s}\,\mathrm{d}s < +\infty, \qquad M_0 := \max_{\overline{D}} u_0(x); \tag{3.3}$$

(iv)

$$\beta := \min_{\overline{D}} \frac{\nabla \cdot (g(u_0)\nabla u_0) + f(x,u_0,q_0,0)}{\mathrm{e}^{u_0}} > 0, \qquad q_0 := |\nabla u_0|^2. \tag{3.4}$$

Then the solution u of problem (1.1) *must blow up in finite time T, and*

$$T \leq \frac{1}{\beta}\int_{M_0}^{+\infty} \frac{b'(s)}{\mathrm{e}^s}\,\mathrm{d}s, \tag{3.5}$$

$$u(x,t) \leq G^{-1}\big(\beta(T-t)\big), \qquad (x,t) \in \overline{D} \times [0,T), \tag{3.6}$$

where

$$G(z) := \int_{z}^{+\infty} \frac{b'(s)}{\mathrm{e}^s}\,\mathrm{d}s, \qquad z > 0, \tag{3.7}$$

and G^{-1} is the inverse function of G.

Proof Construct the following auxiliary function:

$$Q(x,t) := b'(u)u_t - \beta\mathrm{e}^u. \tag{3.8}$$

Replacing P and α with Q and β in (2.17), respectively, we get

$$\begin{aligned}
&\frac{g}{b'}\Delta Q + \left[2\left(\frac{g}{b'}\right)' + 2\frac{f_q}{b'}\right]\nabla u\cdot\nabla Q \\
&\quad + \left\{\left[g\left(\frac{1}{g}\left(\frac{g}{b'}\right)'\right)' + 2f_q\left(\frac{1}{b'}\right)'\right]|\nabla u|^2 + \frac{g}{(b')^2}\left(\frac{fb'}{g}\right)_u\right\}Q - Q_t \\
&\quad = -\beta e^u\left\{g\left[\left(\frac{1}{g}\left(\frac{g}{b'}\right)' + \frac{1}{b'}\right)' + \frac{1}{g}\left(\frac{g}{b'}\right)' + \frac{1}{b'}\right] + 2f_q\left[\left(\frac{1}{b'}\right)' + \frac{1}{b'}\right]\right\}|\nabla u|^2 \\
&\quad\quad - \frac{(b')^2}{g}\left(\frac{g}{b'}\right)'(u_t)^2 - \beta\frac{ge^u}{(b')^2}\left[\left(\frac{fb'}{g}\right)_u - \frac{fb'}{g}\right] - f_t.
\end{aligned} \tag{3.9}$$

Assumptions (3.1) and (3.2) imply that the right-hand side in equality (3.9) is nonpositive, *i.e.*,

$$\begin{aligned}
&\frac{g}{b'}\Delta Q + \left[2\left(\frac{g}{b'}\right)' + 2\frac{f_q}{b'}\right]\nabla u\cdot\nabla Q \\
&\quad + \left\{\left[g\left(\frac{1}{g}\left(\frac{g}{b'}\right)'\right)' + 2f_q\left(\frac{1}{b'}\right)'\right]|\nabla u|^2 + \frac{g}{(b')^2}\left(\frac{fb'}{g}\right)_u\right\}Q - Q_t \\
&\quad \le 0 \quad \text{in } D\times(0,T).
\end{aligned} \tag{3.10}$$

With (3.4), we have

$$\begin{aligned}
\min_{\overline{D}} Q(x,0) &= \min_{\overline{D}}\left\{b'(u_0)(u_0)_t - \beta e^{u_0}\right\} = \min_{\overline{D}}\left\{\nabla\cdot\left(g(u_0)\nabla u_0\right) + f(x,u_0,q_0,0) - \beta e^{u_0}\right\} \\
&= \min_{\overline{D}}\left\{e^{u_0}\left[\frac{\nabla\cdot(g(u_0)\nabla u_0) + f(x,u_0,q_0,0)}{e^{u_0}} - \beta\right]\right\} = 0.
\end{aligned} \tag{3.11}$$

Substituting P and α with Q and β in (2.20), respectively, we have

$$\frac{\partial Q}{\partial n} = -\gamma\frac{(ub')'}{b'}Q + \gamma\beta e^u\frac{ub' - (ub')'}{b'} \quad \text{on } \partial D\times(0,T). \tag{3.12}$$

Combining (3.10)-(3.12) with the fact that $(sb'(s))' \ge 0$, $sb'(s) - (sb'(s))' \ge 0$ for any $s \in \mathbb{R}^+$, and applying the maximum principles again, it follows that the minimum of Q in $\overline{D}\times[0,T)$ is zero. Thus

$$Q \ge 0 \quad \text{in } \overline{D}\times[0,T),$$

and

$$\frac{b'(u)}{e^u}u_t \ge \beta. \tag{3.13}$$

At the point $x^* \in \overline{D}$, where $u_0(x^*) = M_0$, integrate (3.13) over $[0,t]$ to get

$$\int_0^t \frac{b'(u)}{e^u}u_t\,dt = \int_{M_0}^{u(x^*,t)} \frac{b'(s)}{e^s}\,ds \ge \beta t, \tag{3.14}$$

which implies that u must blow up in finite time. Actually, if u is a global solution of (1.1), then for any $t > 0$, (3.14) shows

$$\int_{M_0}^{+\infty} \frac{b'(s)}{e^s}\,ds \geq \int_{M_0}^{u(x^*,t)} \frac{b'(s)}{e^s}\,ds \geq \beta t. \tag{3.15}$$

Letting $t \to +\infty$ in (3.15), we have

$$\int_{M_0}^{+\infty} \frac{b'(s)}{e^s}\,ds = +\infty,$$

which contradicts with assumption (3.3). This shows that u must blow up in finite time $t = T$. Furthermore, letting $t \to T$ in (3.14), we get

$$T \leq \frac{1}{\beta}\int_{M_0}^{+\infty} \frac{b'(s)}{e^s}\,ds.$$

By integrating inequality (3.13) over $[t, s]$ $(0 < t < s < T)$, for each fixed x, we obtain

$$\begin{aligned} G\bigl(u(x,t)\bigr) &\geq G\bigl(u(x,t)\bigr) - G\bigl(u(x,s)\bigr) = \int_{u(x,t)}^{+\infty} \frac{b'(s)}{e^s}\,ds - \int_{u(x,s)}^{+\infty} \frac{b'(s)}{e^s}\,ds \\ &= \int_{u(x,t)}^{u(x,s)} \frac{b'(s)}{e^s}\,ds = \int_t^s \frac{b'(u)}{e^u} u_t\,dt \geq \beta(s - t). \end{aligned}$$

Hence, by letting $s \to T$, we have

$$G\bigl(u(x,t)\bigr) \geq \beta(T - t).$$

Since G is a decreasing function, we obtain

$$u(x,t) \leq G^{-1}\bigl(\beta(T - t)\bigr).$$

The proof is complete. □

4 Applications

When $b(u) \equiv u$ and $f(x,u,q,t) \equiv f(u)$, the results stated in Theorem 3.1 are valid. When $g(u) \equiv 1$ and $f(x,u,q,t) \equiv f(u)$ or $f(x,u,q,t) \equiv f(u)$, the conclusions of Theorems 2.1 and 3.1 still hold true. In this sense, our results extend and supplement the results of [21–23].

In what follows, we present several examples to demonstrate the applications of the abstract results.

Example 4.1 Let u be a solution of the following problem:

$$\begin{cases} u_t = \Delta u + \frac{2+u}{1+u}|\nabla u|^2 + \frac{e^{-u}(e^{-u}+e^q)}{1+u}(e^{-t} + |x|^2) & \text{in } D \times (0,T), \\ \frac{\partial u}{\partial n} + 2u = 0 & \text{on } \partial D \times (0,T), \\ u(x,0) = 2 - |x|^2 & \text{in } \overline{D}, \end{cases}$$

where $q = |\nabla u|^2$, $D = \{x = (x_1, x_2, x_3) \mid |x|^2 < 1\}$ is the unit ball of $\mathbb{R}^3$. The above problem can be transformed into the following problem:

$$\begin{cases} (ue^u)_t = \nabla \cdot ((1+u)e^u \nabla u) + (e^{-u} + e^q)(e^{-t} + |x|^2) & \text{in } D \times (0,T), \\ \frac{\partial u}{\partial n} + 2u = 0 & \text{on } \partial D \times (0,T), \\ u(x,0) = 2 - |x|^2 & \text{in } \overline{D}. \end{cases}$$

Now

$$b(u) = ue^u, \qquad g(u) = (1+u)e^u, \qquad f(x,u,q,t) = \left(e^{-u} + e^q\right)\left(e^{-t} + |x|^2\right),$$
$$u_0(x) = 2 - |x|^2, \qquad \gamma = 2.$$

In order to determine the constant α, we assume

$$s := |x|^2,$$

then $0 \leq s \leq 1$ and

$$\begin{aligned} \alpha &= \max_{\overline{D}} \frac{\nabla \cdot (g(u_0)\nabla u_0) + f(x,u_0,q_0,0)}{e^{u_0}} \\ &= \max_{\overline{D}} \left\{32|x|^2 - 4|x|^4 - 18 + \left(1 + |x|^2\right)\left[\exp\left(-4 + 2|x|^2\right) + \exp\left(-2 + 5|x|^2\right)\right]\right\} \\ &= \max_{0 \leq s \leq 1} \left\{32s - 4s^2 - 18 + (1+s)\left[\exp(-4+2s) + \exp(-2+5s)\right]\right\} \\ &= 50.4417. \end{aligned}$$

It is easy to check that (2.1)-(2.3) hold. By Theorem 2.1, u must be a global solution, and

$$\begin{aligned} u(x,t) &\leq H^{-1}\left(\alpha t + H\left(u_0(x)\right)\right) = -1 + \sqrt{50.4417t + \left(1 + u_0(x)\right)^2} \\ &= -1 + \sqrt{50.4417t + \left(3 - |x|^2\right)^2}. \end{aligned}$$

Example 4.2 Let u be a solution of the following problem:

$$\begin{cases} u_t = \Delta u - \frac{1}{u(1+u)}|\nabla u|^2 + \frac{u(e^u - e^{-q})}{1+u}(6 + t|x|^2) & \text{in } D \times (0,T), \\ \frac{\partial u}{\partial n} + 2u = 0 & \text{on } \partial D \times (0,T), \\ u(x,0) = 2 - |x|^2 & \text{in } \overline{D}, \end{cases}$$

where $q = |\nabla u|^2$, $D = \{x = (x_1, x_2, x_3) \mid |x|^2 < 1\}$ is the unit ball of $\mathbb{R}^3$. The above problem may be turned into the following problem:

$$\begin{cases} (u + \ln u)_t = \nabla \cdot ((1 + \frac{1}{u})\nabla u) + (e^u - e^{-q})(6 + t|x|^2) & \text{in } D \times (0,T), \\ \frac{\partial u}{\partial n} + 2u = 0 & \text{on } \partial D \times (0,T), \\ u(x,0) = 2 - |x|^2 & \text{in } \overline{D}. \end{cases}$$

Now we have

$$b(u) = u + \ln u, \qquad g(u) = 1 + \frac{1}{u}, \qquad f(x,u,q,t) = \left(\mathrm{e}^{u} - \mathrm{e}^{-q}\right)\left(6 + t|x|^2\right),$$
$$u_0(x) = 2 - |x|^2, \qquad \gamma = 2.$$

By setting

$$s := |x|^2,$$

we have $0 \le s \le 1$ and

$$\begin{aligned}
\beta &= \min_{\overline{D}} \frac{\nabla \cdot (g(u_0)\nabla u_0) + f(x,u_0,q_0,0)}{\mathrm{e}^{u_0}} \\
&= \min_{\overline{D}} \left\{ \frac{-6|x|^4 + 26|x|^2 - 36}{(2-|x|^2)^2 \exp(2-|x|^2)} + 6\left[1 - \exp\left(-3|x|^2 - 2\right)\right] \right\} \\
&= \min_{0\le s\le 1} \left\{ \frac{-6s^2 + 26s - 36}{(2-s)^2 \exp(2-s)} + 6\left[1 - \exp(-3s-2)\right] \right\} \\
&= 0.0735.
\end{aligned}$$

Again it is easy to check that (3.1)-(3.3) hold. By Theorem 3.1, u must blow up in finite time T, and

$$\begin{aligned}
&T \le \frac{1}{\beta} \int_{M_0}^{+\infty} \frac{b'(s)}{\mathrm{e}^s}\,\mathrm{d}s = \frac{1}{0.0735} \int_2^{+\infty} \left(1 + \frac{1}{s}\right) \frac{1}{\mathrm{e}^s}\,\mathrm{d}s = 2.5066, \\
&u(x,t) \le G^{-1}\left(\beta(T-t)\right) = G^{-1}\left(0.0735(T-t)\right),
\end{aligned}$$

where

$$G(z) = \int_z^{+\infty} \frac{b'(s)}{\mathrm{e}^s}\,\mathrm{d}s = \int_z^{+\infty} \left(1 + \frac{1}{s}\right) \frac{1}{\mathrm{e}^s}\,\mathrm{d}s, \quad z \ge 0,$$

and G^{-1} is the inverse function of G.

Remark 4.1 We can see from Example 4.1 that when the equation has a gradient term with exponential increase, the functions g and b increase exponentially to ensure that the solution of (1.1) blows up. It follows from Example 4.2 that when the equation has a gradient term with exponential decay, the appropriate assumptions on the functions g and b can guarantee the solution of (1.1) to be global.

Competing interests
The author declares that he has no competing interests.

Author's contributions
All results belong to Juntang Ding.

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