Reich and Zaslavski *Fixed Point Theory and Applications* (2015) 2015:195
DOI 10.1186/s13663-015-0444-9

Fixed Point Theory and Applications
a SpringerOpen Journal

REVIEW
Open Access

Genericity and porosity in fixed point theory: a survey of recent results

Simeon Reich* and Alexander J Zaslavski

*Correspondence:
sreich@tx.technion.ac.il
Department of Mathematics, The Technion - Israel Institute of Technology, Haifa, 32000, Israel

Abstract
We survey recent results regarding genericity and porosity in fixed point theory. These results concern, *inter alia*, infinite products, nonexpansive mappings, approximate fixed points, contractive mappings, bounded linear regularity, set-valued mappings, holomorphic mappings and weak ergodic theorems. We consider both normed linear spaces and hyperbolic metric spaces.

MSC: 37C20; 46G20; 46T25; 47H09; 47H10; 54E50; 54E52; 54H25

Keywords: approximate fixed point; Baire category; Banach space; bounded linear regularity; complete metric space; contractive mapping; fixed point; holomorphic mapping; hyperbolic space; infinite product; nonexpansive mapping; porous set; set-valued mapping; weak ergodic theorem

1 Genericity, porosity and hyperbolic spaces

A property of elements of a complete metric space Z is said to be *generic* (typical) in Z if the set of all elements of Z which possess this property contains an everywhere dense and G_δ subset of Z. In this case we also say that the property holds for a generic (typical) element of Z or that a generic (typical) element of Z has this property.

In this paper we use the concept of porosity which will enable us to obtain more refined results.

To recall this concept of porosity, let Z be a complete metric space. We denote by $B_Z(y,r)$ the closed ball of center $y \in Z$ and radius $r > 0$. A subset $E \subset Z$ is called *porous* in Z if there exist numbers $\alpha \in (0,1)$ and $r_0 > 0$ such that for each number $r \in (0,r_0]$ and each point $y \in Z$, there exists a point $z \in Z$ for which

$$B_Z(z,\alpha r) \subset B_Z(y,r) \setminus E.$$

A subset of the metric space Z is called *σ-porous* in Z if it is a countable union of porous subsets in Z.

Other notions of porosity can be found in the literature. We use the rather strong concept of porosity which has already found application in, for example, approximation theory, the calculus of variations and nonlinear analysis [1–3].

Since porous sets are nowhere dense, all σ-porous sets are of the first Baire category. If Z is a finite-dimensional Euclidean space, then σ-porous sets are also of Lebesgue measure zero. In fact, the class of σ-porous sets in such a space is much smaller than the class of sets which have measure zero and are of the first Baire category.

Some of our results concern nonlinear mappings acting on closed and convex sets in complete hyperbolic spaces, a class of metric spaces, the definition of which we now recall.

Let (X,ρ) be a metric space and let $\mathbb{R}$ denote the real line. We say that a mapping $c : \mathbb{R} \to X$ is a *metric embedding* of $\mathbb{R}$ into X if $\rho(c(s),c(t)) = |s-t|$ for all real s and t. The image of $\mathbb{R}$ under a metric embedding is called a *metric line*. The image of a real interval $[a,b] = \{t \in \mathbb{R} : a \le t \le b\}$ under such a mapping is called a *metric segment*.

Assume that the metric space (X,ρ) contains a family $\mathcal{L}$ of metric lines such that for each pair of distinct points x and y in X, there is a unique metric line in $\mathcal{L}$ which passes through x and y. This metric line determines a unique metric segment joining x and y. We denote this metric segment by $[x,y]$. For each $0 \le t \le 1$, there is a unique point z in $[x,y]$ such that

$$\rho(x,z) = t\rho(x,y) \quad \text{and} \quad \rho(z,y) = (1-t)\rho(x,y).$$

This point will be denoted by $(1-t)x \oplus ty$. We say that X, or more precisely $(X,\rho,\mathcal{L})$, is a *hyperbolic space* if

$$\rho\left(\frac{1}{2}x \oplus \frac{1}{2}y, \frac{1}{2}x \oplus \frac{1}{2}z\right) \le \frac{1}{2}\rho(y,z)$$

for all x, y and z in X. An equivalent requirement is that

$$\rho\left(\frac{1}{2}x \oplus \frac{1}{2}y, \frac{1}{2}w \oplus \frac{1}{2}z\right) \le \frac{1}{2}\bigl(\rho(x,w) + \rho(y,z)\bigr)$$

for all x, y, z and w in X. This inequality, in its turn, implies that

$$\rho\bigl((1-t)x \oplus ty, (1-t)w \oplus tz\bigr) \le (1-t)\rho(x,w) + t\rho(y,z)$$

for all x, y, z and w in X, and all $0 \le t \le 1$.

A set $K \subset X$ is said to be *ρ-convex* if $[x,y] \subset K$ for all x and y in K.

It is clear that all normed linear spaces are hyperbolic in this sense. A discussion of more examples of hyperbolic spaces and, in particular, of the Hilbert ball, as well as applications to holomorphic fixed point theory, can be found, for instance, in [4–6].

Our purpose in this paper is to survey recent results regarding genericity and porosity in fixed point theory. These results concern, *inter alia*, infinite products, nonexpansive mappings, approximate fixed points, contractive mappings, bounded linear regularity, set-valued mappings, holomorphic mappings and weak ergodic theorems. We consider both normed linear spaces and hyperbolic metric spaces and concentrate on those results which have been obtained since the publication of [1]. We hope our survey leads to further progress in this exciting area of current research interest.

2 Stable convergence of infinite products

The convergence of infinite products of nonexpansive mappings is of major importance in nonlinear functional analysis in general and in fixed point theory in particular. Such products find many and diverse applications in the study of feasibility and optimization problems. In this section the concept of porosity is used in order to study generic stable convergence for infinite products of nonexpansive mappings in Banach spaces. The results of this section were established in [7], where several pertinent references can also be found.

Let $(X, \|\cdot\|)$ be a Banach space and let $K \subset X$ be a nonempty, bounded, closed and convex subset of X. In this section (and in the next one) we denote by $\mathcal{M}$ the set of all sequences $\{A_t\}_{t=1}^{\infty}$ of mappings $A_t : K \to X$, $t = 1, 2, \ldots$, which satisfy the following two conditions:

$$\|A_t x - A_t y\| \le \|x - y\| \quad \text{for all } x, y \in K \text{ and } t = 1, 2, \ldots;$$

for each $\epsilon > 0$, there exists a point $x_\epsilon \in K$ such that

$$\|x_\epsilon - A_t x_\epsilon\| \le \epsilon, \quad t = 1, 2, \ldots.$$

Set

$$\rho(K) := \sup\bigl\{\|z\| : z \in K\bigr\}.$$

We first note the following simple fact.

Proposition 2.1 *Let $\{A_t\}_{t=1}^{\infty} \in \mathcal{M}$. Then, for all $x \in K$ and $t = 1, 2, \ldots$,*

$$\|A_t x\| \le 3\rho(K) + 1.$$

Now let $\{A_t\}_{t=1}^{\infty}, \{B_t\}_{t=1}^{\infty} \in \mathcal{M}$. Define

$$d\bigl(\{A_t\}_{t=1}^{\infty}, \{B_t\}_{t=1}^{\infty}\bigr) := \sup\bigl\{\|A_t x - B_t x\| : x \in K, t = 1, 2, \ldots\bigr\}.$$

By Proposition 2.1, $d(\{A_t\}_{t=1}^{\infty}, \{B_t\}_{t=1}^{\infty})$ is finite. As a matter of fact, it is not difficult to see that d is a metric on $\mathcal{M}$ and that the metric space $(\mathcal{M}, d)$ is complete.

We are now ready to state the main result of this section.

Theorem 2.2 *There exists a set $\mathcal{F} \subset (\mathcal{M}, d)$ such that its complement $\mathcal{M} \setminus \mathcal{F}$ is a σ-porous subset of $(\mathcal{M}, d)$ and each $\{B_t\}_{t=1}^{\infty} \in \mathcal{F}$ has the following properties*:

(1) *there is a unique point $\bar{x} \in K$ such that $B_t \bar{x} = \bar{x}$ for $t = 1, 2, \ldots$;*

(2) *for each $\epsilon > 0$, there exist a number $\delta > 0$, a natural number q and a neighborhood $\mathcal{U}$ of $\{B_t\}_{t=1}^{\infty}$ in $(\mathcal{M}, d)$ such that*:

   (a) *if $\{C_t\}_{t=1}^{\infty} \in \mathcal{U}$, $t \in \{1, 2, \ldots\}$, $y \in K$ and $\|y - C_t y\| \le \delta$, then $\|y - \bar{x}\| \le \epsilon$;*

   (b) *if $\{C_t\}_{t=1}^{\infty} \in \mathcal{U}$, $r : \{1, \ldots, q\} \to \{1, 2, \ldots\}$, and if the elements of $\{x_i\}_{i=0}^{q} \subset K$ satisfy*

$$C_{r(i)} x_{i-1} = x_i, \quad i = 1, \ldots, q,$$

   then $\|x_q - \bar{x}\| \le \epsilon$.

This theorem immediately yields the following corollary regarding the convergence of infinite products.

Corollary 2.3 *Let $\{B_t\}_{t=1}^{\infty} \in \mathcal{F}$, where $\mathcal{F}$ and $\bar{x}$ are as in Theorem 2.2. If the mapping $r: \{1,2,\dots\} \to \{1,2,\dots\}$ and the elements of the sequence $\{x_i\}_{i=0}^{\infty} \subset K$ satisfy*

$$C_{r(i)}x_{i-1} = x_i, \quad i = 1,2,\dots,$$

then $\lim_{i\to\infty} \|x_i - \bar{x}\| = 0$.

3 Contractivity, porosity and infinite products

In this short section we continue to study infinite products of operators and the complete metric space $(\mathcal{M}, d)$, which was introduced in the previous section. We first use the notion of porosity to show that most elements in this space of sequences of nonexpansive mappings are, in fact, contractive (see the definition below). We then present a convergence theorem for infinite products governed by such elements, where we also allow for computational errors. The results of this section were obtained in [8].

A sequence $\{A_t\}_{t=1}^{\infty} \in \mathcal{M}$ is called *contractive* [1, 9] if there exists a decreasing function $\phi: [0,\infty) \to [0,1]$ such that

$$\phi(t) < 1 \quad \text{for all real } t > 0$$

and such that for each $x, y \in K$ and each integer $t \geq 1$,

$$\|A_t x - A_t y\| \leq \phi\bigl(\|x - y\|\bigr)\|x - y\|.$$

Theorem 3.1 *There exists a set $\mathcal{F} \subset (\mathcal{M}, d)$ such that its complement $\mathcal{M} \setminus \mathcal{F}$ is a σ-porous set in $(\mathcal{M}, d)$ and such that each sequence $\{A_t\}_{t=1}^{\infty} \in \mathcal{F}$ is contractive and possesses a unique common fixed point $\bar{x} \in K$, that is, a point $\bar{x} \in K$ such that $A_t\bar{x} = \bar{x}$ for all $t = 1,2,\dots$.*

Now we state a convergence theorem for (random) infinite products governed by contractive sequences. We also allow for computational errors.

Theorem 3.2 *Assume that $\{A_t\}_{t=1}^{\infty} \in \mathcal{M}$ is contractive, $x_* \in K$ satisfies $A_t x_* = x_*$, $t = 1,2,\dots$, and that $\epsilon \in (0,1)$. Then there exist a number $\delta > 0$ and a natural number n_0 such that for each $r: \{1,2,\dots\} \to \{1,2,\dots\}$, each integer $T \geq n_0$ and each sequence $\{x_i\}_{i=1}^{T} \subset K$ such that*

$$\|x_{i+1} - A_{r(i)}x_i\| \leq \delta, \quad i = 1,\dots,T-1,$$

the following inequality holds for all $t = n_0,\dots,T$:

$$\|x_t - x_*\| \leq \epsilon.$$

4 Approximate fixed points of nonexpansive mappings in unbounded sets

In this section we turn our attention to the study of approximate fixed points (see the definition below). It follows from Banach's fixed point theorem that every nonexpansive self-mapping of a bounded, closed and convex set in a Banach space has approximate fixed points. See, for example, [4], Proposition 1.4, p.4. This is no longer true, in general, if the set is unbounded. Nevertheless, as we will see in this section, there exists an open and everywhere dense set in the space of all nonexpansive self-mappings of any closed and convex (not necessarily bounded) set in a Banach space (endowed with the natural metric of uniform convergence on bounded subsets) such that all its elements have approximate fixed points. As a matter of fact, it turns out that our results hold in all complete hyperbolic spaces; see [10], where several relevant references are also mentioned.

Let $(X, \rho, \mathcal{L})$ be a complete hyperbolic space and let K be a nonempty, closed and ρ-convex subset of X.

For each point $x \in K$ and each number $r > 0$, set

$$B(x,r) := \{y \in K : \rho(x,y) \leq r\}.$$

Denote by $\mathcal{A}$ the set of all nonexpansive self-mappings of K, that is, all mappings $A : K \to K$ which satisfy

$$\rho(Ax, Ay) \leq \rho(x,y) \quad \text{for all } x, y \in K.$$

Fix some point $\theta \in K$.

We equip the set $\mathcal{A}$ with the uniformity determined by the base

$$\mathcal{U}(n) := \{(A,B) \in \mathcal{A} \times \mathcal{A} : \rho(Ax, Bx) \leq n^{-1} \text{ for all } x \in B(\theta, n)\},$$

where n is a natural number. It is not difficult to see that the uniform space $\mathcal{A}$ is metrizable and complete.

Let $A \in \mathcal{A}$ and $\epsilon \geq 0$ be given. A point $x \in K$ is called an ϵ*-approximate fixed point* of A if $\rho(x, Ax) \leq \epsilon$.

We say that a mapping A has the *bounded approximate fixed point* property (or the BAFP property, for short) if there is a nonempty and bounded set $K_0 \subset K$ such that for each $\epsilon > 0$, A has an ϵ-approximate fixed point in K_0, that is, a point $x_\epsilon \in K_0$ which satisfies $\rho(x_\epsilon, Ax_\epsilon) \leq \epsilon$.

Proposition 4.1 *Assume that $A \in \mathcal{A}$ and $K_0 \subset K$ is a nonempty, closed, ρ-convex and bounded subset of K such that*

$$A(K_0) \subset K_0.$$

Then A has the BAFP property.

This proposition immediately implies the following result.

Corollary 4.2 *Assume that the set K is bounded. Then any mapping $A \in \mathcal{A}$ has the BAFP property.*

Corollary 4.2 does not, of course, hold if the set K is unbounded. For example, if K is a Banach space and A is a translation mapping, then A does not possess the BAFP property.

We are now ready to state the main result of this section.

Theorem 4.3 *There exists an open and everywhere dense set $\mathcal{F} \subset \mathcal{A}$ such that each $A \in \mathcal{F}$ has the BAFP property.*

5 Generic contractivity for a class of nonlinear mappings

We now continue to study generic contractivity. In our previous work [1], Section 2.1, pp.15-23, we studied a certain class of nonlinear self-mappings of a complete metric space endowed with a natural metric. Using the notion of porosity, we showed there that most elements of this class possess a unique fixed point which attracts all iterates uniformly. In this section we present a variant of this result (proved in [11]), which shows that most mappings in this class are quasi-contractive.

Let K be a nonempty, bounded, closed and convex subset of a Banach space $(X, \|\cdot\|)$. We consider the topological space $K \subset X$ with the relative topology induced by the norm $\|\cdot\|$. Set

$$\operatorname{diam}(K) := \sup\{\|x-y\| : x, y \in K\}.$$

Denote by $\mathcal{A}$ the set of all continuous mappings $A : K \to K$ which have the following property:

(P1) For each $\epsilon > 0$, there exists a point $x_\epsilon \in K$ such that

$$\|Ax - x_\epsilon\| \le \|x - x_\epsilon\| + \epsilon \quad \text{for all } x \in K.$$

For each $A, B \in \mathcal{A}$, set

$$d(A,B) := \sup\{\|A(x) - B(x)\| : x \in K\}.$$

It is clear that the metric space $(\mathcal{A}, d)$ is complete.

Now consider an arbitrary complete metric space (Y, d). We denote by $B(y,r)$ the closed ball of center $y \in Y$ and radius $r > 0$. Fix a real number $q \ge 1$.

We say that a subset $E \subset Y$ is (q)-*porous* (with respect to the metric d) if there exist numbers $\alpha \in (0,1)$ and $r_0 > 0$ such that for each number $r \in (0, r_0]$ and each point $y \in Y$, there exists a point $z \in Y$ for which

$$B(z, \alpha r^q) \subset B(y,r) \setminus E.$$

We say that a subset of the space Y is (σ, q)-*porous* (with respect to the metric d) if it is a countable union of (q)-porous subsets of Y.

Since (q)-porous sets are nowhere dense, all (σ, q)-porous sets are of the first Baire category. To point out the difference between (q)-porous and nowhere dense sets, note that if $E \subset Y$ is nowhere dense, $y \in Y$ and $r > 0$, then there are a point $z \in Y$ and a number $s > 0$ such that $B(z,s) \subset B(y,r) \setminus E$. If, however, E is also (q)-porous, then for small enough r we can choose $s = \alpha r^q$, where $\alpha \in (0,1)$ is a constant which only depends on E.

Theorem 5.1 *There exists a set $\mathcal{F} \subset \mathcal{A}$ such that its complement $\mathcal{A} \setminus \mathcal{F}$ is a $(\sigma, 2)$-porous set in $(\mathcal{A}, d)$ and such that for each mapping $B \in \mathcal{F}$, there exist a unique point $x_B \in K$ satisfying*

$$B(x_B) = x_B,$$

and a decreasing function $\phi : [0, \infty) \to [0, 1]$ *for which*

$$\phi(t) < 1 \quad \text{for all } t > 0$$

and

$$\|B(x) - x_B\| \leq \phi(\|x - x_B\|)\|x - x_B\| \quad \text{for all } x \in K.$$

6 Porosity and the bounded linear regularity property

Bauschke and Borwein [12] showed that in the space of all tuples of bounded, closed and convex subsets of a Hilbert space with a nonempty intersection, a typical tuple has the bounded linear regularity property (see the definition below). This property is important because it leads to the convergence of infinite products of the corresponding nearest point projections to a point in the intersection. This section is devoted to the results which were established in [13]. They show that the subset of all tuples which possess the bounded linear regularity property has a porous complement. Moreover, this result is obtained in all normed spaces and for tuples of closed and convex sets which are not necessarily bounded.

Let $(X, \|\cdot\|)$ be a normed linear space. For each point $x \in X$, each number $r > 0$ and each nonempty set $A \subset X$, we set

$$B(x, r) := \{y \in X : \|x - y\| \leq r\}$$

and

$$d(x, A) := \inf\{\|x - y\| : y \in A\}.$$

Let n be a natural number and let C_i, $i = 1, \dots, n$, be nonempty, closed and convex subsets of X. Set

$$C = \bigcap_{i=1}^{n} C_i.$$

We say that the tuple $(C_1, \dots, C_n)$ is *boundedly linearly regular* [12] if the set C is nonempty, and for each nonempty and bounded set $S \subset X$, there is a number $\kappa_S > 0$ such that for each point $x \in S$,

$$d(x, C) \leq \kappa_S \max\{d(x, C_j) : j = 1, \dots, n\}.$$

By $\mathrm{Int}(A)$ we denote the interior of a set $A \subset X$ with respect to the norm topology.

The following result was established by Bauschke and Borwein in [12], Theorem 5.27, p.399. Recall that a subset of a metric space is said to be *residual* if it contains a countable intersection of open and everywhere dense subsets of the space.

Theorem 6.1 *Let the space X be Hilbert. Suppose that $\mathcal{F}$ is the set of all tuples of the form $(C_1, \dots, C_n)$, where each set C_i is bounded, closed and convex, and the intersection $C = \bigcap_{i=1}^{n} C_i$ is nonempty. Then the subset of all boundedly linearly regular tuples $(C_1, \dots, C_n)$ is residual in $\mathcal{F}$ equipped with the Hausdorff metric.*

Bounded linear regularity is important because it leads to strong convergence theorems (with convergence rates) for infinite products of projections [12].

Next we present a sufficient condition for bounded linear regularity.

Theorem 6.2 *Let $C_1, \dots, C_n$ be n closed and convex subsets of X such that*

$$\operatorname{Int}\left(\bigcap_{i=1}^{n} C_i\right) \neq \emptyset.$$

Then the tuple $(C_1, \dots, C_n)$ is boundedly linearly regular.

In [13] we improved Theorem 6.1 in several directions. Our results are established in any normed space and the sets $C_1, \dots, C_n$ are not necessarily bounded. Moreover, we show that the set of all boundedly linearly regular tuples is not only residual, but has, in fact, a porous complement.

Denote by $S(X)$ the collection of all nonempty and closed subsets of X. For each $A, B \in S(X)$, define

$$H(A,B) := \max\bigl\{\sup\bigl\{d(x,B) : x \in A\bigr\}, \sup\bigl\{d(y,A) : y \in B\bigr\}\bigr\}$$

and

$$\tilde{H}(A,B) := H(A,B)\bigl(1 + H(A,B)\bigr)^{-1}.$$

(Here we use the convention that $\infty/\infty = 1$.)

It is not difficult to see that the space $(S(X), \tilde{H})$ is a metric space.

For each natural number n and each $A, B \in S(X)$, we set

$$h_n(A,B) := \sup\bigl\{\bigl|d(x,A) - d(x,B)\bigr| : x \in B(0,n)\bigr\}$$

and

$$h(A,B) := \sum_{n=1}^{\infty} \bigl[2^{-n} h_n(A,B)\bigl(1 + h_n(A,B)\bigr)^{-1}\bigr].$$

Once again, it is not difficult to see that h is a metric on $S(X)$. Clearly,

$$\tilde{H}(A,B) \geq h(A,B) \quad \text{for all } A, B \in S(X).$$

Denote by $S_{co}(X)$ the set of all convex sets $A \in S(X)$. It is clear that the set $S_{co}(X)$ is closed in $(S(X), \tilde{H})$ and in $(S(X), h)$. In the sequel we use the metric spaces $(S_{co}(X), \tilde{H})$ and $(S_{co}(X), h)$.

Let n be a natural number and consider the product space $(S_{co}(X))^n$ which consists of all tuples $(C_1, \dots, C_n) \in (S_{co}(X))^n$. We equip the set $(S_{co}(X))^n$ with the pair of metrics $\tilde{H}^{(n)}$ and $h^{(n)}$, where for all

$$(C_1, \dots, C_n), \bigl(C_1', \dots, C_n'\bigr) \in \bigl(S_{co}(X)\bigr)^n,$$

$$\tilde{H}^{(n)}\big((C_1,\ldots,C_n),\big(C_1',\ldots,C_n'\big)\big) := \sum_{i=1}^{n} \tilde{H}\big(C_i,C_i'\big)$$

and

$$h^{(n)}\big((C_1,\ldots,C_n),\big(C_1',\ldots,C_n'\big)\big) := \sum_{i=1}^{n} h\big(C_i,C_i'\big).$$

Clearly, both $((S_{co}(X))^n, \tilde{H}^{(n)})$ and $((S_{co}(X))^n, h^{(n)})$ are metric spaces.

Let $S_{b,co}(X)$ stand for the set of all bounded sets $C \in S_{co}(X)$. It is clear that the product space $(S_{b,co}(X))^n$ is a closed subset of $((S_{co}(X))^n, \tilde{H}^{(n)})$.

In this section we denote by $\mathcal{M}$ the set of all tuples $(C_1,\ldots,C_n) \in (S_{co}(X))^n$ which have the following property:

(P) For each $\epsilon > 0$, there is a point $x_\epsilon \in X$ such that

$$d(x_\epsilon, C_i) \le \epsilon, \quad i = 1,\ldots,n.$$

It is natural to say that a point $x_\epsilon \in X$ satisfying the above inequality is an (ϵ)-*approximate solution* of the feasibility problem which corresponds to the tuple $(C_1,\ldots,C_n)$. In other words, the set $\mathcal{M}$ is the set of all tuples $(C_1,\ldots,C_n) \in (S_{co}(X))^n$ such that the corresponding feasibility problem possesses an (ϵ)-approximate solution for any positive number ϵ. Usually, the feasibility problems studied in the literature have exact solutions. Here we prefer to consider a more general case because of the following observation: if a sequence of tuples, for which the feasibility problem has an exact solution, converges in the space $(S_{co}(X))^n$, then it is only guaranteed that the limit tuple belongs to $\mathcal{M}$.

It is not difficult to see that $\mathcal{M}$ is a closed subset of $((S_{co}(X))^n, \tilde{H}^{(n)})$. Indeed, assume that a tuple $(C_1,\ldots,C_n)$ belongs to the closure of $\mathcal{M}$ and ϵ is a positive number. Choose a tuple $(\tilde{C}_1,\ldots,\tilde{C}_n) \in \mathcal{M}$ such that

$$\tilde{H}(C_i,\tilde{C}_i) \le \epsilon/2, \quad i = 1,\ldots,n,$$

and let $x_0 \in X$ be an $(\epsilon/2)$-approximate solution of the feasibility problem corresponding to the tuple $(\tilde{C}_1,\ldots,\tilde{C}_n)$. Then it is easy to see that x_0 is an (ϵ)-approximate solution of the feasibility problem corresponding to the tuple $(C_1,\ldots,C_n)$.

Define

$$\mathcal{F} := \left\{(C_1,\ldots,C_n) \in \mathcal{M} : \operatorname{Int}\left(\bigcap_{i=1}^{n} C_i\right) \ne \emptyset\right\}.$$

We are now ready to state the two main results of this section.

Theorem 6.3 *Let $\mathcal{A}$ be either $\mathcal{M}$ or $\mathcal{M} \cap (S_{b,co}(X))^n$ and let $\mathcal{F}' = \mathcal{F} \cap \mathcal{A}$. Then its complement $\mathcal{A} \setminus \mathcal{F}'$ is porous in $(\mathcal{A}, \tilde{H}^{(n)})$.*

Theorem 6.4 *The set $\mathcal{F}$ is an everywhere dense subset of $\mathcal{M}$ with the topology induced by the metric $\tilde{H}^{(n)}$ and an open subset of $\mathcal{M}$ with the topology induced by the metric $h^{(n)}$.*

Combining Theorems 6.3 and 6.4, respectively, with Theorem 6.2, we conclude that any tuple $(C_1,\dots,C_n)\in\mathcal{F}$ is boundedly linearly regular.

Theorem 6.3 shows that the set of all boundedly linearly regular tuples has a porous complement with respect to the metric $\tilde{H}^{(n)}$, while according to Theorem 6.4, this set contains an everywhere dense (with respect to the metric $\tilde{H}^{(n)}$) and open (with respect to the metric $h^{(n)}$) set.

7 Generic contractivity of nonexpansive mappings with unbounded domains

Given a nonempty, bounded, closed and convex subset K of a Banach space, we studied in [1], Section 3.1, pp.119-121, the class of nonexpansive self-mappings of K endowed with a natural metric. Using the Baire category approach and the notion of porosity, we showed that most elements of this class are contractive. In this section we present a variant of this result for unbounded sets which was established in [14]. Namely, we show that most nonexpansive mappings are contractive on all bounded subsets.

Let $(X,\rho,\mathcal{L})$ be a complete hyperbolic space and let K be a nonempty, closed and ρ-convex subset of X.

For each point $x\in K$ and each number $r>0$, set

$$B(x,r):=\{y\in K:\rho(x,y)\le r\}.$$

In this section we denote by $\mathcal{A}$ the set of all mappings $A:K\to K$ which satisfy

$$\rho(Ax,Ay)\le\rho(x,y)\quad\text{for all } x,y\in K.$$

Fix some point $\theta\in K$.

We equip the set $\mathcal{A}$ with the uniformity determined by the base

$$\mathcal{U}(n):=\{(A,B)\in\mathcal{A}\times\mathcal{A}:\rho(Ax,Bx)\le n^{-1}\text{ for all } x\in B(\theta,n)\},$$

where n is a natural number. Clearly, the uniform space $\mathcal{A}$ is metrizable and complete.

Let $A\in\mathcal{A}$. Recall that the mapping A is called *contractive* if there exists a decreasing function $\phi:[0,\infty)\to[0,1]$ such that

$$\phi(t)<1\quad\text{for all } t>0$$

and

$$\rho(Ax,Ay)\le\phi\big(\rho(x,y)\big)\rho(x,y)\quad\text{for all } x,y\in X.$$

According to Rakotch's theorem [9], every contractive mapping has a unique fixed point.

As we have already mentioned, in our previous work we studied the space $\mathcal{A}$ in the case where the set K is bounded. Using the Baire category approach and the notion of porosity, we showed that most elements of the space $\mathcal{A}$ are contractive. If, however, the set K is *unbounded*, it is known [15] that our results no longer hold. Nevertheless, in this section we present a variant of our results for unbounded sets which shows that most mappings in this class are contractive on all *bounded* subsets.

Theorem 7.1 *There exists a set $\mathcal{F}_* \subset \mathcal{A}$ which is a countable intersection of open and everywhere dense sets in $\mathcal{A}$ such that for each $A \in \mathcal{F}_*$, the following two properties hold*:

(i) *there exists a unique point $x_A \in K$ such that $Ax_A = x_A$*;

(ii) *for each $r > 0$, there exists a decreasing function $\phi : [0, \infty) \to [0, 1]$ such that*

$$\phi(t) < 1 \quad \text{for all } t > 0$$

and

$$\rho(Ax, Ay) \le \phi\big(\rho(x, y)\big)\rho(x, y) \quad \text{for all } x, y \in B(x_A, r).$$

8 Approximate fixed points of nonexpansive set-valued mappings in unbounded sets

In this section we are concerned with nonexpansive *set-valued* mappings. In our previous work [1], certain set-valued dynamical systems induced by such mappings have been investigated and some new iterative methods for approximating the corresponding fixed points have been obtained. Here we present the more recent results of [16].

Let $(X, \rho, \mathcal{L})$ be a complete hyperbolic space and let K be a nonempty, closed and ρ-convex subset of X.

For each point $x \in K$ and each number $r > 0$, we set, once again,

$$B(x, r) := \big\{y \in K : \rho(x, y) \le r\big\}.$$

For each point $x \in X$ and each nonempty set $D \subset X$, we set

$$\rho(x, D) := \inf\big\{\rho(x, y) : y \in D\big\}.$$

In this section we denote by $S(K)$ the family of all nonempty, closed and bounded subsets of K.

For each $C, D \in S(K)$, set

$$H(C, D) := \max\big\{\sup\big\{\rho(x, D) : x \in C\big\}, \sup\big\{\rho(x, C) : x \in D\big\}\big\}.$$

The space $(S(K), H)$ is a metric space and its metric H is called the *Hausdorff metric*. It is known that the metric space $(S(K), H)$ is complete.

We also denote by $\mathcal{M}$ the set of all mappings $A : K \to S(K)$ which satisfy

$$H\big(A(x), A(y)\big) \le \rho(x, y) \quad \text{for all } x, y \in K.$$

Fixing again a point $\theta \in K$, we equip the set $\mathcal{M}$ with the uniformity determined by the following base:

$$\mathcal{U}(n) := \big\{(A, B) \in \mathcal{M} \times \mathcal{M} : H\big(A(x), B(x)\big) \le n^{-1} \text{ for all } x \in B(\theta, n)\big\},$$

where n is a natural number. It is clear that the uniform space $\mathcal{M}$ is metrizable and complete.

Let $A \in \mathcal{M}$ and $\epsilon \geq 0$ be given. A point $x \in K$ is called an *ϵ-approximate fixed point* of A if $\rho(x, A(x)) \leq \epsilon$.

We say that the mapping A has the *bounded approximate fixed point property* (or the BAFP property, for short) if there is a nonempty and bounded set $K_0 \subset K$ such that for each $\epsilon > 0$, the mapping A has an ϵ-approximate fixed point in K_0, that is, a point $x_\epsilon \in K_0$ which satisfies $\rho(x_\epsilon, A(x_\epsilon)) \leq \epsilon$.

For each $D \subset X$, we denote by $\mathrm{cl}(D)$ the closure of D.

For each nonempty set $D \subset X$ and each mapping $A \in \mathcal{M}$, set

$$A(D) := \bigcup \bigl\{A(x) : x \in D\bigr\}.$$

Theorem 8.1 *Assume that $A \in \mathcal{M}$ and that $K_0 \subset K$ is a nonempty, closed, ρ-convex and bounded subset of K such that*

$$A(K_0) \subset K_0.$$

Then, for each $\epsilon > 0$, there is a point $x_\epsilon \in K_0$ such that $\rho(x_\epsilon, A(x_\epsilon)) \leq \epsilon$.

Thus every mapping satisfying the assumptions of Theorem 8.1 has the BAFP property.

Theorem 8.2 *There exists an open and everywhere dense set $\mathcal{F} \subset \mathcal{M}$ such that each $A \in \mathcal{F}$ has the BAFP property.*

This theorem follows from Theorem 8.1 when it is combined with the following result.

Theorem 8.3 *There exists an open and everywhere dense set $\mathcal{F} \subset \mathcal{M}$ such that for each mapping $A \in \mathcal{F}$, there exists a nonempty, closed, ρ-convex and bounded set $K_A \subset K$ such that*

$$A(K_A) \subset K_A.$$

9 Contractivity and genericity results for a class of nonlinear mappings

In this section we present certain contractivity and genericity results regarding a class of generalized nonexpansive self-mappings of a bounded, closed and convex subset of a Banach space. These results were obtained in [17]. They show that in the sense of Baire category, most mappings in this class are contractive and that every contractive mapping has a unique fixed point which uniformly attracts all the iterates of the mapping.

Let $(X, \|\cdot\|)$ be a Banach space and let K be a bounded, closed and convex subset of X. Let $f : X \to [0, \infty)$ be a continuous function such that $f(0) = 0$, the set $f(K - K)$ is bounded, and the following three properties hold:

(P1) for each $\epsilon > 0$, there exists $\delta > 0$ such that if $x, y \in K$ satisfy $f(x - y) \leq \delta$, then $\|x - y\| \leq \epsilon$;

(P2) for each $\lambda \in (0, 1)$, there is $\phi(\lambda) \in (0, 1)$ such that

$$f\bigl(\lambda(x - y)\bigr) \leq \phi(\lambda) f(x - y) \quad \text{for all } x, y \in K;$$

(P3) the function $(x, y) \mapsto f(x - y)$, $x, y \in K$, is uniformly continuous on $K \times K$.

In this section we denote by $\mathcal{A}$ the set of all continuous mappings $A : K \to K$ such that

$$f(Ax - Ay) \leq f(x - y) \quad \text{for all } x, y \in K.$$

Set

$$\operatorname{diam}(K) := \sup\bigl\{\|x - y\| : x, y \in K\bigr\}$$

and for each mapping $A : K \to K$, let A^0 denote the identity operator.

For each $A, B \in \mathcal{A}$, set

$$d(A, B) := \sup\bigl\{\|Ax - Bx\| : x \in K\bigr\}.$$

It is clear that $(\mathcal{A}, d)$ is a complete metric space.

This class of mappings was introduced in [18], where the authors studied the existence of fixed points of these mappings by using the Baire category approach. They established the following result.

Theorem 9.1 *There exists a set $\mathcal{F}$ which is a countable intersection of open and everywhere dense subsets of $\mathcal{A}$ such that*:

(1) *Each mapping $C \in \mathcal{F}$ has a unique fixed point $x_C \in K$, that is, a unique point satisfying $Cx_C = x_C$.*

(2) *For each mapping $C \in \mathcal{F}$ and each $\epsilon > 0$, there exist a neighborhood $\mathcal{U}$ of C in $\mathcal{A}$ and a natural number n_ϵ such that for each mapping $B \in \mathcal{U}$ and each integer $n \geq n_\epsilon$,*

$$\bigl\|B^n x - x_C\bigr\| \leq \epsilon$$

for all $x \in K$.

Note that the classical result of De Blasi and Myjak [19] is a particular case of our result where $f = \|\cdot\|$. As a matter of fact, the mappings studied here can be considered generalized nonexpansive mappings with respect to f. Such an approach, where in some problems of functional analysis the norm is replaced by a general function, was used in [1] in the study of generalized best approximation problems, which we now recall.

Given a closed subset S of a Banach space X and a point $x \in X$, we considered in [1], Chapter 7, the minimization problem

$$\min\bigl\{f(x - y) : y \in S\bigr\}. \tag{P}$$

This problem was studied by many mathematicians mostly in the case where $f(x) = \|x\|$. In this special case it is well known that if S is convex and X is reflexive, then problem (P) always has at least one solution. This solution is unique when X is strictly convex. In [1] we established the generic solvability and well-posedness of problem (P) for a general function f.

Now set

$$D_f := \sup\bigl\{f(x - y) : x, y \in K\bigr\}.$$

A mapping $A \in \mathcal{A}$ is called (f)-*contractive* if there exists a decreasing function $\psi : [0,\infty) \to [0,1]$ such that

$$\psi(t) < 1 \quad \text{for all } t > 0$$

and

$$f(Ax - Ay) \leq \psi\bigl(f(x-y)\bigr)f(x-y) \quad \text{for all } x, y \in K.$$

In the case where $f(x) = \|x\|$, our definition coincides with the classical definition of a contractive mapping used in the literature [9]. In this case it is known that a contractive mapping has a unique fixed point which attracts uniformly all the iterates of the mapping. In [17] we extended this result to the general case. We also showed there that a generic (typical) mapping belonging to the space $\mathcal{A}$ is (f)-contractive.

More precisely, in [17] we established the following results.

Theorem 9.2 *Let a mapping $A \in \mathcal{A}$ be (f)-contractive. Then there exists a unique point $\bar{x} \in K$ satisfying $A\bar{x} = \bar{x}$.*

Theorem 9.3 *Let a mapping $A \in \mathcal{A}$ be (f)-contractive, assume that a point $\bar{x} \in K$ satisfies*

$$A\bar{x} = \bar{x}$$

and let $\epsilon > 0$. Then there exist $\delta > 0$ and a natural number $n_0 > 2$ such that for each integer $n \geq n_0$ and each sequence $\{x_i\}_{i=1}^n \subset K$ which satisfies

$$\|x_{i+1} - Ax_i\| \leq \delta$$

for all $i \in \{1,\dots,n-1\}$, the following inequality holds:

$$\|x_i - \bar{x}\| \leq \epsilon, \quad i = n_0,\dots,n.$$

Corollary 9.4 *Let a mapping $A \in \mathcal{A}$ be (f)-contractive and assume that $\bar{x} \in K$ satisfies*

$$A\bar{x} = \bar{x}.$$

Then $A^i x \to \bar{x}$ as $i \to \infty$ for all $x \in K$, uniformly on K.

Theorem 9.5 *Let a mapping $A \in \mathcal{A}$ be (f)-contractive, assume that $\bar{x} \in K$ satisfies*

$$A\bar{x} = \bar{x}$$

and let $\epsilon > 0$. Then there exist a number $\delta > 0$ and a natural number $n_0 > 2$ such that for each integer $n \geq n_0$ and each sequence $\{x_i\}_{i=1}^n \subset K$ which satisfies

$$\|x_{i+1} - Ax_i\| \leq \delta$$

for all $i \in \{1, \dots, n-1\}$, *the following inequality holds*:

$$f(x_i - \bar{x}) \le \epsilon, \quad i = n_0, \dots, n.$$

Theorem 9.6 *There exists a set* $\mathcal{F}$ *which contains a countable intersection of open and everywhere dense subsets of* $\mathcal{A}$ *such that each element in* $\mathcal{F}$ *is an* (f)*-contractive mapping.*

10 Generic results in holomorphic fixed point theory

In this section we begin our discussion of holomorphic fixed point theory. First we recall the three generic theorems which were established in [20]. These results show that most of the bounded holomorphic self-mappings of a star-shaped domain in a complex Banach space map it strictly inside itself (see the definition below). According to the Earle-Hamilton fixed point theorem [21], each such mapping has a unique fixed point.

Let $(X, \|\cdot\|)$ be a normed linear space over the complex field $\mathbb{C}$. For each point $x \in X$ and each number $r > 0$, set

$$B(x, r) := \{y \in X : \|x - y\| \le r\}.$$

Let Ω be a domain in X (that is, a nonempty, open and connected subset of X). Suppose that for some point $x_* \in \Omega$ and some number $r_* > 0$, we have

$$B(x_*, r_*) \subset \Omega,$$

and that the following assumption holds:

for each point $x \in B(x_*, r_*)$, each number $t \in [0, 1]$ and each point $y \in \Omega$,

$$tx + (1 - t)y \in \Omega.$$

In other words, Ω is a star-shaped (in a certain strong sense) set. In this section we denote by $\mathcal{M}$ the set of all holomorphic mappings $f : \Omega \to \Omega$ for which

$$\sup\{\|f(z)\| : z \in \Omega\} < \infty.$$

For each $f, g \in \mathcal{M}$, set

$$\rho_{\mathcal{M}}(f, g) := \sup\{\|f(z) - g(z)\| : z \in \Omega\}.$$

It is clear that $(\mathcal{M}, \rho_{\mathcal{M}})$ is a metric space.

Denote by $\mathcal{F}$ the set of all $f \in \mathcal{M}$ for which there is a number $r_f > 0$ such that

$$B(f(x), r_f) \subset \Omega \quad \text{for all } x \in \Omega.$$

We say that those mappings which satisfy the above relation map the domain Ω *strictly inside* itself. This concept is of interest in view of the Earle-Hamilton fixed point theorem [21].

Theorem 10.1 $\mathcal{F}$ *is an open and everywhere dense subset of* $\mathcal{M}$.

Theorem 10.2 *Let*

$$M > \|x_*\| + r_* + 1$$

and let $\mathcal{M}_M$ *be the set of all* $f \in \mathcal{M}$ *for which*

$$\|f(x)\| \leq M \quad \text{for all } x \in \Omega.$$

Put

$$\mathcal{F}_M = \mathcal{F} \cap \mathcal{M}_M.$$

Then $\mathcal{M}_M \setminus \mathcal{F}_M$ *is a porous set in* $(\mathcal{M}_M, \rho)$.

Theorem 10.2 immediately yields the third and last result of this section.

Theorem 10.3 *Assume that* Ω *is bounded. Then* $\mathcal{M} \setminus \mathcal{F}$ *is porous in* $(\mathcal{M}, \rho)$.

11 Sequences of holomorphic mappings

In this section we define and study a certain metric space the elements of which are sequences of holomorphic mappings. This metric space was introduced in [22]. We keep the notations, definitions and assumptions of the previous section.

Denote by $\mathfrak{M}$ the set of all sequences of holomorphic mappings $f_i : \Omega \to \Omega$, $i = 1, 2, \ldots$, such that

$$\sup\{\|f_i(z)\| : z \in \Omega, i = 1, 2, \ldots\} < \infty.$$

For each $\{f_i\}_{i=1}^{\infty}, \{g_i\}_{i=1}^{\infty} \in \mathfrak{M}$, set

$$\rho_{\mathfrak{M}}(\{f_i\}_{i=1}^{\infty}, \{g_i\}_{i=1}^{\infty}) := \sup\{\|f_i(z) - g_i(z)\| : z \in \Omega, i = 1, 2, \ldots\}.$$

Clearly, $(\mathfrak{M}, \rho_{\mathfrak{M}})$ is a metric space.

Denote by $\mathcal{G}$ the set of all $\{f_i\}_{i=1}^{\infty} \in \mathfrak{M}$ for which there is a number $r(\{f_i\}_{i=1}^{\infty}) > 0$ such that

$$B(f_i(x), r(\{f_i\}_{i=1}^{\infty})) \subset \Omega \quad \text{for all } x \in \Omega \text{ and all } i = 1, 2, \ldots.$$

Theorem 11.1 $\mathcal{G}$ *is an open and everywhere dense subset of* $\mathfrak{M}$.

Theorem 11.2 *Let*

$$M > \|x_*\| + r_* + 1$$

and let $\mathfrak{M}_M$ *be the set of all* $\{f_i\}_{i=1}^{\infty} \in \mathfrak{M}$ *such that*

$$\|f_i(x)\| \leq M \quad \text{for all } x \in \Omega \text{ and all integers } i \geq 1.$$

Put

$$\mathcal{G}_M := \mathcal{G} \cap \mathfrak{M}_M.$$

Then $\mathfrak{M}_M \setminus \mathcal{G}_M$ *is a porous set in* $(\mathfrak{M}_M, \rho_{\mathfrak{M}})$.

It is clear that Theorem 11.2 immediately implies the following result.

Theorem 11.3 *Assume that* Ω *is bounded. Then* $\mathfrak{M} \setminus \mathcal{G}$ *is a porous set in* $(\mathfrak{M}, \rho_{\mathfrak{M}})$.

12 The Kobayashi metric

In this section we assume that X is a complex Banach space and that Ω is a bounded domain in X. Let k_Ω denote the Kobayashi metric on Ω [23]. Set

$$\operatorname{diam}(\Omega) := \sup\bigl\{\|x-y\| : x, y \in \Omega\bigr\},$$

denote by $\partial\Omega$ the boundary of Ω and for all $x \in X$, put

$$\operatorname{dist}(x, \partial\Omega) := \inf\bigl\{\|x-y\| : y \in \partial\Omega\bigr\}.$$

The following two results are known (*cf.* [4]). The second one is the above-mentioned Earle-Hamilton fixed point theorem.

Theorem 12.1 *For all* $x, y \in \Omega$,

$$\operatorname{argtanh}\bigl(\|x-y\|/\operatorname{diam}(\Omega)\bigr) \le k_\Omega(x,y)$$

and if $\|x-y\| < \operatorname{dist}(x, \partial\Omega)$, *then*

$$k_\Omega(x,y) \le \operatorname{argtanh}\bigl(\|x-y\|/\operatorname{dist}(x, \partial\Omega)\bigr).$$

Theorem 12.2 *Assume that* $f : \Omega \to \Omega$ *is a holomorphic mapping such that* $\operatorname{dist}(f(x), \partial\Omega) \ge \epsilon$ *for some* $\epsilon > 0$ *and all* $x \in \Omega$. *Let* $0 < \tau < \epsilon/\operatorname{diam}(\Omega)$. *Then*

$$k_\Omega\bigl(f(x), f(y)\bigr) \le k_\Omega(x,y)/(1+\tau)$$

for all $x, y \in \Omega$ *and* f *has a unique fixed point.*

13 A weak ergodic theorem

In this section we present a weak ergodic theorem for infinite products of holomorphic mappings [22]. Here we use the notion of weak ergodicity in the sense of population biology (see, for example, [24], p.565, and the references therein). Another result in this direction is [25], Theorem 3.1, p.331.

We keep the notations, definitions and assumptions used in the two previous sections. In particular, Ω is a bounded domain in the complex Banach space X and the set $\mathcal{G}$ is the one defined in Section 11.

Theorem 13.1 *Let $\{f_i\}_{i=1}^{\infty} \in \mathcal{G}$. Then there exist a neighborhood $\mathcal{U}$ of $\{f_i\}_{i=1}^{\infty}$ in $\mathfrak{M}$, $c \in (0,1)$ and $M > 0$ such that the following properties hold*:

(i) *for each $\{g_i\}_{i=1}^{\infty} \in \mathcal{U}$, each integer $i \geq 1$ and each $x, y \in \Omega$,*

$$k_{\Omega}\big(g_i(x), g_i(y)\big) \leq c k_{\Omega}(x, y);$$

(ii) *for each $\{g_i\}_{i=1}^{\infty} \in \mathcal{U}$, each $x, y \in \Omega$, each integer $n \geq 1$ and each mapping $r : \{1, 2, \ldots\} \to \{1, 2, \ldots\}$, we have*

$$k_{\Omega}\big(g_{r(n)} \cdots g_{r(1)}(x), g_{r(n)} \cdots g_{r(1)}(y)\big) \leq c^n k_{\Omega}(x, y);$$

(iii) *for each $\{g_i\}_{i=1}^{\infty} \in \mathcal{U}$, each $x, y \in \Omega$, each integer $n \geq 1$ and each mapping $r : \{1, 2, \ldots\} \to \{1, 2, \ldots\}$,*

$$k_{\Omega}\big(g_{r(n)} \cdots g_{r(1)}(x), g_{r(n)} \cdots g_{r(1)}(y)\big) \leq c^{n-1} M;$$

(iv) *for each $\epsilon > 0$, there exists a natural number n_{ϵ} such that for each $\{g_i\}_{i=1}^{\infty} \in \mathcal{U}$, each $x, y \in \Omega$, each integer $n \geq n_{\epsilon}$ and each mapping $r : \{1, 2, \ldots\} \to \{1, 2, \ldots\}$,*

$$\big\| \big(g_{r(n)} \cdots g_{r(1)}(x) - g_{r(n)} \cdots g_{r(1)}(y)\big) \big\| \leq \epsilon.$$

14 Infinite products of nonexpansive mappings with unbounded domains

In this section we return to the study of infinite products of nonexpansive mappings. More precisely, we study here the generic convergence of infinite products of nonexpansive mappings with unbounded domains in hyperbolic spaces [26].

Let $(X, \rho, \mathcal{L})$ be a complete hyperbolic space, and let $K \subset X$ be a nonempty, closed and ρ-convex subset of (X, ρ). For each $C : K \to K$, we set $C^0(x) := x$ for all $x \in K$. In this section we denote by $\mathcal{M}$ the set of all sequences $\{A_t\}_{t=1}^{\infty}$ of mappings $A_t : K \to K$, $t = 1, 2, \ldots$, such that for all integers $t \geq 1$,

$$\rho\big(A_t(x), A_t(y)\big) \leq \rho(x, y) \quad \text{for all } x, y \in K.$$

For each $x \in X$ and each $r > 0$, set

$$B(x, r) := \big\{y \in X : \rho(x, y) \leq r\big\} \quad \text{and} \quad B_K(x, r) := B(x, r) \cap K.$$

Fix a point $\theta \in K$. For each $M, \epsilon > 0$, set

$$\begin{aligned} \mathcal{U}(M, \epsilon) := \big\{ & \big(\{A_t\}_{t=1}^{\infty}, \{B_t\}_{t=1}^{\infty}\big) \in \mathcal{M} \times \mathcal{M} : \\ & \rho\big(A_t(x), B_t(x)\big) \leq \epsilon \text{ for all } x \in B_K(\theta, M) \text{ and all integers } t \geq 1\big\}. \end{aligned}$$

We equip the set $\mathcal{M}$ with the uniformity which has the base

$$\big\{\mathcal{U}(M, \epsilon) : M, \epsilon > 0\big\}.$$

It is not difficult to see that the uniform space $\mathcal{M}$ is metrizable (by a metric d) and complete.

Denote by $\mathcal{M}_*$ the set of all $\{A_t\}_{t=1}^{\infty} \in \mathcal{M}$ for which there exists a point $\tilde{x} \in K$ satisfying

$$A_t(\tilde{x}) = \tilde{x} \quad \text{for all integers } t \geq 1.$$

Denote by $\bar{\mathcal{M}}_*$ the closure of the set $\mathcal{M}_*$ in the uniform space $\mathcal{M}$. We consider the topological subspace $\bar{\mathcal{M}}_* \subset \mathcal{M}$ equipped with the relative topology and the metric d.

In [26] we studied the asymptotic behavior of (unrestricted) infinite products of generic sequences of mappings belonging to the space $\bar{\mathcal{M}}_*$ and obtained convergence to a unique common fixed point. More precisely, we established there the following result.

Theorem 14.1 *There exists a set $\mathcal{F} \subset \bar{\mathcal{M}}_*$, which is a countable intersection of open and everywhere dense subsets of the complete metric space $(\bar{\mathcal{M}}_*, d)$ such that for each $\{B_t\}_{t=1}^{\infty} \in \mathcal{F}$, the following properties hold*:

(a) *there exists a unique point $\bar{x} \in K$ such that $B_t(\bar{x}) = \bar{x}$ for all integers $t \geq 1$;*

(b) *if $t \geq 1$ is an integer and $y \in K$ satisfies $B_t(y) = y$, then $y = \bar{x}$;*

(c) *for each $\epsilon > 0$ and each $M > 0$, there exist a number $\delta > 0$ and a neighborhood $\mathcal{U}$ of $\{B_t\}_{t=1}^{\infty}$ in the metric space $\bar{\mathcal{M}}_*$ such that if $\{C_t\}_{t=1}^{\infty} \in \mathcal{U}$, $t \in \{1, 2, \ldots\}$, and if a point $y \in B_K(\theta, M)$ satisfies $\rho(y, C_t(y)) \leq \delta$, then $\rho(y, \bar{x}) \leq \epsilon$;*

(d) *for each $\epsilon > 0$ and each $M > 0$, there exist a neighborhood $\mathcal{U}$ of $\{B_t\}_{t=1}^{\infty}$ in the metric space $\bar{\mathcal{M}}_*$, a number $\delta > 0$ and a natural number q such that if $\{C_t\}_{t=1}^{\infty} \in \mathcal{U}$, $m \geq q$ is an integer, $r : \{1, \ldots, m\} \to \{1, 2, \ldots\}$, and if $\{x_i\}_{i=0}^{m} \subset K$ satisfies*

$$\rho(x_0, \theta) \leq M$$

and

$$\rho\big(C_{r(i)}(x_{i-1}), x_i\big) \leq \delta, \quad i = 1, \ldots, m,$$

then

$$\rho(x_i, \bar{x}) \leq \epsilon, \quad i = q, \ldots, m.$$

15 Porosity theorems for nonexpansive mappings in hyperbolic spaces

In this section we continue to study nonexpansive mappings with unbounded domains and the issue of contractivity.

Let $(X, \rho, \mathcal{L})$ be a complete hyperbolic space, and let K be a nonempty, closed and ρ-convex subset of X. Fix a point $\theta \in K$.

As in Section 7, in this section we denote by $\mathcal{A}$ the set of all mappings $A : K \to K$ for which

$$\rho\big(A(x), A(y)\big) \leq \rho(x, y) \quad \text{for all } x, y \in K.$$

By the above relation, for each pair $A, B \in \mathcal{A}$ and each $x \in K$, we have

$$\begin{aligned}\rho\big(A(x), B(x)\big) &\leq \rho\big(A(x), A(\theta)\big) + \rho\big(A(\theta), B(\theta)\big) + \rho\big(B(\theta), B(x)\big)\\ &\leq \rho(x, \theta) + \rho\big(A(\theta), B(\theta)\big) + \rho(x, \theta)\end{aligned}$$

and

$$\rho\big(A(x),B(x)\big) \le 2\rho(x,\theta) + \rho\big(A(\theta),B(\theta)\big).$$

For each $A, B \in \mathcal{A}$, set

$$d(A,B) := \inf\big\{\lambda > 0 : \rho\big(A(x),B(x)\big) \le \lambda\big(\rho(x,\theta)+1\big) \text{ for all } x \in K\big\}.$$

It is not difficult to see that for each $A, B \in \mathcal{A}$, $d(A,B)$ is well defined,

$$d(A,B) = \sup\big\{\rho\big(A(x),B(x)\big)\big(\rho(x,\theta)+1\big)^{-1} : x \in K\big\},$$

$d : \mathcal{A} \times \mathcal{A} \to [0,\infty)$ is a metric on $\mathcal{A}$ and the metric space $(\mathcal{A}, d)$ is complete. Clearly, the topology induced by the metric d is stronger than the topology of uniform convergence on bounded sets, but weaker than the topology of uniform convergence on all of K. It does not depend on the choice of θ. Once again, for each point $x \in K$ and each number $r > 0$, we set

$$B(x,r) := \big\{y \in K : \rho(x,y) \le r\big\}.$$

The following two results were established in [27].

Theorem 15.1 *There exists a set $\mathcal{F}_* \subset \mathcal{A}$ such that its complement $\mathcal{A} \setminus \mathcal{F}_*$ is σ-porous in $(\mathcal{A}, d)$ and for each $A \in \mathcal{F}_*$, there exists a number $M_A > 0$ such that for each $M \ge M_A$, we have the inclusion*

$$A\big(B(\theta,M)\big) \subset B(\theta,M),$$

and there exists a decreasing function $\phi : [0,\infty) \to [0,1]$ such that

$$\phi(t) < 1 \quad \text{for all } t > 0$$

and

$$\rho\big(A(x),A(y)\big) \le \phi\big(\rho(x,y)\big)\rho(x,y) \quad \text{for all } x,y \in B(\theta,M).$$

In particular, each $A \in \mathcal{F}_*$ has a unique fixed point in K. This fixed point is the limit of each sequence of iterates of A.

Theorem 15.2 *There exists a set $\mathcal{G} \subset \mathcal{A}$ such that its complement $\mathcal{A} \setminus \mathcal{G}$ is σ-porous in $(\mathcal{A}, d)$ and each mapping $A \in \mathcal{G}$ has the following two properties*:

(1) *There exists a unique fixed point $x_A \in K$ of A, that is, a unique point $x_A \in K$ such that $A(x_A) = x_A$.*
(2) *There exists a decreasing function $\phi : [0,\infty) \to [0,1]$ such that*

$$\phi(t) < 1 \quad \text{for all } t > 0$$

and

$$\rho\big(A(x),x_A\big) \le \phi\big(\rho(x,x_A)\big)\rho(x,x_A) \quad \textit{for all } x \in K.$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Acknowledgements
The first author was partially supported by the Israel Science Foundation (Grant No. 389/12), by the Fund for the Promotion of Research at the Technion and by the Technion General Research Fund.

Received: 14 October 2015 Accepted: 19 October 2015 Published online: 29 October 2015

References

1. Reich, S, Zaslavski, AJ: Genericity in Nonlinear Analysis. Developments in Mathematics, vol. 34. Springer, New York (2014)
2. Zaslavski, AJ: Optimization on Metric and Normed Spaces. Springer, New York (2010)
3. Zaslavski, AJ: Nonconvex Optimal Control and Variational Problems. Springer, New York (2013)
4. Goebel, K, Reich, S: Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings. Dekker, New York (1984)
5. Reich, S: The alternating algorithm of von Neumann in the Hilbert ball. Dyn. Syst. Appl. **2**, 21-25 (1993)
6. Reich, S, Shafrir, I: Nonexpansive iterations in hyperbolic spaces. Nonlinear Anal. **15**, 537-558 (1990)
7. Reich, S, Zaslavski, AJ: A stable convergence theorem for infinite products of nonexpansive mappings in Banach spaces. J. Fixed Point Theory Appl. **8**, 395-403 (2010)
8. Reich, S, Zaslavski, AJ: Contractivity, porosity and infinite products. Contemp. Math. **636**, 203-209 (2015)
9. Rakotch, E: A note on contractive mappings. Proc. Am. Math. Soc. **13**, 459-465 (1962)
10. Reich, S, Zaslavski, AJ: Approximate fixed points of nonexpansive mappings in unbounded sets. J. Fixed Point Theory Appl. **13**, 627-632 (2013)
11. Reich, S, Zaslavski, AJ: Generic contractivity for a class of nonlinear mappings. Libertas Math. **33**, 15-20 (2013)
12. Bauschke, HH, Borwein, JM: On projection algorithms for solving convex feasibility problems. SIAM Rev. **38**, 367-426 (1996)
13. Reich, S, Zaslavski, AJ: Porosity and the bounded linear regularity property. J. Appl. Anal. **20**, 1-6 (2014)
14. Reich, S, Zaslavski, AJ: Generic contractivity of nonexpansive mappings with unbounded domains. J. Nonlinear Convex Anal. **16**, 1-7 (2015)
15. Strobin, F: Some porous and meager sets of continuous mappings. J. Nonlinear Convex Anal. **13**, 351-361 (2012)
16. Reich, S, Zaslavski, AJ: Approximate fixed points of nonexpansive set-valued mappings in unbounded sets. J. Nonlinear Convex Anal. **16**, 1707-1716 (2015)
17. Reich, S, Zaslavski, AJ: Contractivity and genericity results for a class of nonlinear mappings. J. Nonlinear Convex Anal. **16**, 1113-1122 (2015)
18. Gabour, M, Reich, S, Zaslavski, AJ: A generic fixed point theorem. Indian J. Math. **56**, 25-32 (2014)
19. De Blasi, FS, Myjak, J: Sur la convergence des approximations successives pour les contractions non linéaires dans un espace de Banach. C. R. Acad. Sci. Paris Sér. A-B **283**, 185-187 (1976)
20. Reich, S, Zaslavski, AJ: Three generic results in holomorphic fixed point theory. Complex Anal. Oper. Theory **8**, 51-56 (2014)
21. Earle, CJ, Hamilton, RS: A fixed point theorem for holomorphic mappings. Proc. Symp. Pure Math. **16**, 61-65 (1970)
22. Reich, S, Zaslavski, AJ: A weak ergodic theorem for infinite products of holomorphic mappings. Contemp. Math. (accepted for publication)
23. Kobayashi, S: Invariant distances on complex manifolds and holomorphic mappings. J. Math. Soc. Jpn. **19**, 460-480 (1967)
24. Reich, S, Zaslavski, AJ: Generic aspects of metric fixed point theory. In: Handbook of Metric Fixed Point Theory, pp. 557-575. Kluwer Academic, Dordrecht (2001)
25. Budzyńska, M, Reich, S: Infinite products of holomorphic mappings. Abstr. Appl. Anal. **2005**, 327-341 (2005)
26. Reich, S, Zaslavski, AJ: Generic convergence of infinite products of nonexpansive mappings with unbounded domains. Front. Appl. Math. Stat. **1**, 1-5 (2015)
27. Reich, S, Zaslavski, AJ: Two porosity theorems for nonexpansive mappings in hyperbolic spaces. J. Math. Anal. Appl. **433**, 1220-1229 (2016)