Shen et al. *Boundary Value Problems* (2015) 2015:241
DOI 10.1186/s13661-015-0509-z

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

On the existence of solution to a boundary value problem of fractional differential equation on the infinite interval

Chunfang Shen[1], Hui Zhou[1] and Liu Yang[1,2*]

*Correspondence: xjiangfeng@163.com
[1]College of Mathematics and Statistics, Hefei Normal University, Hefei, 230061, P.R. China
[2]College of Mathematical Science, University of Science and Technology of China, Hefei, 230000, P.R. China

Abstract

This work deals with a boundary value problem for a nonlinear multi-point fractional differential equation on the infinite interval. By constructing the proper function spaces and the norm, we overcome the difficulty following from the noncompactness of $[0,\infty)$. By using the Schauder fixed point theorem, we show the existence of one solution with suitable growth conditions imposed on the nonlinear term.

MSC: 34B10; 34B15

Keywords: fractional differential equation; boundary value problem; infinite interval; fixed point theorem

1 Introduction

In this paper, we consider the existence of solution of boundary value problem for a nonlinear multi-point fractional differential equation,

$$D_{0+}^{\alpha}u(t)=f\bigl(t,u(t),D_{0+}^{\alpha-1}u(t)\bigr),\quad t\in J:=[0,+\infty), \tag{1.1}$$

$$u(0)=0,\qquad u'(0)=0,\qquad D_{0+}^{\alpha-1}u(+\infty)=\sum_{i=1}^{m-2}\beta_i u(\xi_i), \tag{1.2}$$

where $2<\alpha\le 3$ is a real number, $f\in C(J\times R\times R,R)$ and $\Gamma(\alpha)-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha-1}\neq 0$.

Due to the intensive development of the theory of fractional calculus itself as well as its applications, such as in the fields of physics, chemistry, aerodynamics, polymer rheology, *etc.*, many papers and books on fractional calculus, fractional differential equations have appeared (see [1–16]).

For example, Bai [11] established the existence results of positive solutions for the problem

$$D_{0+}^{\alpha}u(t)+f\bigl(t,u(t)\bigr)=0,\quad 0\le t\le 1,$$
$$u(0)=0,\qquad u(1)=\beta u(\eta),\quad \eta\in(0,1).$$

In [13], the authors considered the three-point boundary value problem of a coupled system of the nonlinear fractional differential equation

$$
\begin{aligned}
&D_{0+}^{\alpha}u(t) = f\bigl(t, v(t), D^{p}v(t)\bigr), \quad 0 \le t \le 1, \\
&D_{0+}^{\beta}v(t) = f\bigl(t, u(t), D^{q}u(t)\bigr), \quad 0 \le t \le 1, \\
&u(0) = v(0) = 0, \qquad u(1) = \gamma u(\eta), \qquad v(1) = \gamma u(\eta),
\end{aligned}
$$

under the conditions $0 < \gamma\eta^{\alpha-1} < 1$, $0 < \gamma\eta^{\beta-1} < 1$. By using the Schauder fixed point theorem, they obtained at least one solution of this problem.

The theory of boundary value problems on infinite intervals arises naturally and has many applications; see [17]. The existence and multiplicity of solutions to boundary value problems of fractional differential equations on the infinite interval have been investigated in recent years [18–21].

Agarwal *et al.* [22] established existence results of solutions for a class of boundary value problems involving the Riemann-Liouville fractional derivative on the half line by using the nonlinear alternative of Leray-Schauder type combined with the diagonalization process.

Arara *et al.* [23] considered boundary value problems involving the Caputo fractional derivative on the half line,

$$
{}^{c}D^{\alpha}u(t) = f\bigl(t, u(t)\bigr), \quad t \in J := [0, \infty), u(0) = u_0, u \text{ is bounded on } J.
$$

By using fixed point theorem combined with the diagonalization process, they obtained the existence of solutions.

Liang and Zhang [24] consider the m-point boundary value problem of fractional differential equation on the infinite interval

$$
\begin{aligned}
&D_{0+}^{\alpha}u(t) + a(t)f\bigl(t, u(t)\bigr) = 0, \quad 0 < t < +\infty, \\
&u(0) = 0, \qquad u'(0) = 0, \qquad D_{0+}^{\alpha-1}u(+\infty) = \sum_{i=1}^{m-2}\beta_i u(\xi_i),
\end{aligned}
$$

where $2 < \alpha \le 3$, D_{0+}^{α} is the standard Riemann-Liouville derivative. Using a fixed point theorem for operators on a cone, sufficient conditions for the existence of multiple positive solutions were established. We point out that the nonlinear term of the equation does not depend on the lower order derivative of the unknown function.

In this paper, by constructing the proper function spaces and the norm to overcome the difficulty of the noncompactness of $[0, \infty)$ and using the Schauder fixed point theorem, we show the existence of one solution with suitable growth conditions imposed on the nonlinear term. Our method is different from [22, 23] in essence.

2 Preliminaries and lemmas

For convenience of the reader, we present the necessary definitions from fractional calculus theory [1].

Definition 2.1 The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $u(t) : R \to R$ is given by

$$I_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\, ds$$

provided the right side is point-wise defined on $(0, \infty)$.

Definition 2.2 The fractional derivative of order $\alpha > 0$ of a continuous function $u(t) : R \to R$ is given by

$$D_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{u(s)}{(t-s)^{\alpha-n+1}}\, ds$$

where $n = [\alpha] + 1$, provided that the right side is point-wise defined on $(0, \infty)$.

Lemma 2.1 *Assume that $u \in C(0,1) \cup L(0,1)$, and $D_{0+}^{\alpha} \in C(0,1) \cup L(0,1)$. Then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_N t^{\alpha-N},$$

for some $C_i \in R$, $i = 1, 2, \ldots, N$, where N is the smallest integer greater than or equal to α.

Lemma 2.2 *Given $y(t) \in L[0, \infty)$. The problem*

$$D_{0+}^{\alpha} u(t) = y(t), \quad 0 < t < \infty, 2 < \alpha < 3,$$
$$u(0) = u'(0) = 0, \qquad D^{\alpha-1} u(+\infty) = \sum_{i=1}^{m-2} \beta_i u(\xi_i),$$

is equivalent to

$$\begin{aligned} u(t) = {} & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\, ds - \frac{t^{\alpha-1}}{\Gamma(\alpha) - \sum_{i=1}^{m-2} \beta_i \xi_i^{\alpha-1}} \int_0^{\infty} y(s)\, ds \\ & + \frac{\sum_{i=1}^{m-2} \beta_i t^{\alpha-1}}{\Gamma(\alpha) - \sum_{i=1}^{m-2} \beta_i \xi_i^{\alpha-1}} \int_0^{\xi_i} \frac{(\xi_i - s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\, ds. \end{aligned}$$

Proof By Lemma 2.1, we have

$$u(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\, ds + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3}.$$

The boundary condition $u(0) = u'(0) = 0$ implies that $c_2 = c_3 = 0$.

Considering the boundary condition $D^{\alpha-1} u(+\infty) = \sum_{i=1}^{m-2} \beta_i u(\xi_i)$, we have

$$c_1 = \frac{-\int_0^{\infty} y(s)\, ds + \sum_{i=1}^{m-2} \beta_i \int_0^{\xi_i} \frac{(\xi_i - s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\, ds}{\Gamma(\alpha) - \sum_{i=1}^{m-2} \beta_i \xi_i^{\alpha-1}}.$$

The proof is completed. □

Define the function spaces

$$X = \left\{ u(t) \in C(J,R) : \sup_{t\in J} \frac{|u(t)|}{1+t^{\alpha-1}} < +\infty \right\}$$

with the norm

$$\|u\|_X = \sup_{t\in J} \frac{|u(t)|}{1+t^{\alpha-1}}$$

and

$$Y = \left\{ u(t) \in X : u'(t), D^{\alpha-1}u(t) \in C(J,R), \sup_{t\in J} \frac{|u'(t)|}{1+t^{\alpha-2}} < +\infty, \sup_{t\in J}\left|D^{\alpha-1}u(t)\right| < +\infty \right\}$$

with the norm

$$\|u\|_Y = \max\left\{ \sup_{t\in J} \frac{|u(t)|}{1+t^{\alpha-1}}, \sup_{t\in J} \frac{|u'(t)|}{1+t^{\alpha-2}}, \sup_{t\in J}\left|D^{\alpha-1}u(t)\right| \right\}.$$

Lemma 2.3 $(X, \|\cdot\|_X)$ *is a Banach space.*

Proof Let $\{u_n\}_{n=1}^{\infty}$ be a Cauchy sequence in the space $(X, \|\cdot\|_X)$, then $\forall \varepsilon > 0$, $\exists N > 0$ such that

$$\left| \frac{u_n(t)}{1+t^{\alpha-1}} - \frac{u_m(t)}{1+t^{\alpha-1}} \right| < \varepsilon$$

for any $t \in J$ and $n, m > N$. Thus, $\{u_n\}_{n=1}^{\infty}$ converges uniformly to a function $\frac{v(t)}{1+t^{\alpha-1}}$ and we can verify easily that $v(t) \in X$. Then $(X, \|\cdot\|_X)$ is a Banach space. □

Lemma 2.4 $(Y, \|\cdot\|_Y)$ *is a Banach space.*

Proof Let $\{u_n\}_{n=1}^{\infty}$ be a Cauchy sequence in the space $(Y, \|\cdot\|_Y)$, then $\{u_n\}_{n=1}^{\infty}$ is also a Cauchy sequence in $(X, \|\cdot\|_X)$. Thus there exists a function $v(t) \in X$ such that

$$\lim_{n\to+\infty} \frac{u_n(t)}{1+t^{\alpha-1}} = \frac{u(t)}{1+t^{\alpha-1}}.$$

Moreover,

$$\lim_{n\to+\infty} \frac{u_n'(t)}{1+t^{\alpha-2}} = \frac{v(t)}{1+t^{\alpha-2}}, \qquad \lim_{n\to+\infty} D^{\alpha-1}u_n = w(t),$$

and

$$\sup_{t\in J} \frac{|v(t)|}{1+t^{\alpha-2}} < +\infty, \qquad \sup_{t\in J}\left|D^{\alpha-1}u(t)\right| < +\infty.$$

It is easy to check that $v = u'(t)$. Next we need to ensure that $w = D^{\alpha-1}u(t)$.

In view of the Lebesgue dominated convergence theorem and the uniform convergence of $\{D^{\alpha-1}u_n(t)\}_{n=1}^{\infty}$, there exists a positive constant $M > 0$ such that $\frac{|u_n(t)|}{1+t^{\alpha-1}} \leq M$, $n = 1, 2, \ldots$. Then

$$w(t) = \lim_{n\to+\infty} D^{\alpha-1}u_n(t) = \frac{1}{\Gamma(2-\alpha)} \lim_{n\to+\infty} \frac{d}{dt}\int_0^t (t-s)^{1-\alpha}(1+s)^{\alpha+1}\frac{u_n(s)}{1+s^{\alpha}}\,ds$$

together with

$$\begin{aligned}
&\int_0^t (t-s)^{1-\alpha}(1+s)^{\alpha+1}\frac{u_n(s)}{1+s^{\alpha}} \\
&\quad\leq M\int_0^t (t-s)^{1-\alpha}\left(1+s^{\alpha-1}\right)ds \\
&\quad= M\left[t^{2-\alpha}\int_0^1 (1-\tau)^{1-\alpha}\,d\tau + t\int_0^1 \tau^{\alpha-1}(1-\tau)^{1-\alpha}\,d\tau\right] = \frac{M}{2-\alpha}t^{2-\alpha} + B(\alpha,\ 2-\alpha)Mt
\end{aligned}$$

ensures that $w = D^{\alpha-1}u(t)$.

Thus $(Y\|\cdot\|_Y)$ is a Banach space. □

Because the Arzela-Ascoli theorem fails to work in Y, we need a modified compactness criterion to prove the compactness of the operator.

Lemma 2.5 *Let $Z \subseteq Y$ be a bounded set and the following conditions hold*:

(i) *for any $u(t) \in Z$, $\frac{u(t)}{1+t^{\alpha-1}}$, $\frac{u'(t)}{1+t^{\alpha-2}}$ and $D^{\alpha-1}u(t)$ are equicontinuous on any compact interval of J*;

(ii) *given $\varepsilon > 0$, there exists a constant $T = T(\varepsilon) > 0$ such that*

$$\left|\frac{u(t_1)}{1+t_1^{\alpha-1}} - \frac{u(t_2)}{1+t_2^{\alpha-1}}\right| < \varepsilon, \qquad \left|\frac{u'(t_1)}{1+t_1^{\alpha-2}} - \frac{u'(t_2)}{1+t_2^{\alpha-2}}\right| < \varepsilon, \quad \textit{and}$$
$$\left|D^{\alpha-1}u(t_1) - D^{\alpha-1}u(t_2)\right| < \varepsilon$$

for any $t_1, t_2 > T$ and $u(t) \in Z$. Then Z is relatively compact in Y.

Proof We need to prove that Z is totally bounded. First we consider the case $t \in [0, T]$. Define

$$Z_{[0,T]} = \left\{u(t) : u(t) \in Z, t \in [0, T]\right\}.$$

It is easy to check that $Z_{[0,T]}$ with the norm $\|u\|_{\infty} = \sup_{t\in[0,T]} |\frac{u(t)}{1+t^{\alpha-1}}|$ is a Banach space. Then condition (i) combined with the Arzela-Ascoli theorem indicates that $Z_{[0,T]}$ is relatively compact. Thus for any positive number ε, there exist finitely many balls $B_{\varepsilon}(u_i)$ such that

$$Z_{[0,T]} \subset \bigcup_{i=1}^{n} B_{\varepsilon}(u_i),$$

where

$$B_{\varepsilon}(u_i) = \left\{u(t) \in Z_{[0,T]} : \|u - u_i\|_{\infty} = \sup_{t\in[0,T]}\left|\frac{u(t)}{1+t^{\alpha-1}} - \frac{u_i(t)}{1+t^{\alpha-1}}\right| < \varepsilon\right\}.$$

Similarly, the space

$$Z^1_{[0,T]} = \{u'(t) : u(t) \in Z, t \in [0,T]\}$$

with the norm $\|u'\| = |\frac{u'(t)}{1+t^{\alpha-2}}|$ and

$$Z^{\alpha-1}_{[0,T]} = \{D^{\alpha-1}u(t) : u(t) \in Z, t \in [0,T]\}$$

with the norm

$$\|D^{\alpha-1}u\| = \sup_{t\in[0,T]} |D^{\alpha-1}u(t)|$$

are Banach spaces. Then

$$Z^1_{[0,T]} \subset \bigcup_{j=1}^{m} B_\varepsilon(v'_j),$$
$$Z^{\alpha-1}_{[0,T]} \subset \bigcup_{p=1}^{k} B_\varepsilon(D^{\alpha-1}w_p),$$

where

$$B_\varepsilon(v'_j) = \{u'(t) \in Z^1_{[0,T]} : \|u' - v'_j\| < \varepsilon\},$$
$$B_\varepsilon(D^{\alpha-1}w_p) = \{D^{\alpha-1}u - w \in Z^{\alpha 1}_{[0,T]} : \|D^{\alpha-1}u - D^{\alpha-1}w_j\| < \varepsilon\}.$$

Next we define

$$Z_{ijp} = \{u(t) \in Z, u_{[0,T]} \in B_\varepsilon(u_i), u'_{[0,T]} \in B_\varepsilon(v'_j), D^{\alpha-1}u_{[0,T]} \in B_\varepsilon(D^{\alpha-1}w_p)\}.$$

Now we take $u_{ijp} \in Z_{ijp}$. Then Z can be covered by the balls $B_{5\varepsilon}(u_{ijp})$, $i = 1,2,\ldots,n$, $j = 1,2,\ldots,m$, $p = 1,2,\ldots,k$, where

$$B_{5\varepsilon}(u_{ijp}) = \{u(t) \in Z : \|u - u_{ijp}\|_Y < 5\varepsilon\}.$$

In fact, for $t \in [0,T]$,

$$\begin{aligned}
&\left|\frac{u(t)}{1+t^{\alpha-1}} - \frac{u_{ijp}(t)}{1+t^{\alpha-1}}\right| \\
&\quad\le \left|\frac{u(t)}{1+t^{\alpha-1}} - \frac{u_i(t)}{1+t^{\alpha-1}}\right| + \left|\frac{u_i(t)}{1+t^{\alpha-1}} - \frac{u_{ij}(t)}{1+t^{\alpha-1}}\right| + \left|\frac{u_{ij}(t)}{1+t^{\alpha-1}} - \frac{u_{ijp}(t)}{1+t^{\alpha-1}}\right| \\
&\quad< \varepsilon + \varepsilon + \varepsilon = 3\varepsilon, \\
&\left|\frac{u'(t)}{1+t^{\alpha-2}} - \frac{u'_{ijp}(t)}{1+t^{\alpha-2}}\right| \\
&\quad\le \left|\frac{u'(t)}{1+t^{\alpha-2}} - \frac{u'_i(t)}{1+t^{\alpha-2}}\right| + \left|\frac{u'_i(t)}{1+t^{\alpha-2}} - \frac{u'_{ij}(t)}{1+t^{\alpha-2}}\right| + \left|\frac{u'_{ij}(t)}{1+t^{\alpha-2}} - \frac{u'_{ijp}(t)}{1+t^{\alpha-2}}\right| \\
&\quad< \varepsilon + \varepsilon + \varepsilon = 3\varepsilon,
\end{aligned}$$

and

$$\begin{aligned}&\left|D^{\alpha-1}u(t)-D^{\alpha-1}u_{ijp}(t)\right|\\&\quad\le\left|D^{\alpha-1}u(t)-D^{\alpha-1}u_{ijp}(t)\right|+\left|D^{\alpha-1}u_i(t)-D^{\alpha-1}u_{ij}(t)\right|+\left|D^{\alpha-1}u_{ij}(t)-D^{\alpha-1}u_{ijp}(t)\right|\\&\quad<\varepsilon+\varepsilon+\varepsilon=3\varepsilon.\end{aligned}$$

For $t\in[T,+\infty]$, we have

$$\begin{aligned}&\left|\frac{u(t)}{1+t^{\alpha-1}}-\frac{u_{ijp}(t)}{1+t^{\alpha-1}}\right|\\&\quad\le\left|\frac{u(t)}{1+t^{\alpha-1}}-\frac{u(T)}{1+t^{\alpha-1}}\right|+\left|\frac{u(T))}{1+t^{\alpha-1}}-\frac{u_{ijp}(T)}{1+t^{\alpha-1}}\right|+\left|\frac{u_{ijp}(T)}{1+t^{\alpha-1}}-\frac{u_{ijp}(t)}{1+t^{\alpha-1}}\right|\\&\quad<\varepsilon+\varepsilon+3\varepsilon=5\varepsilon,\\&\left|\frac{u'(t)}{1+t^{\alpha-2}}-\frac{u'_{ijp}(t)}{1+t^{\alpha-2}}\right|\\&\quad\le\left|\frac{u'(t)}{1+t^{\alpha-2}}-\frac{u'(T)}{1+t^{\alpha-2}}\right|+\left|\frac{u'(T)}{1+t^{\alpha-2}}-\frac{u'_{ijp}(T)}{1+t^{\alpha-2}}\right|+\left|\frac{u'_{ijp}(T)}{1+t^{\alpha-2}}-\frac{u'_{ijp}(t)}{1+t^{\alpha-2}}\right|\\&\quad<\varepsilon+\varepsilon+3\varepsilon=5\varepsilon,\end{aligned}$$

and

$$\begin{aligned}&\left|D^{\alpha-1}u(t)-D^{\alpha-1}u_{ijp}(t)\right|\\&\quad\le\left|D^{\alpha-1}u(t)-D^{\alpha-1}u(T)\right|+\left|D^{\alpha-1}u(T)-D^{\alpha-1}u_{ijp}(T)\right|+\left|D^{\alpha-1}u_{ijp}(T)-D^{\alpha-1}u_{ijp}(t)\right|\\&\quad<\varepsilon+\varepsilon+3\varepsilon=5\varepsilon.\end{aligned}$$

These ensure that

$$\left\|u(t)-u_{ijp}(t)\right\|_Y<5\varepsilon.$$

□

3 Main results

Define the operator T by

$$\begin{aligned}Tu(t)={}&\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f\bigl(s,u(s),D^{\alpha-1}u(s)\bigr)\,ds\\&+\frac{-\int_0^\infty f(s,u(s),D^{\alpha-1}u(s))\,ds+\sum_{i=1}^{m-2}\beta_i\int_0^{\xi_i}\frac{(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(s),D^{\alpha-1}u(s))\,ds}{\Gamma(\alpha)-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha-1}}t^{\alpha-1}.\end{aligned}$$

Theorem 3.1 *Assume that $f:J\times R\times R\to R$ is continuous. Then problem* (1.1)-(1.2) *has at least one solution under the assumption that*

(H) *there exist nonnegative functions $a(t)(1+t^{\alpha-1}),b(t),c(t)\in L^1(J)$, such that*

$$\left\|f(t,x,y)\right\|\le a(t)|x|+b(t)|y|+c(t),$$

where $\int_0^\infty c(t)\,dt<+\infty$.

Proof First of all, in view of

$$Tu'(t) = \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha)} f\bigl(s,u,D^{\alpha-1}u\bigr)\,ds + \frac{-\int_0^\infty f(s,u,D^{\alpha-1}u)\,ds + \sum_{i=1}^{m-2}\beta_i\int_0^{\xi_i}\frac{(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u,D^{\alpha-1}u)\,ds}{\Gamma(\alpha)-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha-1}}(\alpha-1)t^{\alpha-2},$$

$$D^{\alpha-1}Tu(t) = \int_0^t f\bigl(s,u(s),D^{\alpha-1}u(s)\bigr)\,ds + \frac{-\int_0^\infty f(s,u(s),D^{\alpha-1}u(s))\,ds + \sum_{i=1}^{m-2}\beta_i\int_0^{\xi_i}\frac{(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(s),D^{\alpha-1}u(s))\,ds}{\Gamma(\alpha)-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha-1}}\Gamma(\alpha),$$

together with the continuity of f, we see that $T'u(t)$ and $D^{\alpha-1}Tu(t)$ are continuous on J.

In the following we divide the proof into several steps.

Step 1 Choose the positive number

$$R > \max\{R_1,R_2,R_3\},$$

where

$$R_1 = \frac{\frac{1}{\Gamma(\alpha)}\int_0^1 c(t)\,dt + \frac{1}{\Gamma(\alpha)\Lambda}\sum_{i=1}^{m-2}\beta_i\int_0^{\xi_i}(\xi_i-t)^{\alpha-1}c(t)\,dt + \frac{1}{\Lambda}\int_0^\infty c(t)\,dt}{1-\frac{1}{\Gamma(\alpha)}\int_0^1(a(t)+b(t))\,dt - \frac{1}{\Gamma(\alpha)\Lambda}\sum_{i=1}^{m-2}\beta_i\int_0^{\xi_i}(\xi_i-t)^{\alpha-1}(a(t)+b(t))\,dt - \frac{1}{\Lambda}\int_0^\infty(a(t)+b(t))\,dt},$$

$$R_2 = \frac{\frac{1}{\Gamma(\alpha)}\int_0^1 c(t)\,dt + \frac{\alpha-1}{\Gamma(\alpha)\Lambda}\sum_{i=1}^{m-2}\beta_i\int_0^{\xi_i}(\xi_i-t)^{\alpha-1}c(t)\,dt + \frac{\alpha-1}{\Lambda}\int_0^\infty c(t)\,dt}{1-\frac{1}{\Gamma(\alpha)}\int_0^1(a(t)+b(t))\,dt - \frac{\alpha-1}{\Gamma(\alpha)\Lambda}\sum_{i=1}^{m-2}\beta_i\int_0^{\xi_i}(\xi_i-t)^{\alpha-1}(a(t)+b(t))\,dt - \frac{\alpha-1}{\Lambda}\int_0^\infty(a(t)+b(t))\,dt},$$

$$R_3 = \frac{\int_0^1 c(t)\,dt - + \frac{1}{\Lambda}\sum_{i=1}^{m-2}\beta_i\int_0^{\xi_i}(\xi_i-t)^{\alpha-1}c(t)\,dt + \frac{\Gamma(\alpha)}{\Lambda}\int_0^\infty c(t)\,dt}{1-\int_0^1(a(t)+b(t))\,dt - \frac{1}{\Lambda}\sum_{i=1}^{m-2}\beta_i\int_0^{\xi_i}(\xi_i-t)^{\alpha-1}(a(t)+b(t))\,dt - \frac{\Gamma(\alpha)}{\Lambda}\int_0^\infty(a(t)+b(t))\,dt},$$

and $\Lambda = \Gamma(\alpha) - \sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha-1}$.

Let set

$$U = \bigl\{u(t)\in Y : \|u(t)\|_Y \le R\bigr\}.$$

Then, $A : U \to U$. In fact, for any $u(t)\in U$, we have

$$\begin{aligned}
\frac{|Tu(t)|}{1+t^{\alpha-1}} &= \Biggl|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)(1+t^{\alpha-1})}f\bigl(s,\ u(s),D^{\alpha-1}u(s)\bigr)\,ds \\
&\quad + \frac{-\int_0^\infty f(s,u(s),D^{\alpha-1}u(s))\,ds + \sum_{i=1}^{m-2}\beta_i\int_0^{\xi_i}\frac{(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(s),D^{\alpha-1}u(s))\,ds}{\Gamma(\alpha)-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha-1}}\frac{t^{\alpha-1}}{(1+t^{\alpha-1})}\Biggr| \\
&\le \frac{1}{\Gamma(\alpha)}\int_0^t \bigl(a(s)\bigl|u(s)\bigr| + b(s)\bigl|D^{\alpha-1}u(s)\bigr| + c(s)\bigr)\,ds \\
&\quad + \frac{1}{\Lambda}\int_0^\infty \bigl(a(s)\bigl|u(s)\bigr| + b(s)\bigl|D^{\alpha-1}u(s)\bigr| + c(s)\bigr)\,ds
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{\sum_{i=1}^{m-2}\beta_i}{\Lambda}\int_0^{\xi_i}\frac{(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)}\bigl(a(s)\bigl|u(s)\bigr| + b(s)\bigl|D^{\alpha-1}u(s)\bigr| + c(s)\bigr)\,ds \\
&\le \frac{\|u\|_Y}{\Gamma(\alpha)}\int_0^1\bigl(a(t)+b(t)\bigr)\,dt + \frac{1}{\Gamma(\alpha)}\int_0^1 c(t)\,dt \\
&+ \frac{\|u\|_Y}{\Lambda}\int_0^\infty\bigl(a(t)+b(t)\bigr)\,dt + \frac{1}{\Lambda}\int_0^\infty c(t)\,dt \\
&+ \frac{\|u\|_Y}{\Lambda}\sum_{i=1}^{m-2}\beta_i\int_0^{\xi_i}\frac{(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)}\bigl(a(t)+b(t)\bigr)\,dt + \frac{\sum_{i=1}^{m-2}\beta_i}{\Lambda}\int_0^{\xi_i}\frac{(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)}c(t)\,dt \\
&\le R,
\end{aligned}
$$

$$
\begin{aligned}
&\frac{|T'u(t)|}{1+t^{\alpha-2}} \\
&= \int_0^t\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha)(1+t^{\alpha-2})}f\bigl(s,u(s),D^{\alpha-1}u(s)\bigr)\,ds \\
&+ \frac{-\int_0^\infty f(s,u(s),D^{\alpha-1}u(s))\,ds + \sum_{i=1}^{m-2}\beta_i\int_0^{\xi_i}\frac{(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(s),D^{\alpha-1}u(s))\,ds}{\Gamma(\alpha)-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha-1}} \\
&\times \frac{(\alpha-1)t^{\alpha-2}}{(1+t^{\alpha-2})} \\
&\le \frac{1}{\Gamma(\alpha)}\int_0^t\bigl(a(s)\bigl|u(s)\bigr| + b(s)\bigl|D^{\alpha-1}u(s)\bigr| + c(s)\bigr)\,ds \\
&+ \frac{\alpha-1}{\Lambda}\int_0^\infty\bigl(a(s)\bigl|u(s)\bigr| + b(s)\bigl|D^{\alpha-1}u(s)\bigr| + c(s)\bigr)\,ds \\
&+ (\alpha-1)\frac{\sum_{i=1}^{m-2}\beta_i}{\Lambda}\int_0^{\xi_i}\frac{(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)}\bigl(a(s)\bigl|u(s)\bigr| + b(s)\bigl|D^{\alpha-1}u(s)\bigr| + c(s)\bigr)\,ds \\
&\le \frac{\|u\|_Y}{\Gamma(\alpha)}\int_0^1\bigl(a(t)+b(t)\bigr)\,dt + \frac{1}{\Gamma(\alpha)}\int_0^1 c(t)\,dt \\
&+ \frac{(\alpha-1)\|u\|_Y}{\Lambda}\int_0^\infty\bigl(a(t)+b(t)\bigr)\,dt + \frac{\alpha-1}{\Lambda}\int_0^\infty c(t)\,dt \\
&+ \frac{(\alpha-1)\|u\|_Y}{\Lambda}\sum_{i=1}^{m-2}\beta_i\int_0^{\xi_i}\frac{(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)}\bigl(a(t)+b(t)\bigr)\,dt \\
&+ \frac{(\alpha-1)\sum_{i=1}^{m-2}\beta_i}{\Lambda}\int_0^{\xi_i}\frac{(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)}c(t)\,dt \\
&\le R,
\end{aligned}
$$

$$
\begin{aligned}
&\bigl|D^{\alpha-1}Tu(t)\bigr| \\
&\le \int_0^t\bigl|f\bigl(s,u(s),D^{\alpha-1}u(s)\bigr)\bigr|\,ds + \frac{\Gamma(\alpha)}{\Gamma(\alpha)-\sum_{k=1}^{m-2}\beta_i\xi_i^{\alpha-1}}\int_0^t\bigl|f\bigl(s,u(s),D^{\alpha-1}u(s)\bigr)\bigr|\,ds \\
&+ \frac{\sum_{i=1}^{m-2}\beta_i}{\Gamma(\alpha)-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha-1}}\int_0^{\xi_i}(\xi_i-s)^{\alpha-1}\bigl|f\bigl(s,u(s),D^{\alpha-1}u(s)\bigr)\bigr|\,ds \\
&\le R*\int_0^1\bigl(a(t)+b(t)\bigr)\,dt + \int_0^1 c(t)\,dt \\
&+ \frac{\Gamma(\alpha)R}{\Lambda}\int_0^\infty\bigl(a(t)+b(t)\bigr)\,dt + \frac{\Gamma(\alpha)}{\Lambda}\int_0^\infty c(t)\,dt
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\sum_{i=1}^{m-2}\beta_i R}{\Lambda}\int_0^{\xi_i}(\xi_i-s)^{\alpha-1}\bigl(a(t)+b(t)\bigr)\,dt+\frac{\sum_{i=1}^{m-2}\beta_i}{\Lambda}\int_0^{\xi_i}(\xi_i-s)^{\alpha-1}c(t)\,dt\\
&\le R.
\end{aligned}
$$

Hence, $\|Tu(t)\|_Y \le R$, which shows that $A : U \to U$.

Step 2 Let V be a nonempty subset of U. We will show that TV is relative compact. Let $I \subset J$ be a compact interval, $t_1, t_2 \in I$ and $t_1 < t_2$. Then for any $u(t) \in V$, we have

$$
\begin{aligned}
&\left|\frac{Tu(t_2)}{1+t_2^{\alpha-1}}-\frac{Tu(t_1)}{1+t_1^{\alpha-1}}\right|\\
&\quad=\biggl|\int_0^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)(1+t_2^{\alpha-1})}f\bigl(s,u,\ D^{\alpha-1}u\bigr)\,ds\\
&\qquad+\frac{-\int_0^{\infty}f(s,u,D^{\alpha-1}u)\,ds+\sum_{i=1}^{m-2}\beta_i\int_0^{\xi_i}\frac{(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u,D^{\alpha-1}u)\,ds}{\Gamma(\alpha)-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha-1}}\frac{t_2^{\alpha-1}}{(1+t_2^{\alpha-1})}\\
&\qquad-\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)(1+t_1^{\alpha-1})}f\bigl(s,u,D^{\alpha-1}u\bigr)\,ds\\
&\qquad-\frac{-\int_0^{\infty}f(s,u,D^{\alpha-1}u)\,ds+\sum_{i=1}^{m-2}\beta_i\int_0^{\xi_i}\frac{(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u,D^{\alpha-1}u)\,ds}{\Gamma(\alpha)-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha-1}}\frac{t_1^{\alpha-1}}{(1+t_1^{\alpha-1})}\biggr|\\
&\quad\le\int_0^{t_1}\left|\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)(1+t_2^{\alpha-1})}-\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)(1+t_1^{\alpha-1})}\right|\bigl|f\bigl(s,u,D^{\alpha-1}u\bigr)\bigr|\,ds\\
&\qquad+\int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}\bigl|f\bigl(s,\ u,D^{\alpha-1}u\bigr)\bigr|\,ds\\
&\qquad+\left|\frac{-\int_0^{\infty}f(s,u(s),D^{\alpha-1}u(s))\,ds+\sum_{i=1}^{m-2}\beta_i\int_0^{\xi_i}\frac{(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u,D^{\alpha-1}u)\,ds}{\Gamma(\alpha)-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha-1}}\right|\\
&\qquad\times\left|\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}-\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}\right|,\\
&\left|\frac{T'u(t_2)}{1+t^{\alpha-2}}-\frac{T'u(t)_1}{1+t^{\alpha-2}}\right|\\
&\quad\le\left|\int_0^{t_2}\frac{(t_2-s)^{\alpha-2}}{\Gamma(\alpha)(1+t_2^{\alpha-2})}f\bigl(s,u,D^{\alpha-1}u\bigr)\,ds-\int_0^{t_1}\frac{(t_1-s)^{\alpha-2}}{\Gamma(\alpha)(1+t_1^{\alpha-2})}f\bigl(s,u,D^{\alpha-1}u\bigr)\,ds\right|\\
&\qquad+(\alpha-1)\left|\frac{-\int_0^{\infty}f(s,u,D^{\alpha-1}u)\,ds+\sum_{i=1}^{m-2}\beta_i\int_0^{\xi_i}\frac{(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u,D^{\alpha-1}u)\,ds}{\Gamma(\alpha)-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha-1}}\right|\\
&\qquad\times\left|\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}-\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}\right|
\end{aligned}
$$

and

$$
\begin{aligned}
&\bigl|D^{\alpha-1}Tu(t_2)-D^{\alpha-1}Tu(t_1)\bigr|\\
&\quad\le\left|\int_0^{t_2}f\bigl(s,u(s),D^{\alpha-1}u(s)\bigr)\,ds-\int_0^{t_1}f\bigl(s,u(s),D^{\alpha-1}u(s)\bigr)\,ds\right|\\
&\quad\le\int_{t_1}^{t_2}\bigl|f\bigl(s,u(s),D^{\alpha-1}u(s)\bigr)\,ds\bigr|.
\end{aligned}
$$

Note that for any $u(t) \in V$, we have $f(t, u(t), D^{\alpha-1}u(t))$ is bounded on I. Then it is easy to see that $\frac{|Tu(t)|}{1+t^{\alpha-1}}$, $\frac{|T'u(t)|}{1+t^{\alpha-2}}$, and $D^{\alpha-1}Tu(t)$ are equicontinuous on I.

Considering the condition H, for given $\varepsilon > 0$, there exists a constant $L > 0$ such that

$$\int_L^{+\infty} \left|f\left(t, u(t), D^{\alpha-1}u(t)\right)\right| < \varepsilon.$$

On the other hand, since $\lim_{t\to+\infty} \frac{t^{\alpha-1}}{1+t^{\alpha-1}} = 1$, there exists a constant $T_1 > 0$ such that $t_1, t_2 \geq T_1$,

$$\left|\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}} - \frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}\right| < \varepsilon.$$

Similarly, in view of $\lim_{t\to+\infty} \frac{(t-L)^{\alpha-1}}{1+t^{\alpha-1}} = 1$, there exists a constant $T_2 > L > 0$ such that $t_1, t_2 \geq T_2$ and $0 \leq s \leq L$,

$$\left|\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}} - \frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}\right| < \varepsilon.$$

In view of $\lim_{t\to+\infty} \frac{(t-L)^{\alpha-2}}{1+t^{\alpha-2}} = 1$, there exists a constant $T_3 > L > 0$ such that $t_1, t_2 \geq T_3$, and $0 \leq s \leq L$,

$$\left|\frac{(t_1-s)^{\alpha-2}}{1+t_1^{\alpha-2}} - \frac{(t_2-s)^{\alpha-2}}{1+t_2^{\alpha-2}}\right| < \varepsilon.$$

Now choose $T > \max\{T_1, T_2, T_3\}$. Then for $t_1, t_2 \geq T$, we have

$$\begin{aligned}
&\left|\frac{Tu(t_2)}{1+t_2^{\alpha-1}} - \frac{Tu(t_1)}{1+t_1^{\alpha-1}}\right| \\
&\quad\leq \frac{\max_{t\in[0,L],u\in V}|f(t,u,D^{\alpha-1}u)|}{\Gamma(\alpha)} L\varepsilon + \frac{2}{\Gamma(\alpha)}\varepsilon \\
&\qquad + \left|\frac{-\int_0^\infty f(s,u,D^{\alpha-1}u)\,ds + \sum_{i=1}^{m-2}\beta_i\int_0^{\xi_i}\frac{(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u,D^{\alpha-1}u)\,ds}{\Gamma(\alpha) - \sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha-1}}\right|\varepsilon, \\
&\left|\frac{T'u(t_2)}{1+t_2^{\alpha-1}} - \frac{T'u(t_1)}{1+t_1^{\alpha-1}}\right| \\
&\quad\leq \frac{\max_{t\in[0,L],u\in V}|f(t,u,D^{\alpha-1}u)|}{\Gamma(\alpha)} L\varepsilon + \frac{2}{\Gamma(\alpha)}\varepsilon \\
&\qquad + (\alpha-1)\left|\frac{-\int_0^\infty f(s,u,D^{\alpha-1}u)\,ds + \sum_{i=1}^{m-2}\beta_i\int_0^{\xi_i}\frac{(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u,D^{\alpha-1}u)\,ds}{\Gamma(\alpha) - \sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha-1}}\right|\varepsilon,
\end{aligned}$$

and

$$\left|D^{\alpha-1}Tu(t_2) - D^{\alpha-1}Tu(t_1)\right| \leq \int_{t_1}^{t_2}\left|f\left(s, u(s), D^{\alpha-1}u(s)\right)ds\right| < \varepsilon.$$

Consequently, Lemma 2.5 shows that TV is relative compact.

Step 3 $T: U \to U$ is a continuous operator.

Let $u_n, u \in U$, $n = 1, 2, \dots$, and $\|u_n - u\|_Y \to 0$ as $n \to \infty$. Then, we have

$$\begin{aligned}
&\left|\frac{Tu_n(t)}{1+t^{\alpha-1}} - \frac{Tu(t)}{1+t^{\alpha-1}}\right| \\
&\quad\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)(1+t^{\alpha-1})} \left|f\left(s, u_n(s), D^{\alpha-1}u_n(s)\right) - f\left(s, u_n(s), D^{\alpha-1}u_n(s)\right)\right| ds \\
&\qquad + \frac{t^{\alpha-1}}{(1+t^{\alpha-1})\Lambda} \sum_{i=1}^{m-2} \beta_i \int_0^{\xi_i} \frac{(\xi_i - s)^{\alpha-1}}{\Gamma(\alpha)} \left|f\left(s, u_n(s), D^{\alpha-1}u_n(s)\right)\right. \\
&\qquad \left. - f\left(s, u_n(s), D^{\alpha-1}u_n(s)\right)\right| ds \\
&\qquad + \frac{t^{\alpha-1}}{(1+t^{\alpha-1})\Lambda} \int_0^\infty \left|f\left(s, u_n(s), D^{\alpha-1}u_n(s)\right) - f\left(s, u_n(s), D^{\alpha-1}u_n(s)\right)\right| ds \\
&\quad\leq \left(\frac{2}{\Gamma(\alpha)} + \frac{4}{\Lambda}\right) \int_0^\infty \left|f\left(s, u_n(s), D^{\alpha-1}u_n(s)\right) - f\left(s, u_n(s), D^{\alpha-1}u_n(s)\right)\right| ds \\
&\quad\leq \left(\frac{4}{\Gamma(\alpha)} + \frac{8}{\Lambda}\right) R \int_0^\infty \left[\left(1+t^{\alpha-1}\right)a(t) + b(t)\right] dt \\
&\qquad + \left(\frac{4}{\Gamma(\alpha)} + \frac{8}{\Lambda}\right) \int_0^\infty c(t)\, dt, \\
&\left|\frac{T'u_n(t)}{1+t^{\alpha-2}} - \frac{T'u(t)}{1+t^{\alpha-2}}\right| \\
&\quad\leq \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha)(1+t^{\alpha-1})} \left|f\left(s, u_n(s), D^{\alpha-1}u_n(s)\right) - f\left(s, u_n(s), D^{\alpha-1}u_n(s)\right)\right| ds \\
&\qquad + \frac{(\alpha-1)t^{\alpha-2}}{(1+t^{\alpha-2})\Lambda} \sum_{i=1}^{m-2} \beta_i \int_0^{\xi_i} \frac{(\xi i - s)^{\alpha-1}}{\Gamma(\alpha)} \left|f\left(s, u_n(s), D^{\alpha-1}u_n(s)\right)\right. \\
&\qquad \left. - f\left(s, u_n(s), D^{\alpha-1}u_n(s)\right)\right| ds \\
&\qquad + \frac{(\alpha-1)t^{\alpha-2}}{(1+t^{\alpha-2})\Lambda} \int_0^\infty \left|f\left(s, u_n(s), D^{\alpha-1}u_n(s)\right) - f\left(s, u_n(s), D^{\alpha-1}u_n(s)\right)\right| ds \\
&\quad\leq \left(\frac{2}{\Gamma(\alpha)} + \frac{4(\alpha-1)}{\Lambda}\right) \int_0^\infty \left|f\left(s, u_n(s), D^{\alpha-1}u_n(s)\right) - f\left(s, u_n(s), D^{\alpha-1}u_n(s)\right)\right| ds \\
&\quad\leq \left(\frac{4}{\Gamma(\alpha)} + \frac{8(\alpha-1)}{\Lambda}\right) R \int_0^\infty \left[\left(1+t^{\alpha-1}\right)a(t) + b(t)\right] dt \\
&\qquad + \left(\frac{4}{\Gamma(\alpha)} + \frac{8(\alpha-1)}{\Lambda}\right) \int_0^\infty c(t)\, dt, \\
&\left|D^{\alpha-1}Tu_n(t) - D^{\alpha-1}Tu(t)\right| \\
&\quad\leq \int_0^\infty \left|f\left(s, u_n(s), D^{\alpha-1}u_n(s)\right) - f\left(s, u_n(s), D^{\alpha-1}u_n(s)\right)\right| ds \\
&\qquad + \frac{\Gamma(\alpha)}{\Lambda} \sum_{k=1}^{m-2} \beta_i \int_0^{\xi_i} \frac{(\xi_i - s)^{\alpha-1}}{\Gamma(\alpha)} \left|f\left(s, u_n(s), D^{\alpha-1}u_n(s)\right) - f\left(s, u_n(s), D^{\alpha-1}u_n(s)\right)\right| ds \\
&\qquad + \frac{\Gamma(\alpha)}{\Lambda} \int_0^\infty \left|f\left(s, u_n(s), D^{\alpha-1}u_n(s)\right) - f\left(s, u_n(s), D^{\alpha-1}u_n(s)\right)\right| ds \\
&\quad\leq \left(\frac{2}{\Gamma(\alpha)} + \frac{2+2\Gamma(\alpha)}{\Lambda}\right) \int_0^\infty \left|f\left(s, u_n(s), D^{\alpha-1}u_n(s)\right) - f\left(s, u_n(s), D^{\alpha-1}u_n(s)\right)\right| ds
\end{aligned}$$

$$\leq \left(\frac{4}{\Gamma(\alpha)} + \frac{4 + 4\Gamma(\alpha)}{\Lambda}\right) R \int_0^\infty \left[\left(1 + t^{\alpha-1}\right)a(t) + b(t)\right] dt$$
$$+ \left(\frac{4}{\Gamma(\alpha)} + \frac{4 + 4\Gamma(\alpha)}{\Lambda}\right) \int_0^\infty c(t)\, dt.$$

Then the operator T is continuous in view of the Lebesgue dominated convergence theorem. Thus by Schauder's fixed point theorem we conclude that the problem (1.1)-(1.2) has at least one solution in U and the proof is completed. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that the work was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

Acknowledgements
The work is sponsored by the NSFC (11201109), Anhui Provincial Natural Science Foundation (1408085QA07), the Higher School Natural Science Project of Anhui Province (KJ2014A200), and the outstanding talents plan of Anhui High school.

Received: 9 October 2015 Accepted: 8 December 2015 Published online: 29 December 2015

References

1. Delbosco, D: Fractional calculus and function spaces. J. Fract. Calc. **6**, 45-53 (1994)
2. Miller, KS, Ross, B: An Introduction to the Fractional Calculus and Fractional Differential Equations. Wiley, New York (1993)
3. Lakshmikantham, V, Leela, S: Theory of fractional differential inequalities and applications. Commun. Appl. Anal. **11**, 395-402 (2007)
4. Lakshmikantham, V, Devi, J: Theory of fractional differential equations in a Banach space. Eur. J. Pure Appl. Math. **1**, 38-45 (2008)
5. Lakshmikantham, V, Leela, S: Nagumo-type uniqueness result for fractional differential equations. Nonlinear Anal. TMA **71**, 2886-2889 (2009)
6. Lakshmikantham, V, Leela, S: A Krasnoselskii-Krein-type uniqueness result for fractional differential equations. Nonlinear Anal. TMA **71**, 3421-3424 (2009)
7. Lakshmikantham, V: Theory of fractional differential equations. Nonlinear Anal. TMA **69**, 3337-3343 (2008)
8. Agarwal, RP, Lakshmikantham, V, Nieto, JJ: On the concept of solution for fractional differential equations with uncertainty. Nonlinear Anal. **72**, 2859-2862 (2010)
9. Benchohra, M, Henderson, J, Ntouyas, SK, Ouahab, A: Existence results for fractional order functional differential equations with infinite delay. J. Math. Anal. Appl. **338**, 1340-1350 (2008)
10. Zhou, Y: Existence and uniqueness of fractional functional differential equations with unbounded delay. Int. J. Dyn. Syst. Differ. Equ. **1**, 239-244 (2008)
11. Bai, Z: On positive solutions of a nonlocal fractional boundary value problem. Nonlinear Anal. TMA **72**, 916-924 (2010)
12. Xu, XJ, Jiang, DQ, Yuan, CJ: Multiple positive solutions for the boundary value problem of a nonlinear fractional differential equations. Nonlinear Anal. TMA **71**, 4676-4688 (2009)
13. Ahmad, B, Nieto, J: Existence results for a coupled system of nonlinear fractional differential equations with three-point boundary conditions. Comput. Math. Appl. **58**, 1838-1843 (2009)
14. Jia, M, Liu, X: Multiplicity of solutions for integral boundary value problems of fractional differential equations with upper and lower solutions. Appl. Math. Comput. **232**, 313-323 (2014)
15. Liu, X, Jia, M: Multiple solutions for fractional differential equations with nonlinear boundary conditions. Comput. Math. Appl. **59**, 2880-2886 (2010)
16. Liu, X, Jia, M, Xiang, X: On the solvability of fractional differential equation model involving the *p*-Laplacian operator. Comput. Math. Appl. **64**, 3267-3275 (2012)
17. Agarwal, RP, O'Regan, D: Infinite Interval Problems for Differential, Difference and Integral Equations. Kluwer Academic, Dordrecht (2001)
18. Yan, B: Multiple unbounded solutions of boundary value problems for second-order differential equations on the half-line. Nonlinear Anal. **51**, 1031-1044 (2002)
19. Su, X, Zhang, S: Unbounded solutions to a boundary value problem of fractional order on the half-line. Comput. Math. Appl. **61**, 1079-1087 (2011)
20. Zima, M: On positive solution of boundary value problems on the half-line. J. Math. Anal. Appl. **259**, 127-136 (2001)
21. Lian, H, Pang, H, Ge, W: Triple positive solutions for boundary value problems on infinite intervals. Nonlinear Anal. **67**, 2199-2207 (2007)
22. Agarwal, RP, Benchohra, M, Hamani, S, Pinelas, S: Boundary value problem for differential equations involving Riemann-Liouville fractional derivative on the half line. Dyn. Contin. Discrete Impuls. Syst., Ser. A Math. Anal. **18**(1), 235-244 (2011)
23. Arara, A, Benchohra, M, Hamidi, N, Nieto, J: Fractional order differential equations on an unbounded domain. Nonlinear Anal. **72**, 580-586 (2010)
24. Liang, S, Zhang, J: Existence of multiple positive solutions for m-point fractional boundary value problems on an infinite interval. Math. Comput. Model. **54**, 1334-1346 (2011)