Kamel *Boundary Value Problems* 2011, **2011**:50
http://www.boundaryvalueproblems.com/content/2011/1/50

Boundary Value Problems
a SpringerOpen Journal

RESEARCH **Open Access**

Nonexistence of nontrivial solutions for the *p*(*x*)-Laplacian equations and systems in unbounded domains of $\mathbb{R}^n$

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Abstract

In this paper, we are interested on the study of the nonexistence of nontrivial solutions for the $p(x)$-Laplacian equations, in unbounded domains of $\mathbb{R}^n$. This leads us to extend these results to m-equations systems. The method used is based on pohozaev type identities.

1 Introduction

Several works have been reported by many authors, comprise results of nonexistence of nontrivial solutions of the semilinear elliptic equations and systems, under various situations, see [1-8]. The Pohozaĕv identity [1] published in 1965 for solutions of the Dirichlet problem proved absence of nontrivial solutions for some elliptic equations of the form

$$\begin{cases} -\Delta u + f(u) = 0 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

when Ω is a star shaped bounded open domain in $\mathbb{R}^n$ and f is a continuous function on $\mathbb{R}$ satisfying

$$(n-2)F(u) - 2nuf(u) > 0,$$

A. Hareux and B. Khodja [2] established under the assumption

$$f(0) = 0,$$
$$2F(u) - uf(u) \leq 0.$$

that the problems

$$\begin{cases} -\Delta u + f(u) = 0 \text{ in } J \times \omega, \\ \left(u \text{ or } \frac{\partial u}{\partial n}\right) = 0 \text{ on } \partial(J \times \omega). \end{cases}$$

admit only the null solution in $H^2(J \times \omega) \cap L^\infty(J \times \omega)$. where J is an interval of $\mathbb{R}$ and ω is a connected unbounded domain of $\mathbb{R}^N$ such as

$$\exists \Lambda \in \mathbb{R}^N, ||\Lambda|| = 1, \langle n(x), \Lambda \rangle \geq 0 \text{ on } \partial\omega, \langle n(x), \Lambda \rangle \not\equiv 0,$$

($n(x)$ is the outward normal to $\partial\omega$ at the point x)

In this work we are interested in the study of the nonexistence of nontrivial solutions for the $p(x)$-laplacian problem

$$\begin{cases} -\Delta_{p(x)}u = H(x)f(u) \text{in } \Omega \\ Bu = 0 \text{ on } \partial\Omega \end{cases} \tag{1.1}$$

with

$$Bu = \begin{cases} u \quad \text{Dirichlet condition} \quad (1.2) \\ \frac{\partial u}{\partial \nu} \quad \text{Neumann condition} \quad (1.3) \end{cases}$$

where

$$\Delta_{p(x)}u = div\left(|\nabla u|^{p(x)-2}\nabla u\right)$$

Ω is bounded or unbounded domains of $\mathbb{R}^n$, f is a locally lipshitzian function, H and p are given continuous real functions of $C(\bar{\Omega})$ verifying

$$\begin{aligned} &F(t) = \int_0^t f(\sigma)d\sigma, f(0) = 0, \\ &H(x) > 0, (x, \nabla H(x)) \neq 0 \text{ and } \lim_{|x|\to+\infty} H(x) = 0, \\ &p(x) > 1, (x, \nabla p(x)) \geq 0, \forall x \in \bar{\Omega}, \\ &a = \sup_{x\in\bar{\Omega}}\left(1 - \frac{n}{p(x)} + \frac{(x, \nabla p(x))}{p^2(x)}\right). \end{aligned} \tag{1.4}$$

$(.,.)$ is the inner product in $\mathbb{R}^n$.

We extend this technique to the system of m-equations

$$\begin{cases} -\Delta_{p_k(x)}u = H(x)f_k(u_1, \ldots, u_m) \text{ in } \Omega, 1 \leq k \leq m, \\ Bu_k = 0 \text{ on } \partial\Omega, 1 \leq k \leq m, \end{cases} \tag{1.6}$$

with

$$Bu_k = \begin{cases} u_k \quad \text{Dirichlet condition} \quad (1.7) \\ \frac{\partial u_k}{\partial \nu} \text{ Neumann condition} \quad (1.8) \end{cases}$$

Where $\{f_k\}$ are locally lipshitzian functions verify

$$\begin{aligned} &f_k(s_1, \ldots, s_{k-1}, 0, u_{k+1}, \ldots, s_m) = 0, (0 \leq k \leq m), \\ &\exists F_m : \mathbb{R}^m \to \mathbb{R} : \frac{\partial F_m}{\partial s_k}(s_1, \ldots, s_m) = f_k(s_1, \ldots, s_m). \end{aligned}$$

H is previously defined and p_k functions of $C^1(\overline{\Omega})$ class, verify

$$\begin{aligned} &p_k(x) > 1, (x, \nabla p_k(x)) \geq 0, \forall x \in \bar{\Omega}. \\ &a_k = \sup_{x\in\bar{\Omega}}\left(1 - \frac{n}{p_k(x)} + \frac{(x, \nabla p_k(x))}{p_k^2(x)}\right) \end{aligned} \tag{1.9}$$

2 Integral identities

Let

$$L^{p(x)}(\Omega) = \left\{ u \text{ measurable real function} : \int_\Omega |u(x)|^{p(x)} dx < +\infty \right\},$$

with the norm

$$|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf\left\{ \lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \le 1 \right\},$$

and

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\},$$

with the norm

$$||u||_{W^{1,p(x)}(\Omega)} = |u|_{L^{p(x)}(\Omega)} + |\nabla u|_{L^{p(x)}(\Omega)}.$$

Denote $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,\,p(x)}$ (Ω),

Lemma 1 *Let* $u \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\bar\Omega)$ *solution of the equation* (1.1) - (1.2), *we have*

$$\begin{aligned}&\int_\Omega \Big[\Big(1 - \frac{n}{p(x)} + \frac{(x,\nabla p(x))}{p^2(x)}\Big(1 - \ln\Big(|\nabla u|^{p(x)}\Big)\Big) - a\Big)|\nabla u|^{p(x)}\,dx \\ &+H(x)(nF(u) - auf(u)) + (x,\nabla H(x))F(u)\Big]\,dx \\ &= \int_{\partial\Omega}\Big(1 - \frac{1}{p(x)}\Big)|\nabla u|^{p(x)}(x,\nu)ds\end{aligned} \tag{2.1}$$

Lemma 2 *Let* $u \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\bar\Omega)$ *solution of the equation* (1.1) - (1.3), *we have*

$$\begin{aligned}&\int_\Omega \Big[\Big(1 - \frac{n}{p(x)} + \frac{(x,\nabla p(x))}{p^2(x)}\Big(1 - \ln\Big(|\nabla u|^{p(x)}\Big)\Big) - a\Big)|\nabla u|^{p(x)}\,dx \\ &+H(x)(nF(u) - auf(u)) + (x,\nabla H(x))F(u)\Big]\,dx \\ &= \int_{\partial\Omega}\Big(\Big(1 - \frac{1}{p(x)}\Big)|\nabla u|^{p(x)} + H(x)F(u)\Big)(x,\nu)ds\end{aligned} \tag{2.2}$$

Proof Multiplying the equation (1.1) by $\sum_{j=1}^n x_i \frac{\partial u}{\partial x_i}$ and integrating the new equation by parts in $\Omega \cap B_R$, $B_R = B\,(0, R)$

$$\begin{aligned}&- \int_{\Omega\cap B_R} div\left(|\nabla u|^{p(x)-2}\nabla u\right)\left(\sum_{j=1}^n x_j \frac{\partial u}{\partial x_j}\right)dx \\ &= -\sum_{i,j=1}^n \int_{\Omega\cap B_R} \frac{\partial}{\partial x_i}\left(|\nabla u|^{p(x)-2}\frac{\partial u}{\partial x_i}\right)x_j\frac{\partial u}{\partial x_j}dx \\ &= \int_{\Omega\cap B_R}\left[|\nabla u|^{p(x)} + |\nabla u|^{p(x)-2}\sum_{i,j=1}^n x_j\frac{\partial u}{\partial x_i}\frac{\partial^2 u}{\partial x_i\partial x_j}\right]dx \\ &-\sum_{i,j=1}^n \int_{\partial(\Omega\cap B_R)} |\nabla u|^{p(x)-2}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}x_j\nu_i ds\end{aligned}$$

Introducing the following result

$$|\nabla u|^{p(x)-2}\sum_{i=1}^{n}\frac{\partial u}{\partial x_i}\frac{\partial^2 u}{\partial x_i\partial x_j}=\frac{1}{p(x)}\frac{\partial}{\partial x_j}\left(|\nabla u|^{p(x)}\right)-\frac{\frac{\partial p}{\partial x_j}}{p^2(x)}|\nabla u|^{p(x)}\ln\left(|\nabla u|^{p(x)}\right)$$

we have

$$\begin{aligned}&\int_{\Omega\cap B_R}\left[|\nabla u|^{p(x)}+\sum_{j=1}^{n}\frac{x_j}{p(x)}\frac{\partial}{\partial x_j}\left(|\nabla u|^{p(x)}\right)-\sum_{j=1}^{n}\frac{(x,\nabla p(x))}{p^2(x)}|\nabla u|^{p(x)}\ln\left(|\nabla u|^{p(x)}\right)\right]dx\\&-\int_{\partial(\Omega\cap B_R)}\sum_{i,j=1}^{n}\int_{\partial(\Omega\cap B_R)}|\nabla u|^{p(x)-2}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}x_j\nu_i ds\\&=\int_{\Omega\cap B_R}\left[1-\frac{n}{p(x)}+\frac{(x,\nabla p(x))}{p^2(x)}\left(1-\ln\left(|\nabla u|^{p(x)}\right)\right)\right]|\nabla u|^{p(x)}dx\\&-\int_{\partial(\Omega\cap B_R)}\left(\sum_{i,j=1}^{n}|\nabla u|^{p(x)-2}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}x_j\nu_i-\sum_{j=1}^{n}\frac{1}{p(x)}|\nabla u|^{p(x)}x_j\nu_j\right)ds\end{aligned}$$

On the other hand

$$\begin{aligned}&\int_{\Omega\cap B_R}H(x)f(u)\left(\sum_{j=1}^{n}x_j\frac{\partial u}{\partial x_j}\right)dx=\sum_{j=1}^{n}\int_{\Omega\cap B_R}x_jH(x)\frac{\partial}{\partial x_j}(F(u))dx\\&=-\int_{\Omega\cap B_R}(nH(x)+(x,\nabla H(x)))F(u)dx+\sum_{j=1}^{n}\int_{\partial(\Omega\cap B_R)}H(x)F(u)x_j\nu_j ds\end{aligned}$$

these results conduct to the following formula

$$\begin{aligned}&\int_{\Omega\cap B_R}\left[\left(1-\frac{n}{p(x)}+\frac{(x,\nabla p(x))}{p^2(x)}\left(1-\ln\left(|\nabla u|^{p(x)}\right)\right)\right)|\nabla u|^{p(x)}dx\right.\\&\left.+(nH(x)+(x,\nabla H(x)))F(u)\right]dx\\&=\int_{\partial(\Omega\cap B_R)}\left[\sum_{i,j=1}^{n}|\nabla u|^{p(x)-2}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}x_j\nu_i\right.\\&\left.-\sum_{j=1}^{n}\left(\frac{1}{p(x)}|\nabla u|^{p(x)}-H(x)F(u)\right)x_j\nu_j\right]ds\end{aligned}\tag{2.3}$$

Multiplying the present equation (1.1) by au and integrating the obtained equation by parts in Ω, we obtain

$$\int_{(\Omega\cap B_R)}\left[a|\nabla u|^{p(x)}-auH(x)f(u)\right]dx=\int_{\partial(\Omega\cap B_R)}a|\nabla u|^{p(x)}\frac{\partial u}{\partial\nu}uds=0,\tag{2.4}$$

Combining (2.3) and (2.4) we obtain

$$\int_{\Omega\cap B_R} \Big[\Big(1 - \tfrac{n}{p(x)} + \tfrac{(x,\nabla p(x))}{p^2(x)}\Big(1 - \ln\Big(|\nabla u|^{p(x)}\Big)\Big) - a\Big)|\nabla u|^{p(x)}\,dx$$

$$+H(x)(nF(u) - auf(u)) + (x, \nabla H(x))F(u)\Big]\,dx$$

$$= \int_{\partial(\Omega\cap B_R)} \Bigg[\sum_{i,j=1}^{n} |\nabla u|^{p(x)-2}\tfrac{\partial u}{\partial x_i}\tfrac{\partial u}{\partial x_j}x_j\nu_i$$

$$-\sum_{j=1}^{n}\Big(\tfrac{1}{p(x)}|\nabla u|^{p(x)} - H(x)F(u)\Big)x_j\nu_j\Bigg]\,ds$$

$$= \int_{\partial\Omega\cap B_R} \Bigg[\sum_{i,j=1}^{n} |\nabla u|^{p(x)-2}\tfrac{\partial u}{\partial x_i}\tfrac{\partial u}{\partial x_j}x_j\nu_i$$

$$-\sum_{j=1}^{n}\Big(\tfrac{1}{p(x)}|\nabla u|^{p(x)} - H(x)F(u)\Big)x_j\nu_j\Bigg]\,ds$$

$$+ \int_{\Omega\cap\partial B_R} \Bigg[\sum_{i,j=1}^{n} |\nabla u|^{p(x)-2}\tfrac{\partial u}{\partial x_i}\tfrac{\partial u}{\partial x_j}x_j\nu_i$$

$$-\sum_{j=1}^{n}\Big(\tfrac{1}{p(x)}|\nabla u|^{p(x)} - H(x)F(u)\Big)x_j\nu_j\Bigg]\,ds$$

On $(\Omega \cap \partial B_R)$ we have $n_i = \frac{x_i}{|x|}$
so the last integral is major by

$$M(R) = R\int_{\Omega\cap\partial B_R}\Big(\Big(1 + \tfrac{1}{p(x)}\Big)|\nabla u|^{p(x)} + |H(x)|\,|F(u)|\Big)ds$$

We remark that if Ω in bounded, so for R is little greater, we get $\Omega \cap \partial B_R = \varphi$, then $M\ (R) = 0$.

If Ω is not bounded, such as $|\nabla u| \in W^{1,\ p(x)}\ (\Omega)$, $F(u) \in L^1\ (\Omega)$ and $\lim_{|x|\to+\infty} H(x) \to 0$, we should see

$$\int_0^{+\infty} dr \int_{\Omega\cap\partial B_R}\Big(\Big(1 + \tfrac{1}{p(x)}\Big)|\nabla u|^{p(x)} + |H(x)||F(u)|\Big)\,ds < +\infty$$

consequently we can always find a sequence $(R_n)_n$, such as

$$\lim_{n\to+\infty} R_n \to +\infty \quad \text{and} \quad \lim_{n\to+\infty} M(R_n) \to 0.$$

In the problem (1.1) - (1.2), $u|_{\partial\Omega} = 0$. Then, $\nabla u = \frac{\partial u}{\partial \nu}n$, we obtain the identity (2.1).

In the problem (1.1) - (1.3), $\frac{\partial u}{\partial \nu}\big|_{\partial\Omega} = 0$, we obtain the identity (2.2). ■

Lemma 3 *Let* $u_k \in W_0^{1,p_k(x)}(\Omega) \cap L^\infty(\bar{\Omega})(1 \le k \le m)$, *solution of the system* (1.6) - (1.7). *Then for the constants* a_k *of* $\mathbb{R}$, *we have*

$$\begin{aligned}
&\int_{\Omega}\Bigg[\sum_{k=1}^{m}\Big(1-\tfrac{n}{p_k(x)}+\tfrac{(x,\nabla p_k(x))}{p_k^2(x)}\Big(1-\ln\Big(|\nabla u_k|^{p_k(x)}\Big)\Big)-a_k\Big)|\nabla u_k|^{p_k(x)}\\
&+H(x)\left(nF_m(u_1,\ldots,u_m)-\sum_{k=1}^{m}a_k u_k f_k(u_1,\ldots,u_m)\right)+\\
&+(x,\nabla H(x))F_m(u_1,\ldots,u_m)\Bigg]dx\\
&=\int_{\partial\Omega}\sum_{k=1}^{m}\Big(1-\tfrac{1}{p_k(x)}\Big)|\nabla u_k|^{p_k(x)}(x,\nu)ds
\end{aligned}\tag{2.5}$$

Lemma 4 *Let* $u_k \in W_0^{1,p}(\Omega)\cap L^{\infty}(\bar{\Omega})(1\le k\le m)$*, solutions of the system* (1.6) - (1.8). *Then for the constants* a_k *of* $\mathbb{R}$*, we have*

$$\begin{aligned}
&\int_{\Omega}\Bigg[\sum_{k=1}^{m}\Big(1-\tfrac{n}{p_k(x)}+\tfrac{(x,\nabla p_k(x))}{p_k^2(x)}\Big(1-\ln\Big(|\nabla u_k|^{p_k(x)}\Big)\Big)-a_k\Big)|\nabla u_k|^{p_k(x)}\\
&+H(x)\left(nF_m(u_1,\ldots,u_m)-\sum_{k=1}^{m}a_k u_k f_k(u_1,\ldots,u_m)\right)+\\
&+(x,\nabla H(x))F_m(u_1,\ldots,u_m)\Bigg]dx\\
&=\int_{\partial\Omega}\left[\sum_{k=1}^{m}\Big(1-\tfrac{1}{p_k(x)}\Big)|\nabla u_k|^{p_k(x)}+H(x)F_m(u_1,\ldots,u_m)\right](x,\nu)ds
\end{aligned}\tag{2.6}$$

Proof Multiplying the equation (1.6) by $\sum_{j=1}^{n} x_i \frac{\partial u_k}{\partial x_i}$ and integrating the new equation by part in $\Omega\cap B_R$, $B_R = B\ (0, R)$, we get

$$\begin{aligned}
&\int_{\Omega\cap B_R}\left[1-\tfrac{n}{p_k(x)}+\tfrac{(x,\nabla p_k(x))}{p_k^2(x)}\Big(1-\ln\Big(|\nabla u_k|^{p_k(x)}\Big)\Big)\right]|\nabla u_k|^{p_k(x)}dx\\
&=\int_{\partial(\Omega\cap B_R)}\left(\sum_{i,j=1}^{n}|\nabla u_k|^{p_k(x)-2}\tfrac{\partial u_k}{\partial x_i}\tfrac{\partial u_k}{\partial x_j}x_j\nu_i-\sum_{j=1}^{n}\tfrac{1}{p_k(x)}|\nabla u_k|^{p_k(x)}x_j\nu_j\right)ds
\end{aligned}$$

On the other hand

$$\begin{aligned}
&\int_{\Omega\cap B_R}H(x)f_k(u_1,\ldots,u_m)\left(\sum_{j=1}^{n}x_j\tfrac{\partial u_k}{\partial x_j}\right)dx\\
&=\sum_{j=1}^{n}\int_{\Omega\cap B_R}x_j H(x)\tfrac{\partial u_k}{\partial x_j}\tfrac{\partial}{\partial u_k}(F_m(u_1,\ldots,u_m))dx
\end{aligned}$$

These results conduct to the following formula

$$\int\limits_{\Omega\cap B_R} \left[\left(1-\frac{n}{p_k(x)}+\frac{(x,\nabla p_k(x))}{p_k^2(x)}\left(1-\ln\left(|\nabla u_k|^{p_k(x)}\right)\right)\right)|\nabla u_k|^{p_k(x)} + \sum_{j=1}^{n} x_j H(x)\frac{\partial u_k}{\partial x_j}\frac{\partial}{\partial u_k}(F_m(u_1,\dots,u_m))\right]dx$$
$$= \int\limits_{\partial(\Omega\cap B_R)} \left[\sum_{i,j=1}^{n} |\nabla u_k|^{p_k(x)-2}\frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}x_j\nu_i - \sum_{j=1}^{n}\frac{1}{p_k(x)}|\nabla u_k|^{p_k(x)}x_j\nu_j\right]ds$$

Doing the sum on k of 1 to m, we obtain

$$\int\limits_{\Omega\cap B_R} \sum_{k=1}^{m}\left[\left(1-\frac{n}{p_k(x)}+\frac{(x,\nabla p_k(x))}{p_k^2(x)}\left(1-\ln\left(|\nabla u_k|^{p_k(x)}\right)\right)\right)|\nabla u_k|^{p_k(x)} + \sum_{j=1}^{n} x_j H(x)\frac{\partial}{\partial x_j}F_m(u_1,\dots,u_m)\right]dx$$
$$= \int\limits_{\partial(\Omega\cap B_R)} \left[\sum_{k=1}^{m}\sum_{i,j=1}^{n} |\nabla u_k|^{p_k(x)-2}\frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}x_j\nu_i + \sum_{k=1}^{m}\sum_{j=1}^{n}\frac{1}{p_k(x)}|\nabla u_k|^{p_k(x)}x_j\nu_j\right]ds$$

which leads to the following identity

$$\int\limits_{\Omega\cap B_R} \sum_{k=1}^{m}\left[\left(1-\frac{n}{p_k(x)}+\frac{(x,\nabla p_k(x))}{p_k^2(x)}\left(1-\ln\left(|\nabla u_k|^{p_k(x)}\right)\right)\right)|\nabla u_k|^{p_k(x)} - (nH(x)+(x,\nabla H(x)))F_m(u_1,\dots,u_m)\right]dx$$
$$= \int\limits_{\partial(\Omega\cap B_R)} \left[\sum_{k=1}^{m}\sum_{i,j=1}^{n} |\nabla u_k|^{p_k(x)-2}\frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}x_j\nu_i + \left(\sum_{k=1}^{m}\frac{1}{p_k(x)}|\nabla u_k|^{p_k(x)} + H(x)F_m(u_1,\dots,u_m)\right)(x,\nu)\right]ds \tag{2.7}$$

Now, multiply the equation (1.1) by au and integrating the obtained equation by parts in $\Omega\cap B_R$

$$\int\limits_{(\Omega\cap B_R)} \left[a_k|\nabla u|^{p_k(x)} - a_k u_k H(x) f_k(u_1,\dots,u_m)\right]dx = 0 \tag{2.8}$$

Combining (2.7) and (2.8), we get the identities (2.5) and (2.6).

The rest of the proof is similar to the that of lemma 1. ■

3 Principal Result

theorem 3.1 *If* $u \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\bar{\Omega})$*be a solution of the problem* (1.1) - (1.2), Ω *is star shaped and that* a, H, f *and* F *verify the following assumptions*

$$nF(u) - auf(u) \leq 0, \forall x \in \Omega, \tag{3.1}$$

$$(x, \nabla H(x))F(u) \leq 0, \forall x \in \Omega. \tag{3.2}$$

Then, the problem admits only the null solution.

Proof Ω is star shaped, imply that

$$\int_{\partial\Omega} \left(1 - \frac{1}{p(x)}\right)|\nabla u|^{p(x)}(x, \nu)ds \geq 0. \tag{3.3}$$

On the other hand, the condition (3.1) give

$$\begin{aligned}\int_\Omega \Big[\Big(1 - \frac{n}{p(x)} + \frac{(x, \nabla p(x))}{p^2(x)}\left(1 - \ln\left(|\nabla u|^{p(x)}\right)\right) - a\Big)|\nabla u|^{p(x)}\,dx \\ + H(x)(nF(u) - auf(u)) + (x, \nabla H(x))F(u)\Big]\,dx \leq 0\end{aligned} \tag{3.4}$$

(1.4), (3.3) and (3.4), allow to get

$$F(u) = 0 \text{ in } \Omega.$$

So, the problem (1.1) - (1.2) becomes

$$\begin{cases} -div\left(|\nabla u|^{p(x)-2}\nabla u\right) = 0 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{3.5}$$

Multiplying the equation (3.5) by u and integrating over Ω, we get

$$\int_\Omega |\nabla u|^{p(x)}dx = 0.$$

So

$$|\nabla u| = 0,$$

Hence $u = cte = 0$, because $u|_{\partial\Omega} = 0$. ■

theorem 3.2 *If* $u \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\bar{\Omega})$*solution of the problem* (1.1) - (1.3), Ω *is a star shaped and that* a, H, F *and* F *verify the following conditions*

$$nF(u) - auf(u) \leq 0, \forall x \in \Omega, \tag{3.6}$$

$$(x, \nabla H(x))F(u) \leq 0, \forall x \in \Omega. \tag{3.7}$$

$$H(x)F(u) \geq 0, \forall x \in \partial\Omega. \tag{3.8}$$

Therefore, the problem admits only the null solution.

Proof Similar to the proof of theorem 1. ■

theorem 3.3 *If* $u_k \in W_0^{1,p_k(x)}(\Omega) \cap L^\infty(\bar{\Omega})$*solution of the system* (1.6) - (1.7), Ω *is a star shaped and that* a_k, H, f_k *and* F_m *verify the following conditions*

$$nF_m(u_1, ..., u_m) - \sum_{k=1}^{m} a_k u_k f_k(u_1, ..., u_m) \leq 0, \forall x \in \Omega, \tag{3.9}$$

$$(x, \nabla H(x))F_m(u_1, ..., u_m) \leq 0, \forall x \in \Omega. \tag{3.10}$$

So, the system admits only the null solutions.

Proof Ω is a star shaped, implies that

$$\int_{\partial\Omega} \sum_{k=1}^{m} \left(1 - \frac{1}{p_k(x)}\right)|\nabla u_k|^{p_k(x)}(x, \nu)ds \geq 0 \ . \tag{3.11}$$

On the other hand, the conditions (3.9) and (3.10), give

$$\begin{aligned}&\int_\Omega \Bigg[\sum_{k=1}^{m} \left(1 - \frac{n}{p_k(x)} + \frac{(x, \nabla p_k(x))}{p_k^2(x)}\left(1 - \ln\left(|\nabla u_k|^{p_k(x)}\right)\right) - a_k\right)|\nabla u_k|^{p_k(x)} \\ &+H(x)\left(nF_m(u_1, ..., u_m) - \sum_{k=1}^{m} a_k u_k f_k(u_1, ..., u_m)\right) + \\ &+(x, \nabla H(x))F_m(u_1, ..., u_m)\Bigg]dx \leq 0.\end{aligned} \tag{3.12}$$

(1.4), (3.11) and (3.12), allow to have

$$F_m(u_1, ..., u_m) = 0 \text{ in } \Omega.$$

So the system (1.6) - (1.7) becomes

$$\begin{cases} -div\left(|\nabla u_k|^{p_k(x)-2}\nabla u_k\right) = 0 \text{ in } \Omega, 1 \leq k \leq m, \\ u_k = 0 \text{ on } \partial\Omega,\ 1 \leq k \leq m. \end{cases} \tag{3.13}$$

Multiplying (3.13) by u_k and integrating on Ω, we have

$$\int_\Omega |\nabla u_k|^{p_k(x)}dx = 0$$

So

$$|\nabla u_k| = 0$$

Therefore $u_k = cte = 0$, $\forall 1 \leq k \leq m$, because $u_k|_{\partial\Omega} = 0$. ■

theorem 3.4 *If* $u_k \in W_0^{1,p_k(x)}(\Omega) \cap L^\infty(\bar{\Omega})$*solution of the system* (1.6) - (1.8), Ω *is a star shaped and that* a_k, H, f_k *and* F_m *verify the following conditions*

$$nF_m(u_1, ..., u_m) - \sum_{k=1}^{m} a_k u_k f_k(u_1, ..., u_m) \leq 0, \forall x \in \Omega, \tag{3.14}$$

$$(x, \nabla H(x))F_m(u_1, ..., u_m) \leq 0, \forall x \in \Omega, \tag{3.15}$$

$$H(x)F_m(u_1, ..., u_m) \geq 0, \forall x \in \partial\Omega. \tag{3.16}$$

So, the problem admit only the null solution.

Proof Similar to the that of theorem 3. ■

4 Examples

Example 1 *Considering in* $W_0^{1,p(x)}(\Omega) \cap W_0^{1,q}(\bar{\Omega})$*the following problem*

$$\begin{cases} -div\left(|\nabla u|^{p(x)-2}\nabla u\right) = \frac{c}{(1+|x|)^{\mu}}u|u|^{q-1} \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{4.1}$$

where Ω *is a bounded domain of* $\mathbb{R}^n$, c, $\mu > 0$, $q > 1$ *and* $p(x) = \sqrt{1 + |x|^2} > 1$.
By choosing

$$a = \sup_{\Omega}\left(1 - \frac{n+(n-1)|x|^2}{(1+|x|^2)\sqrt{1+|x|^2}}\right),$$

we obtain

$$(x, \nabla H(x))F(u) = \frac{-c\mu|x|}{q(1+|x|)^{\mu+1}}|u|^{q+1} < 0,$$
$$(x, \nabla p(x)) = \frac{|x|^2}{\sqrt{1+|x|^2}} \geq 0,$$
$$nF(u) - auf(u) = \left(\frac{n}{q+1} - a\right)|u|^{q+1} \leq 0 \text{ if } q \geq \frac{n-a}{a}.$$

So, the problem (4.1) *doesn't admit non trivial solutions if*

$$q \geq \frac{n-a}{a}.$$

Example 2 *Considering in* $W_0^{1,p(x)}(\Omega) \cap W_0^{1,\gamma}(\bar{\Omega})$,*the following elliptic system*

$$\begin{cases} -\Delta_{p(x)}u = \frac{c\gamma}{(1+|x|)^{\mu}}u|u|^{\gamma-1}|v|^{\delta} \text{ in } \Omega, \\ -\Delta_{q(x)}v = \frac{c\delta}{(1+|x|)^{\mu}}v|v|^{\delta-1}|u|^{\gamma} \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega \end{cases} \tag{4.2}$$

where Ω *is a bounded domain of* $\mathbb{R}^n$, c, μ, γ, $\delta > 0$ *and* p, $q > 1$.
By choosing

$$a_1 = \sup_{x\in\bar{\Omega}}\left(1 - \frac{n}{p(x)} + \frac{(x,\nabla p(x))}{p^2(x)}\right)$$
and
$$a_2 = \sup_{x\in\bar{\Omega}}\left(1 - \frac{n}{p(x)} + \frac{(x,\nabla q(x))}{q^2(x)}\right)$$

we obtain

$$(x, \nabla H(x))F(u, v) = \frac{-c\mu}{(1+|x|)^{\mu+1}}|u|^{\gamma}|v|^{\delta} < 0,$$
$$nF(u, v) - a_1uf_1(u, v) - a_2vf_2(u, v) = (n - \gamma a_1 - \delta a_2)|u|^{\gamma}|v|^{\delta}$$

So, the system (4.2) *doesn't admit non trivial solutions if*

$$\gamma a_1 + \delta a_2 \geq n$$

Competing interests
The author declares that they have no competing interests.

Received: 12 January 2011 Accepted: 30 November 2011 Published: 30 November 2011

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doi:10.1186/1687-2770-2011-50
Cite this article as: Kamel: Nonexistence of nontrivial solutions for the ***p*(*x*)-**Laplacian equations and systems in unbounded domains of $\mathbb{R}^n$. *Boundary Value Problems* 2011 **2011**:50.