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On positive solutions to equations involving the one-dimensional p -Laplacian

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Abstract

We consider equations involving the one-dimensional p -Laplacian

$$\left(|u'(t)|^{p-2}u'(t)\right)' + \lambda f(u(t)) = 0, \quad t \in (0, 1)$$

with the Dirichlet boundary conditions. By using time map methods, we show how changes of the sign of $f(\cdot)$ lead to multiple positive solutions of the problem for sufficiently large λ .

MSC: 34B10; 34B18

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1 Introduction

Let $f : [0, \infty) \rightarrow \mathbb{R}$ be continuous and change its sign. Let Ω be an open subset of \mathbb{R}^N with smooth boundary $\partial\Omega$. The semi-positone problems and their special cases

$$\Delta u + \lambda f(u) = 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega \quad (1.1)$$

and

$$u''(t) + \lambda f(u(t)) = 0, \quad t \in (0, 1), \quad u(0) = u(1) = 0 \quad (1.2)$$

(and their finite difference analogues) have been extensively studied since early 1980s. Several different approaches such as variational methods, bifurcation theory, lower and upper solutions method and quadrature arguments have been successfully applied to show the existence of multiple solutions. See Brown and Budin [1], Peitgen *et al.* [2], Peitgen and Schmitt [3], Hess [4], Ambrosetti and Hess [5], Cosner and Schmitt [6], Dancer and Schmitt [7], Espinoza [8], Anuradha and Shivaji [9], Anuradha *et al.* [10], de Figueiredo [11], Lin and Pai [12], Clément and Sweers [13] and the references therein.

Very recently, Loc and Schmitt [14] considered the problem

$$\Delta_p u + \lambda f(u) = 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \quad (1.3)$$

where Δ_p is the p -Laplace operator for $p \in (1, \infty)$. They assumed that the nonlinearity f is a continuous function on \mathbb{R} , $f(0) \geq 0$, and there exist $0 < a_1 < b_1 < a_2 < b_2 < \dots < b_{m-1} < a_m$

such that $f \leq 0$ on (a_k, b_k) and $f \geq 0$ on (b_k, a_{k+1}) for every $k = 1, \dots, m-1$. They proved that, for λ sufficiently large, if

$$\int_{a_k}^{a_{k+1}} f(s) ds > 0 \quad \text{for all } k \in \{1, \dots, m-1\}, \quad (1.4)$$

then the problem (1.3) has at least $m-1$ positive bounded solutions u_1, \dots, u_{m-1} which belong to the Sobolev space $W_0^{1,p}(\Omega)$ and are such that $\|u\|_\infty \in (a_k, a_{k+1}]$ for each $k \in \{1, \dots, m-1\}$, where

$$\|u\|_\infty = \max\{|u(x)| \mid x \in \bar{\Omega}\}.$$

In the special case that $p = 2$ and $N = 1$, Brown and Budin [1] applied the quadrature arguments to get the following more detailed results.

Theorem A [1, Theorem 3.8] *Assume that*

(H1) $f \in C^1[0, \infty)$;

(H2) $f(0) > 0$;

(H3) *There exists $a_1, \dots, a_n \in \mathbb{R}$ such that $0 < a_1 < a_2 < \dots < a_n$ and $f(a_i) \leq 0$ for $i = 1, 2, \dots, n$;*

(H4) *If $F(u) = \int_0^u f(s) ds$, there exist $b_1, \dots, b_{n-1} \in \mathbb{R}$ with $a_1 < b_1 < a_2 < b_2 < \dots < a_{n-1} < b_{n-1} < a_n$ such that $f(b_i) > 0$ and $F(b_i) > F(u)$ for $0 \leq u \leq b_i$, $i = 1, 2, \dots, n-1$.*

Then:

(a) *For all $\lambda > 0$, there exists a solution (λ, u) of (1.2).*

(b) *If $\lambda > \inf\{\lambda(\rho) : \rho \in (\alpha_i, \beta_i)\}$, there exist at least two solutions (λ, u) of (1.2) such that*

$$\alpha_i < \|u\|_\infty < \beta_i, \quad i = 1, 2, \dots, n-1,$$

where

$$\beta_i = \inf\{u > b_i \mid f(u) = 0\}, \quad \alpha_i = \inf\{u \mid (u, \beta_i) \subseteq S\} \quad (1.5)$$

and

$$S = \{u \mid u > 0, f(u) > 0, F(u) > F(s) \text{ for all } s : 0 \leq s < u\}. \quad (1.6)$$

(c) *If (λ, u) is any solution of (1.2) such that $\alpha_i < \|u\|_\infty < \beta_i$, then*

$$\lambda > 4\alpha_i k^{-1},$$

where $k = \sup\{|f(u)| : 0 \leq u \leq \beta_i\}$.

Of course the natural question is whether or not the similar results still hold for the corresponding problem involving the one-dimensional p -Laplacian

$$\begin{aligned} &(|u'(t)|^{p-2} u'(t))' + \lambda f(u(t)) = 0, \quad t \in (0, 1), \\ &u(0) = u(1) = 0. \end{aligned} \quad (1.7)$$

We shall answer these questions in the affirmative if $p \in (1, 2]$. More precisely, we get the following theorem.

Theorem 1.1 *Let $p \in (1, 2]$ and let (H1), (H3), (H4) hold. Assume that*

(H2') either $f(0) > 0$ or $f(0) = 0$ and

$$f_0 = \lim_{s \rightarrow 0^+} \frac{f(s)}{s^{p-1}} > 0. \quad (1.8)$$

Then:

- (a) *For all $\lambda > \frac{\lambda_1}{f_0}$, there exists a solution (λ, u) of (1.7), and λ_1 is the least eigenvalue of BVP*

$$\begin{aligned} & \left(|u'(t)|^{p-2} u'(t) \right)' + \lambda |u(t)|^{p-2} u(t) = 0, \quad t \in (0, 1), \\ & u(0) = u(1) = 0. \end{aligned} \quad (1.9)$$

- (b) *If $\lambda > \inf\{\lambda(\rho) : \rho \in (\alpha_i, \beta_i)\}$, there exist at least two solutions (λ, u) of (1.7) such that*

$$\alpha_i < \|u\|_\infty < \beta_i, \quad i = 1, 2, \dots, n-1.$$

- (c) *If (λ, u) is any solution of (1.7) such that $\alpha_i < \|u\|_\infty < \beta_i$, then*

$$\lambda > \left(\frac{\alpha_i}{C_*} \right)^{p-1},$$

where

$$C_* = \frac{p-1}{p} \cdot \left(\frac{1}{2} \right)^{\frac{p}{p-1}} \left(\sup_{s \in [0, \beta_i]} |f(s)| \right)^{\frac{1}{p-1}}. \quad (1.10)$$

We shall apply the time map method to show how changes of the sign of $f(\cdot)$ lead to multiple positive solutions of (1.7) for sufficiently large λ .

In the following, we extend f so that $f(u) > 0$ for all $u < 0$, then all the solutions of (1.7) are positive on $(0, 1)$.

2 Preliminaries

To prove our main results, we use the uniqueness results due to Reichel and Walter [15] on the initial value problem

$$\begin{aligned} & \left(|u'(t)|^{p-2} u'(t) \right)' + \lambda f(u(t)) = 0, \quad t \in (0, 1), \\ & u(a) = b, \quad u'(a) = d, \end{aligned} \quad (2.1)$$

where $a \in [0, 1]$ and $b, d \in \mathbb{R}$.

Lemma 2.1 *Let (H1) hold. If $a \in (0, 1]$ and $d \neq 0$, then the initial value problem (2.1) has a unique local solution. The extension $u(t)$ remains unique as long as $u'(t) \neq 0$.*

Proof It is an immediate consequence of Reichel and Walter [15, Theorem 2]. \square

Lemma 2.2 Let (H1) hold. Let $a \in (0, 1)$, and let $\rho \in (0, \infty)$ be such that

$$f(\rho) \neq 0.$$

Then the initial value problem

$$\begin{aligned} & \left(|u'(t)|^{p-2} u'(t) \right)' + \lambda f(u(t)) = 0, \quad t \in (0, 1), \\ & u(a) = \rho, \quad u'(a) = 0 \end{aligned} \quad (2.2)$$

has a unique local solution.

Proof (H1) implies that f is locally Lipschitzian. This together with the assumption $f(\rho) \neq 0$ and using [15, (iii) and (v) in the case (β) of Theorem 4] yields that (2.2) has a unique solution in some neighborhood of a . \square

Lemma 2.3 Let $g: \mathbb{R} \rightarrow \mathbb{R}$ be continuous. Let u be a solution of the equation

$$\left(|u'(t)|^{p-2} u'(t) \right)' + g(u(t)) = 0, \quad t \in (0, 1) \quad (2.3)$$

with $\|u\|_{\infty} = \rho \in \mathcal{S}$. Let $x_0 \in (0, 1)$ be such that $u'(x_0) = 0$. Then

$$u(x_0 - t) \equiv u(x_0 + t), \quad t \in (0, \min\{x_0, 1 - x_0\}). \quad (2.4)$$

Proof Since g is independent of t , both $u(x_0 - t)$ and $u(x_0 + t)$ satisfy the initial value problem

$$\begin{cases} \left(|w'(t)|^{p-2} w'(t) \right)' + g(w(t)) = 0, & t \in (0, \min\{x_0, 1 - x_0\}), \\ w(0) = u(x_0), \\ w'(0) = 0. \end{cases} \quad (2.5)$$

By Lemmas 2.1 and 2.2, (2.5) has a unique solution defined on $t \in (0, \min\{x_0, 1 - x_0\})$. Therefore, (2.4) is true. \square

Lemma 2.4 Let (λ, u) be a positive solution of the problem

$$\left(|u'(t)|^{p-2} u'(t) \right)' + \lambda f(u(t)) = 0, \quad t \in (0, 1), \quad (2.6)$$

$$u(0) = u(1) = 0 \quad (2.7)$$

with $\|u\|_{\infty} = \rho \in \mathcal{S}$ and $\lambda > 0$. Let $x_0 \in (0, 1)$ be such that $u'(x_0) = 0$. Then

- (a) $x_0 = \frac{1}{2}$;
- (b) x_0 is the unique point on which u attains its maximum;
- (c) $u'(t) > 0$, $t \in (0, \frac{1}{2})$.

Proof (a) Suppose on the contrary that $x_0 \neq \frac{1}{2}$, say $x_0 > \frac{1}{2}$, then

$$0 = u(1) = u(1 - 2x_0).$$

However, this is impossible since $1 - 2x_0 \in (0, 1)$ and $u > 0$ in $(0, 1)$. Therefore $x_0 = \frac{1}{2}$.

(b) Suppose on the contrary that there exists $x_1 \in (0, 1)$ with $x_1 \neq x_0$ and

$$u(x_1) = u(x_0) =: \rho.$$

We may assume that $x_1 < x_0$. The other case can be treated in a similar way.

If $u(t) \equiv u(x_0)$ in the interval (x_1, x_0) , then Lemma 2.3 yields that

$$u(t) \equiv u(x_0) = \rho > 0, \quad t \in (0, 1).$$

This contradicts the boundary conditions $u(0) = u(1) = 0$. Therefore, $u(t) \not\equiv u(x_0)$ in any subinterval of $(0, 1)$.

So, there exists $x_* \in (x_1, x_0)$, such that

$$u(x_*) = \min\{u(t) \mid t \in (x_1, x_0)\}.$$

Obviously,

$$0 < u(x_*) < \rho, \quad u'(x_*) = 0.$$

Multiplying both sides of the equation in (2.6) by u' and integrating from t to x_0 , we get that

$$|u'(t)|^p = \lambda \frac{p}{p-1} [F(\rho) - F(u(t))], \quad t \in \left[0, \frac{1}{2}\right] \quad (2.8)$$

and subsequently,

$$0 = |u'(x_*)|^p = \lambda \frac{p}{p-1} [F(\rho) - F(u(x_*))].$$

This contradicts the facts that $\rho \in \mathcal{S}$ and $u(x_*) < \rho$. Therefore,

$$u\left(\frac{1}{2}\right) > u(t), \quad t \in \left[0, \frac{1}{2}\right).$$

Similarly, we can prove that

$$u\left(\frac{1}{2}\right) > u(t), \quad t \in \left(\frac{1}{2}, 1\right].$$

(c) Suppose on the contrary that there exists $\hat{x} \in (0, \frac{1}{2})$ with $u'(\hat{x}) = 0$. Then

$$u(\hat{x}) < \rho.$$

This together with (2.8) implies that

$$0 = |u'(\hat{x})|^p = \lambda \frac{p}{p-1} [F(\rho) - F(u(\hat{x}))].$$

This contradicts the facts that $\rho \in \mathcal{S}$ and $u(\hat{x}) < \rho$. □

3 Proof of the main results

To prove Theorem 1.1, we need the following preliminary results.

Lemma 3.1 *For any $\rho \in \mathcal{S}$, there exists a unique $\lambda > 0$ such that*

$$\left(|u'(t)|^{p-2}u'(t)\right)' + \lambda f(u(t)) = 0, \quad t \in (0, 1), \quad (3.1)$$

$$u(0) = u(1) = 0 \quad (3.2)$$

has a positive solution (λ, u) with $\|u\|_\infty = \rho$. Moreover, $\rho \rightarrow \lambda(\rho)$ is a continuous function on \mathcal{S} .

Proof By Lemma 2.4, (λ, u) is a positive solution of (3.1), (3.2) if and only if (λ, u) is a positive solution of

$$\left(|u'(t)|^{p-2}u'(t)\right)' + \lambda f(u(t)) = 0, \quad t \in \left(0, \frac{1}{2}\right), \quad (3.3)$$

$$u(0) = u'\left(\frac{1}{2}\right) = 0. \quad (3.4)$$

Suppose that (λ, u) is a solution of (3.3), (3.4) with $\|u\|_\infty = \rho$. Then

$$|u'(t)|^p = \lambda \frac{p}{p-1} (F(\rho) - F(u(t))), \quad t \in \left[0, \frac{1}{2}\right]$$

and so

$$t \left(\frac{p}{p-1} \lambda\right)^{1/p} = \int_0^{u(t)} (F(\rho) - F(s))^{-1/p} ds, \quad t \in \left[0, \frac{1}{2}\right]. \quad (3.5)$$

Putting $t = \frac{1}{2}$, we obtain

$$\lambda^{1/p} = 2 \left(\frac{p-1}{p}\right)^{1/p} \int_0^\rho (F(\rho) - F(s))^{-1/p} ds. \quad (3.6)$$

Hence λ (if exists) is uniquely determined by ρ .

If $\rho \in \mathcal{S}$, we define $\lambda(\rho)$ by (3.6) and $u(t)$ by (3.5). It is straightforward to verify that u is twice differentiable, u satisfies (3.3), (3.4), $u > 0$ in $(0, 1)$ and $u(1/2) = \rho$. The continuity of $\lambda(\cdot)$ is implied by (3.6) and this completes the proof. \square

Let

$$r = \inf\{u > 0 : f(u) = 0\}.$$

Then $(0, r) \in \mathcal{S}$.

Lemma 3.2 *Let (H1) and (H2') hold, and let $p \in (1, \infty)$. Then*

$$\lim_{\rho \rightarrow 0} \lambda(\rho) = \frac{\lambda_1}{f_0}, \quad \lim_{\rho \rightarrow r^-} \lambda(\rho) = \infty,$$

where λ_1 is the least eigenvalue of (1.9).

Proof We only deal with $\lim_{\rho \rightarrow 0} \lambda(\rho) = \frac{\lambda_1}{f_0}$. The other one can be treated by the same method.

To this end, we divide the proof into two cases.

Case 1. We show that $f_0 = \infty$ implies $\lim_{\rho \rightarrow 0} \lambda(\rho) = 0$.

In this case, for any $M > 0$, there is a positive number R such that $f(w) > Mw^{p-1}$ for $0 \leq w \leq R$. Thus, if $\rho < R$, then

$$\begin{aligned} F(\rho) - F(w) &= \int_w^\rho f(v) dv \\ &\geq \frac{M}{p} (\rho^p - w^p) \end{aligned}$$

for $0 \leq w \leq \rho$. From (3.6), we have that for any $\rho \leq R$,

$$\begin{aligned} [\lambda(\rho)]^{\frac{1}{p}} &= 2 \left(\frac{p-1}{p} \right)^{\frac{1}{p}} \int_0^\rho \frac{dw}{[F(\rho) - F(w)]^{1/p}} \\ &\leq 2 \left(\frac{p-1}{p} \right)^{1/p} \left(\frac{p}{M} \right)^{1/p} \int_0^\rho \frac{dw}{[\rho^p - w^p]^{1/p}} \\ &\leq 2 \left(\frac{p-1}{M} \right)^{1/p} \int_0^1 \frac{d\frac{w}{\rho}}{[1 - (\frac{w}{\rho})^p]^{1/p}} \\ &\leq 2 \left(\frac{p-1}{M} \right)^{1/p} \left(\frac{1}{p-1} \right)^{1/p} \int_0^{(p-1)^{1/p}} \frac{ds}{[1 - \frac{s^p}{p-1}]^{1/p}} \\ &\leq 2 \left(\frac{1}{M} \right)^{1/p} \int_0^{(p-1)^{1/p}} \frac{ds}{[1 - \frac{s^p}{p-1}]^{1/p}} \\ &= \left(\frac{1}{M} \right)^{1/p} \pi_p, \end{aligned}$$

where

$$\pi_p := 2 \int_0^{(p-1)^{1/p}} \frac{ds}{[1 - \frac{s^p}{p-1}]^{1/p}},$$

see Zhang [16]. Hence

$$\lim_{\rho \rightarrow 0} \lambda(\rho) = 0.$$

Case 2. We show that $f_0 = m$ for some $m \in (0, \infty)$ implies that

$$\lim_{\rho \rightarrow 0} \lambda(\rho) = \frac{p-1}{pm} \tau_p^p = \frac{(\pi_p)^p}{f_0}, \quad (3.7)$$

where

$$\tau_p = 2 \int_0^1 \left[\frac{p}{1-v^p} \right]^{1/p} dv, \quad p > 1. \quad (3.8)$$

In fact, (3.6) yields

$$\begin{aligned} [m\lambda(\rho)]^{1/p} &= 2 \left[\frac{m(p-1)}{p} \right]^{\frac{1}{p}} \int_0^\rho \frac{dw}{[F(\rho) - F(w)]^{1/p}} \\ &= 2 \left(\frac{p-1}{p} \right)^{1/p} \cdot \int_0^1 \left[\frac{p}{1-v^p} \right]^{1/p} dv \\ &\quad - 2 \left(\frac{p-1}{p} \right)^{1/p} \int_0^1 \left[\frac{p}{1-v^p} \right]^{1/p} \frac{[1 + \gamma(\rho, v)]^{1/p} - 1}{[1 + \gamma(\rho, v)]^{1/p}} dv \end{aligned} \quad (3.9)$$

for $p > 1$, where

$$\gamma(\rho, v) = \frac{p}{m} \frac{\int_{\rho v}^\rho [f(w) - mw^{p-1}] dw}{\rho^p(1-v^p)}.$$

We will show that the last integral in (3.9) converges to zeros as $\rho \rightarrow 0$.

For $0 \leq v \leq \frac{1}{2}$, using l'Hospital's rule, it follows that as $\rho \rightarrow 0$,

$$\begin{aligned} |\gamma(\rho, v)| &= \frac{p}{m} \frac{\int_{\rho v}^\rho |f(w) - mw^{p-1}| dw}{\rho^p(1-v^p)} \\ &\leq \frac{p}{m} \frac{\int_0^\rho |f(w) - mw^{p-1}| dw}{\rho^p(1-v^p)} \\ &\rightarrow \frac{p}{m} \frac{|f(\rho) - m\rho^{p-1}|}{p\rho^{p-1}(1-v^p)} \\ &= \frac{p}{m} \frac{|\frac{f(\rho)}{\rho^{p-1}} - m|}{p(1-v^p)} \\ &\rightarrow 0. \end{aligned}$$

For $\frac{1}{2} \leq v \leq 1$,

$$\lim_{\rho \rightarrow 0} |\gamma(\rho, v)| = \limsup_{\rho \rightarrow 0} \left\{ \left| \frac{f(w) - mw^{p-1}}{w^{p-1}} \right| : \frac{1}{2}\rho \leq w \leq \rho \right\} \frac{p}{m\rho^p(1-v^p)} \int_{\rho v}^\rho w^{p-1} dw = 0$$

uniformly in v . Therefore, (3.9) implies

$$\lim_{\rho \rightarrow 0} [m\lambda(\rho)]^{1/p} = 2 \left(\frac{p-1}{p} \right)^{\frac{1}{p}} \int_0^1 \left[\frac{p}{1-v^p} \right]^{1/p} dv. \quad (3.10)$$

Therefore, (3.7) holds. \square

From the definitions of α_i and β_i , we have that $\alpha_i \leq \alpha_i < \beta_i \leq \alpha_{i+1}$ and $(\alpha_i, \beta_i) \subseteq \mathcal{S}$ for $i = 1, 2, \dots, n-1$. Moreover, we have the following.

Lemma 3.3 *Let $p \in (1, 2]$. Then*

- (i) $\lim_{\rho \rightarrow \alpha_i^+} \lambda(\rho) = \infty$;
- (ii) $\lim_{\rho \rightarrow \beta_i^-} \lambda(\rho) = \infty$.

Proof (i) Suppose firstly that $f(\alpha_i) > 0$. Since S is open, $\alpha_i \notin S$ and so there exists $k : 0 < k < \alpha_i$ such that

$$F(\alpha_i) = F(k).$$

Clearly k must be a local maximum for F and so $f(k) = 0$. If $M = \max\{|f'(u)| : 0 \leq u \leq b_i\}$, then

$$f(u) \leq M|u - k|, \quad 0 \leq u \leq b_i.$$

Let

$$N = \max\{|f(u)| : 0 \leq u \leq b_i\}.$$

Then if $\alpha_i < \rho < b_i$,

$$\begin{aligned} F(\rho) - F(u) &= F(\rho) - F(\alpha_i) + F(k) - F(u) \\ &= (\rho - \alpha_i)f(\xi) + (k - u)f(\eta), \\ &\quad \text{where } \xi \in (\alpha_i, \rho) \text{ and } \eta \in (k, u) \\ &\leq N(\rho - \alpha_i) + M(k - u)^2. \end{aligned}$$

Hence

$$\begin{aligned} (\lambda(\rho))^{1/p} &= 2 \left(\frac{p-1}{p} \right)^{1/p} \int_0^\rho (F(\rho) - F(s))^{-1/p} ds \\ &\geq 2 \left(\frac{p-1}{p} \right)^{1/p} \int_0^{\alpha_i} (N(\rho - \alpha_i) + M(k - u)^2)^{-1/p} du \\ &= \int_0^{\alpha_i} H_\rho(u) du. \end{aligned}$$

As $\rho \rightarrow \alpha_i^+$, $H_\rho(u) = 2 \left(\frac{p-1}{p} \right)^{1/p} (N(\rho - \alpha_i) + M(k - u)^2)^{-1/p}$ is a nondecreasing sequence of measurable functions. Therefore, by the monotone convergence theorem and the assumption $p \geq 2$, it follows that

$$\begin{aligned} \lim_{\rho \rightarrow \alpha_i^+} [\lambda(\rho)]^{1/p} &\geq \lim_{\rho \rightarrow \alpha_i^+} \int_0^{\alpha_i} H_\rho(u) du \\ &= \int_0^{\alpha_i} 2 \left(\frac{p-1}{p} \right)^{1/p} M^{-1/p} [k - u]^{-2/p} du = \infty \end{aligned}$$

since $k \in (0, \alpha_i)$.

Suppose next that $f(\alpha_i) = 0$. Then $F'(\alpha_i) = 0$.

Since

$$\begin{aligned} F(\alpha_i) - F(u) &= f(\eta)(\alpha_i - u), \quad \text{where } \eta \in (u, \alpha_i), \\ |f(u)| &= |f(u) - f(\alpha_i)| \leq M|u - \alpha_i|. \end{aligned}$$

Thus

$$\begin{aligned}\int_0^{\alpha_i} [F(\alpha_i) - F(u)]^{-1/p} du &\geq \int_0^{\alpha_i} [M|\alpha_i - u|^2]^{-1/p} du \\ &= \int_0^{\alpha_i} M^{-1/p} |\alpha_i - u|^{-2/p} du = \infty.\end{aligned}$$

(ii) Let $K_1 = \max\{|f(u)| : 0 \leq u \leq \beta_i\}$ and $K_2 = \max\{|f'(u)| : 0 \leq u \leq \beta_i\}$. Since $f(\beta_i) = 0$,

$$f(u) \leq K_2|u - \beta_i|, \quad 0 \leq u < \beta_i. \quad (3.11)$$

Hence, if $0 \leq u \leq \rho < \beta_i$, then it follows from (3.11) that

$$\begin{aligned}F(\rho) - F(u) &= F(\rho) - F(\beta_i) + F(\beta_i) - F(u) \\ &= (\rho - \beta_i)f(\xi) + (\beta_i - u)f(\eta), \\ &\quad \text{where } \xi \in (\rho, \beta_i), \eta \in (u, \beta_i) \\ &\leq K_1(\beta_i - \rho) + K_2(\beta_i - u)^2.\end{aligned}$$

Hence, if $0 < \rho < \beta_i$,

$$\begin{aligned}(\lambda(\rho))^{1/p} &\geq 2\left(\frac{p-1}{p}\right)^{1/p} \int_0^\rho (K_1(\beta_i - \rho) + K_2(\beta_i - u)^2)^{-1/p} du \\ &= \int_0^{\beta_i} G_\rho(u) du,\end{aligned}$$

where $G_\rho(u) = 2\left(\frac{p-1}{p}\right)^{1/p} (K_1(\beta_i - \rho) + K_2(\beta_i - u)^2)^{-1/p} \chi_{[0, \rho]}$ and $\chi_{[0, \rho]}$ denotes the characteristic function of $[0, \rho]$. As G_ρ is a nondecreasing sequence of measurable functions, by the monotone convergence theorem

$$\begin{aligned}\lim_{\rho \rightarrow \beta_i^-} [\lambda(\rho)]^{1/p} &\geq \lim_{\rho \rightarrow \beta_i^-} \int_0^{\beta_i} G_\rho(u) du \\ &= \int_0^{\beta_i} 2\left(\frac{p-1}{p}\right)^{1/p} K_2^{-1/p} |\beta_i - u|^{-2/p} du = \infty.\end{aligned}$$

□

Proof of Theorem 1.1 (a) follows from the continuity of $\rho \rightarrow \lambda(\rho)$ and Lemma 3.2.

(b) follows from the continuity of $\rho \rightarrow \lambda(\rho)$ and Lemma 3.3.

(c) (λ, u) is any solution of (3.1), (3.2) if and only if

$$u(t) = \int_0^t \left(\int_\tau^{1/2} \lambda f(u(s)) ds \right)^{\frac{1}{p-1}} d\tau, \quad t \in \left[0, \frac{1}{2}\right].$$

Hence

$$\begin{aligned}|u(t)| &\leq \lambda^{\frac{1}{p-1}} \int_0^{1/2} \left(\int_\tau^{1/2} |f(u(s))| ds \right)^{\frac{1}{p-1}} d\tau \\ &\leq \lambda^{\frac{1}{p-1}} \frac{p-1}{p} \cdot \left(\frac{1}{2}\right)^{\frac{p}{p-1}} \left(\sup_{y \in [0, 1]} |f(u(y))| \right)^{\frac{1}{p-1}}.\end{aligned}$$

Now, if $\alpha_i < \|u\|_\infty < \beta_i$, then

$$\alpha_i \leq \lambda^{\frac{1}{p-1}} \frac{p-1}{p} \cdot \left(\frac{1}{2}\right)^{\frac{p}{p-1}} \left(\sup_{s \in [0, \beta_i]} |f(s)|\right)^{\frac{1}{p-1}}$$

and so

$$\lambda > \left(\frac{\alpha_i}{C_*}\right)^{p-1}.$$

□

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

RM completed the main study, carried out the results of this article. YL drafted the manuscript. AOMA checked the proofs and verified the calculation. All the authors read and approved the final manuscript.

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