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Multiple positive solutions of nonlinear BVPs for differential systems involving integral conditions

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Abstract

In this paper, we consider the following system of nonlinear third-order nonlocal boundary value problems (BVPs for short):

$$\begin{cases} -u'''(t) = f(t, v(t), v'(t)), & t \in (0,1), \\ -v'''(t) = g(t, u(t), u'(t)), & t \in (0,1), \\ u(0) = 0, \quad au'(0) - bu''(0) = \alpha[u], & cu'(1) + du''(1) = \beta[u], \\ v(0) = 0, \quad av'(0) - bv''(0) = \alpha[v], & cv'(1) + dv''(1) = \beta[v], \end{cases}$$

where $f, g \in C([0,1] \times R^+ \times R^+, R^+)$, $\alpha[u] = \int_0^1 u(t)\,dA(t)$ and $\beta[u] = \int_0^1 u(t)\,dB(t)$ are linear functionals on $C[0,1]$ given by Riemann-Stieltjes integrals and are not necessarily positive functionals; a, b, c, d are nonnegative constants with $\rho := ac + ad + bc > 0$. By using the Guo-Krasnoselskii fixed point theorem, some sufficient conditions are obtained for the existence of at least one or two positive solutions and nonexistence of positive solutions to the above problem. Two examples are also included to illustrate the main results.

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Keywords: differential system; multiple positive solution; integral conditions

1 Introduction

The theory of BVPs with integral boundary conditions for ordinary differential equations arises in different areas of applied mathematics and physics. For example, heat conduction, chemical engineering, underground water flow, thermo-elasticity, and plasma physics can be reduced to nonlocal problems with integral boundary conditions. Moreover, BVPs with Riemann-Stieltjes integral boundary condition (BC for short) have been considered recently as both multipoint and Riemann integral type BCs are treated in a single framework. For more comments on Stieltjes integral BC and its importance, we refer the reader to the papers by Webb and Infante [1–3] and their other related works.

In recent years, third-order nonlocal BVPs have received much attention from many authors; see, for example [4–14]. It is worth mentioning that Sun and Li [12] studied the third-order BVP with integral boundary conditions

$$\begin{cases} u'''(t) + f(t, u(t), u'(t)) = 0, & 0 < t < 1, \\ u(0) = u'(0) = 0, & u'(1) = \int_0^1 g(t)u'(t)\,dt. \end{cases} \tag{1.1}$$

Their main tool was the Guo-Krasnoselskii fixed point theorem. Recently, we [14] were concerned with the existence of a monotone positive solution for the third-order BVP

$$\begin{cases} u'''(t) + f(t, u(t), u'(t)) = 0, & 0 < t < 1, \\ u(0) = 0, \\ au'(0) - bu''(0) = \alpha[u], \\ cu'(1) + du''(1) = \beta[u], \end{cases} \tag{1.2}$$

by applying monotone iterative techniques, where $f \in C([0,1] \times R^+ \times R^+, R^+)$, $\alpha[u] = \int_0^1 u(t)\,dA(t)$ and $\beta[u] = \int_0^1 u(t)\,dB(t)$ are linear functionals on $C[0,1]$ given by Riemann-Stieltjes integrals.

Furthermore, motivated by the wide applications of systems of differential equations in biomathematics, the study of systems of BVPs has received increased interest; see [15–28] and the references therein. In particular, Henderson and Luca [17] established the existence of positive solutions for the system of BVPs with multi-point boundary conditions

$$\begin{cases} -u''(t) + \lambda c(t) f(u(t), v(t)) = 0, & t \in (0, T), \\ -v''(t) + \mu d(t) g(u(t), v(t)) = 0, & t \in (0, T), \\ \alpha u(0) - \beta u'(0) = 0, & u(T) = \sum_{i=1}^{m-2} a_i u(\xi_i), \quad m \geq 3, \\ \gamma v(0) - \delta v'(0) = 0, & v(T) = \sum_{i=1}^{n-2} b_i v(\eta_i), \quad m \geq 3 \end{cases} \tag{1.3}$$

by applying the fixed point index theory.

Yang [27] studied the existence of positive solutions for the system of second-order nonlocal BVPs

$$\begin{cases} -u''(t) = f(t, u, v), \\ -v''(t) = g(t, u, v), \\ u(0) = v(0) = 0, \\ u(1) = H_1(\int_0^1 u(\tau)\,d\alpha(\tau)), \\ v(1) = H_2(\int_0^1 v(\tau)\,d\beta(\tau)) \end{cases} \tag{1.4}$$

by using fixed point index theory in a cone.

Infante and Pietramala [19] studied the existence of positive solutions for a system of perturbed Hammerstein integral equations by fixed point index theory for compact maps and illustrated their theory by studying the following system of BVPs:

$$\begin{cases} u''(t) + g_1(t) f_1(t, u(t), v(t)) = 0, & 0 < t < 1, \\ v''(t) + g_2(t) f_2(t, u(t), v(t)) = 0, & 0 < t < 1, \\ u(0) = H_{11}(\beta_{11}[u]), & u(1) = H_{12}(\beta_{12}[u]), \\ v(0) = H_{21}(\beta_{21}[v]), & v'(1) = H_{22}(\beta_{22}[v]). \end{cases} \tag{1.5}$$

The result was quite general and covered a wide class of systems of BVPs. Here $\beta_{ij}[w]$ was of the form $\beta_{ij}[w] = \int_0^1 w(t)\,dB_{ij}(t)$ involving positive Riemann-Stieltjes measures.

Inspired greatly by the above-mentioned excellent works, in this paper, we are concerned with the following system of third-order BVPs:

$$\begin{cases} -u'''(t) = f(t, v(t), v'(t)), & t \in (0,1), \\ -v'''(t) = g(t, u(t), u'(t)), & t \in (0,1), \\ u(0) = 0, \quad au'(0) - bu''(0) = \alpha[u], & cu'(1) + du''(1) = \beta[u], \\ v(0) = 0, \quad av'(0) - bv''(0) = \alpha[v], & cv'(1) + dv''(1) = \beta[v], \end{cases} \tag{1.6}$$

where $f, g \in C([0,1] \times R^+ \times R^+, R^+)$, $\alpha[u] = \int_0^1 u(t)\,dA(t)$ and $\beta[u] = \int_0^1 u(t)\,dB(t)$ are linear functionals on $C[0,1]$ given by Riemann-Stieltjes integrals with signed measures; a, b, c, d are nonnegative constants with $\rho := ac + ad + bc > 0$. To the best of our knowledge, the study of existence of positive solutions of third-order differential systems (1.6) has not been done.

A vector $(u, v) \in C^2(0,1) \times C^2(0,1)$ is said to be a positive solution of BVP (1.6) if and only if u, v satisfy BVP (1.6) and u, v are positive on $(0,1)$. The proof of our main results is based on the well-known Guo-Krasnoselskii fixed point theorem, which we present now.

Theorem 1.1 *Let E be a Banach space, $K \subset E$ be a cone, and Ω_1 and Ω_2 be bounded open subsets of E with $0 \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$. Assume that $A : K \cap (\overline{\Omega_2} \setminus \Omega_1) \to K$ is a completely continuous operator such that either*

(i) $\|Au\| \le \|u\|$ *for* $u \in K \cap \partial\Omega_1$ *and* $\|Au\| \ge \|u\|$ *for* $u \in K \cap \partial\Omega_2$; *or*

(ii) $\|Au\| \ge \|u\|$ *for* $u \in K \cap \partial\Omega_1$ *and* $\|Au\| \le \|u\|$ *for* $u \in K \cap \partial\Omega_2$.

Then A has a fixed point in $K \cap (\overline{\Omega_2} \setminus \Omega_1)$.

2 Preliminary lemmas

In this section, we adopt the ideas and the method which have been widely used and which are due to Webb and Infante in [1, 2].

In our case, the existence of positive solutions of nonlocal BVP

$$\begin{cases} u'''(t) + f(t, u(t), u'(t)) = 0, & 0 < t < 1, \\ u(0) = 0, \\ au'(0) - bu''(0) = \alpha[u], \\ cu'(1) + du''(1) = \beta[u] \end{cases} \tag{2.1}$$

with two nonlocal boundary terms $\alpha[u]$, $\beta[u]$ can be studied via a perturbed Hammerstein integral equation of the type

$$u(t) = \gamma(t)\alpha[u] + \delta(t)\beta[u] + \int_0^1 G(t,s) f\bigl(s, u(s), u'(s)\bigr)\,ds =: T_1 u(t). \tag{2.2}$$

Here $\gamma(t)$, $\delta(t)$ are linearly independent and given by

$$-\gamma'''(t) = 0, \qquad \gamma(0) = 0, \qquad a\gamma'(0) - b\gamma''(0) = 1, \qquad c\gamma'(1) + d\gamma''(1) = 0,$$

$$-\delta'''(t) = 0, \qquad \delta(0) = 0, \qquad a\delta'(0) - b\delta''(0) = 0, \qquad c\delta'(1) + d\delta''(1) = 1,$$

which imply $\gamma(t) = \frac{2ct+2dt-ct^2}{2\rho}$ and $\delta(t) = \frac{at^2+2bt}{2\rho}$, $t \in [0,1]$. Let $\|\cdot\|_\infty$ be the usual supremum norm in $C[0,1]$. A direct calculation shows that for $t \in [\theta, 1-\theta]$, $0 < \theta < \frac{1}{2}$,

$$\begin{aligned} &\gamma(t) \geq c_1\|\gamma\|_\infty, \qquad \delta(t) \geq c_2\|\delta\|_\infty, \\ &\gamma'(t) \geq d_1\|\gamma'\|_\infty \quad \text{and} \quad \delta'(t) \geq d_2\|\delta'\|_\infty, \end{aligned} \tag{2.3}$$

where $c_1 = \frac{2c\theta+2d\theta-c\theta^2}{c+2d}$, $c_2 = \frac{a\theta^2+2b\theta}{a+2b}$, $d_1 = \frac{c\theta+d}{c+d}$ and $d_2 = \frac{a\theta+b}{a+b}$; $G(t,s)$ is Green's function for the corresponding problem with local terms when $\alpha[u]$ and $\beta[u]$ are identically 0, *i.e.*,

$$G(t,s) = \begin{cases} \frac{(at^2+2bt)(c(1-s)+d)}{2\rho} - \frac{(t-s)^2}{2}, & 0 \leq s \leq t \leq 1, \\ \frac{(at^2+2bt)(c(1-s)+d)}{2\rho}, & 0 \leq t \leq s \leq 1. \end{cases}$$

In the remainder of this paper, we always assume that

(H1) $0 \leq \alpha[\gamma], \beta[\delta] < 1$, $\alpha[\delta], \beta[\gamma] \geq 0$ and $D := (1-\alpha[\gamma])(1-\beta[\delta]) - \alpha[\delta]\beta[\gamma] > 0$;

(H2) A, B are functions of bounded variation, and $\mathcal{K}_A(s), \mathcal{K}_B(s) \geq 0$ for $s \in [0,1]$, where

$$\mathcal{K}_A(s) := \int_0^1 G(t,s)\,dA(t) \quad \text{and} \quad \mathcal{K}_B(s) := \int_0^1 G(t,s)\,dB(t).$$

As shown in Theorem 2.3 in [1], if u is a fixed point of T_1 in (2.2), then u is a fixed point of S, which is now given by

$$\begin{aligned} Su(t) :=\ & \frac{\gamma(t)}{D}\left(\big(1-\beta[\delta]\big)\int_0^1 \mathcal{K}_A(s)f\big(s,u(s),u'(s)\big)\,ds + \alpha[\delta]\int_0^1 \mathcal{K}_B(s)f\big(s,u(s),u'(s)\big)\,ds\right) \\ & + \frac{\delta(t)}{D}\left(\beta[\gamma]\int_0^1 \mathcal{K}_A(s)f\big(s,u(s),u'(s)\big)\,ds\right. \\ & \left. + \big(1-\alpha[\gamma]\big)\int_0^1 \mathcal{K}_B(s)f\big(s,u(s),u'(s)\big)\,ds\right) + \int_0^1 G(t,s)f\big(s,u(s),u'(s)\big)\,ds \\ =:\ & \int_0^1 G_S(t,s)f\big(s,u(s),u'(s)\big)\,ds \end{aligned}$$

in our case. The kernel G_S is Green's function corresponding to BVP (2.1). By Lemma 2.1 and Lemma 2.2 in [14], we can get the following properties of Green's function.

Lemma 2.1 *Let* $\rho := ac + ad + bc > 0$, $c_3 = \frac{\rho\int_0^\theta \Phi(\tau)\,d\tau}{(a+b)(c+d)}$, $0 < \theta < 1$. *Then* $G(t,s)$ *satisfies*

$$\begin{aligned} &G(t,s) \leq \Phi(s) \quad \text{for } t \in [0,1], s \in [0,1], \\ &G(t,s) \geq c_3\Phi(s) \quad \text{for } t \in [\theta, 1-\theta], s \in [0,1], \end{aligned}$$

where $\Phi(s) = \frac{1}{\rho}(b+as)(d+c(1-s))$, $s \in [0,1]$.

Lemma 2.2 *Let* $c_0 = \min\{c_1, c_2, c_3\}$. *Then* $G_S(t,s)$ *satisfies*

$$\begin{aligned} &G_S(t,s) \leq \Phi_1(s) \quad \text{for } t \in [0,1], s \in [0,1], \\ &G_S(t,s) \geq c_0\Phi_1(s) \quad \text{for } t \in [\theta, 1-\theta], s \in [0,1], \end{aligned}$$

where $\Phi_1(s) := \frac{\|\gamma\|_\infty}{D}((1-\beta[\delta])\mathcal{K}_A(s)+\alpha[\delta]\mathcal{K}_B(s)) + \frac{\|\delta\|_\infty}{D}(\beta[\gamma]\mathcal{K}_A(s)+(1-\alpha[\gamma])\mathcal{K}_B(s)) + \Phi(s)$, $s \in [0,1]$.

Lemma 2.3 *Let* $d_0 = \min\{d_1, d_2, d_3\}$, $d_3 = \frac{\rho \min_{t\in[\theta,1-\theta]}\Phi(t)}{(a+b)(c+d)}$, $0 < \theta < 1$, *and*

$$\begin{aligned}\Phi_2(s) := {}& \frac{\|\gamma'\|_\infty}{D}\bigl[\bigl(1-\beta[\delta]\bigr)\mathcal{K}_A(s) + \alpha[\delta]\mathcal{K}_B(s)\bigr] \\ &+ \frac{\|\delta'\|_\infty}{D}\bigl[\beta[\gamma]\mathcal{K}_A(s) + \bigl(1-\alpha[\gamma]\bigr)\mathcal{K}_B(s)\bigr] + \Phi(s).\end{aligned}$$

Then $\frac{\partial G_S(t,s)}{\partial t}$ *satisfies*

$$\begin{aligned}&\frac{\partial G_S(t,s)}{\partial t} \le \Phi_2(s) \quad \textit{for } t \in [0,1], s \in [0,1], \\ &\frac{\partial G_S(t,s)}{\partial t} \ge d_0\Phi_2(s) \quad \textit{for } t \in [\theta, 1-\theta], s \in [0,1].\end{aligned}$$

Proof For $s \in [0,1]$, by the fact

$$\frac{\frac{\partial G(t,s)}{\partial t}}{\Phi(s)} = \begin{cases} \frac{(b+as)(d+c(1-t))}{(b+as)(d+c(1-s))} = \frac{(b+at)(d+c(1-t))}{(b+at)(d+c(1-s))} \ge \frac{\rho\Phi(t)}{(a+b)(c+d)}, & 0 \le s \le t \le 1, \\ \frac{(b+at)(d+c(1-s))}{(b+as)(d+c(1-s))} = \frac{(b+at)(d+c(1-t))}{(b+as)(d+c(1-t))} \ge \frac{\rho\Phi(t)}{(a+b)(c+d)}, & 0 \le t \le s \le 1,\end{cases}$$

so $\frac{\partial G(t,s)}{\partial t} \ge \frac{\rho \min_{t\in[\theta,1-\theta]}\Phi(t)}{(a+b)(c+d)}\Phi(s)$, $t \in [\theta, 1-\theta]$, $s \in [0,1]$, which together with (2.3) shows that $\frac{\partial G_S(t,s)}{\partial t} \ge d_0\Phi_2(s)$ holds. □

Let $E = \{C^1[0,1] : u(0) = 0\}$ equipped with the norm $\|u\| = \max\{\|u\|_\infty, \|u'\|_\infty\}$, where $\|u\|_\infty$ is the usual supremum norm in $C[0,1]$. Similar to Lemma 2.1 in [29], we can get the following lemma.

Lemma 2.4 *If* $u \in E$, *then* $\|u\|_\infty \le \|u'\|_\infty$. *And so, E is a Banach space when it is endowed with the norm* $\|u\| = \|u'\|_\infty$.

Define

$$K = \Bigl\{u \in E : u(t) \ge 0, u'(t) \ge 0, t \in [0,1], \min_{t\in[\theta,1-\theta]} u'(t) \ge d_0\|u\|\Bigr\}.$$

Then it is easy to verify that K is a cone in E.

For $u \in K$, we define

$$\begin{aligned}(Tu)(t) = {}& \int_0^1 G_S(t,s) f\biggl(s, \int_0^1 G_S(s,\tau) g\bigl(\tau, u(\tau), u'(\tau)\bigr)\,d\tau, \\ & \int_0^1 \frac{\partial G_S(s,\tau)}{\partial s} g\bigl(\tau, u(\tau), u'(\tau)\bigr)\,d\tau\biggr)\,ds, \quad t \in [0,1].\end{aligned}$$

It is easy to see that if x is a fixed point of T in K, then BVP (1.6) has one solution (u, v), where

$$\begin{cases} u(t) = x(t), \\ v(t) = \int_0^1 G_S(t,s) g(s, x(s), x'(s))\,ds.\end{cases}$$

Lemma 2.5 $T: K \to K$.

Proof It is obvious that $(Tu)(t) \geq 0$ and $(Tu)'(t) \geq 0$. Moreover, for $t \in [0,1]$, by Lemma 2.3, we have

$$\begin{aligned}(Tu)'(t) &= \int_0^1 \frac{\partial G_S(t,s)}{\partial t} f\biggl(s, \int_0^1 G_S(s,\tau) g\bigl(\tau, u(\tau), u'(\tau)\bigr)\,d\tau, \\ &\qquad \int_0^1 \frac{\partial G_S(s,\tau)}{\partial s} g\bigl(\tau, u(\tau), u'(\tau)\bigr)\,d\tau\biggr)\,ds \\ &\leq \int_0^1 \Phi_2(s) f\biggl(s, \int_0^1 G_S(s,\tau) g\bigl(\tau, u(\tau), u'(\tau)\bigr)\,d\tau, \\ &\qquad \int_0^1 \frac{\partial G_S(s,\tau)}{\partial s} g\bigl(\tau, u(\tau), u'(\tau)\bigr)\,d\tau\biggr)\,ds\end{aligned}$$

and hence

$$\begin{aligned}\|Tu\| &\leq \int_0^1 \Phi_2(s) f\biggl(s, \int_0^1 G_S(s,\tau) g\bigl(\tau, u(\tau), u'(\tau)\bigr)\,d\tau, \\ &\qquad \int_0^1 \frac{\partial G_S(s,\tau)}{\partial s} g\bigl(\tau, u(\tau), u'(\tau)\bigr)\,d\tau\biggr)\,ds.\end{aligned}$$

Moreover, it follows from Lemma 2.2 that for $t \in [\theta, 1-\theta]$,

$$\begin{aligned}(Tu)'(t) &= \int_0^1 \frac{\partial G_S(t,s)}{\partial t} f\biggl(s, \int_0^1 G_S(s,\tau) g\bigl(\tau, u(\tau), u'(\tau)\bigr)\,d\tau, \\ &\qquad \int_0^1 \frac{\partial G_S(s,\tau)}{\partial s} g\bigl(\tau, u(\tau), u'(\tau)\bigr)\,d\tau\biggr)\,ds \\ &\geq d_0 \int_0^1 \Phi_2(s) f\biggl(s, \int_0^1 G_S(s,\tau) g\bigl(\tau, u(\tau), u'(\tau)\bigr)\,d\tau, \\ &\qquad \int_0^1 \frac{\partial G_S(s,\tau)}{\partial s} g\bigl(\tau, u(\tau), u'(\tau)\bigr)\,d\tau\biggr)\,ds \\ &\geq d_0 \|Tu\|.\end{aligned}$$

Then we get

$$\min_{t \in [\theta, 1-\theta]} (Tu)'(t) \geq d_0 \|Tu\|,$$

which shows that $TK \subset K$. □

Similar to the proof of Lemma 2.4 in [28], we can get the following lemma.

Lemma 2.6 $T: K \to K$ *is completely continuous.*

3 Main results

Denote

$$f^0 = \limsup_{x+y \to 0^+} \max_{t \in [0,1]} \frac{f(t,x,y)}{x+y} \quad \text{and} \quad g^0 = \limsup_{x+y \to 0^+} \max_{t \in [0,1]} \frac{g(t,x,y)}{x+y},$$

$$f_0 = \liminf_{x+y\to 0^+} \min_{t\in[\theta,1-\theta]} \frac{f(t,x,y)}{x+y} \quad \text{and} \quad g_0 = \liminf_{x+y\to 0^+} \min_{t\in[\theta,1-\theta]} \frac{g(t,x,y)}{x+y},$$

$$f^\infty = \limsup_{x+y\to +\infty} \max_{t\in[0,1]} \frac{f(t,x,y)}{x+y} \quad \text{and} \quad g^\infty = \limsup_{x+y\to +\infty} \max_{t\in[0,1]} \frac{g(t,x,y)}{x+y},$$

$$f_\infty = \liminf_{x+y\to +\infty} \min_{t\in[\theta,1-\theta]} \frac{f(t,x,y)}{x+y} \quad \text{and} \quad g_\infty = \liminf_{x+y\to +\infty} \min_{t\in[\theta,1-\theta]} \frac{g(t,x,y)}{x+y},$$

$$A_1 = \int_0^1 \Phi_2(s)\,ds, \qquad B_1 = 2\int_0^1 \bigl(\Phi_1(s) + \Phi_2(s)\bigr)\,ds,$$

$$A_2 = d_0\int_\theta^{1-\theta} \Phi_2(s)\,ds, \qquad B_2 = d_0\int_\theta^{1-\theta} \bigl(c_0\Phi_1(s) + d_0\Phi_2(s)\bigr)\,ds.$$

Theorem 3.1 *Assume that $A_1 f^0 < 1 < A_2 f_\infty$ and $B_1 g^0 < 1 < B_2 g_\infty$. Then BVP* (1.6) *has at least one positive solution.*

Proof In view of $A_1 f^0 < 1$ and $B_1 g^0 < 1$, there exists $\varepsilon_1 > 0$ such that

$$A_1\bigl(f^0 + \varepsilon_1\bigr) \le 1, \qquad B_1\bigl(g^0 + \varepsilon_1\bigr) \le 1. \tag{3.1}$$

By the definition of f^0, g^0, we may choose $\sigma_1 > 0$ so that

$$\begin{aligned} &f(t,x,y) \le \bigl(f^0 + \varepsilon_1\bigr)(x+y), \\ &g(t,x,y) \le \bigl(g^0 + \varepsilon_1\bigr)(x+y), \quad t\in[0,1], (x+y)\in[0,\sigma_1]. \end{aligned} \tag{3.2}$$

Set $\Omega_1 = \{u \in E \mid \|u\| < \sigma_1/2\}$. It follows from (3.1), (3.2), Lemmas 2.2 and 2.3 that for any $u \in K \cap \partial\Omega_1$, $s \in [0,1]$,

$$\begin{aligned} &\int_0^1 \biggl(G_S(s,\tau) + \frac{\partial G_S(s,\tau)}{\partial s}\biggr) g\bigl(\tau,u(\tau),u'(\tau)\bigr)\,d\tau \\ &\quad\le \int_0^1 \bigl(\Phi_1(\tau) + \Phi_2(\tau)\bigr)\bigl(g^0 + \varepsilon_1\bigr)\bigl(u(\tau) + u'(\tau)\bigr)\,d\tau \\ &\quad\le 2\|u\|\bigl(g^0 + \varepsilon_1\bigr)\int_0^1 \bigl(\Phi_1(\tau) + \Phi_2(\tau)\bigr)\,d\tau \\ &\quad\le \sigma_1. \end{aligned} \tag{3.3}$$

Then, by (3.1), (3.2) and (3.3), we have

$$\begin{aligned} (Tu)'(t) &= \int_0^1 \frac{\partial G_S(t,s)}{\partial t} f\biggl(s, \int_0^1 G_S(s,\tau) g\bigl(\tau,u(\tau),u'(\tau)\bigr)\,d\tau, \\ &\qquad \int_0^1 \frac{\partial G_S(s,\tau)}{\partial s} g\bigl(\tau,u(\tau),u'(\tau)\bigr)\,d\tau\biggr)\,ds \\ &\le \int_0^1 \Phi_2(s)\bigl(f^0 + \varepsilon_1\bigr)\biggl(\int_0^1 G_S(s,\tau) g\bigl(\tau,u(\tau),u'(\tau)\bigr)\,d\tau \\ &\qquad + \int_0^1 \frac{\partial G_S(s,\tau)}{\partial s} g\bigl(\tau,u(\tau),u'(\tau)\bigr)\,d\tau\biggr)\,ds \\ &\le \int_0^1 \Phi_2(s)\bigl(f^0 + \varepsilon_1\bigr)\biggl(\int_0^1 \bigl(\Phi_1(\tau) + \Phi_2(\tau)\bigr)\bigl(g^0 + \varepsilon_1\bigr)\bigl(u(\tau) + u'(\tau)\bigr)\,d\tau\biggr)\,ds \end{aligned}$$

$$\begin{aligned}
&\le \int_0^1 \Phi_2(s)\,ds\big(f^0+\varepsilon_1\big)\left(\int_0^1 \big(\Phi_1(s)+\Phi_2(s)\big)\,ds\right)\big(g^0+\varepsilon_1\big)2\|u\| \\
&\le \|u\|, \quad t\in[0,1].
\end{aligned}$$

Therefore,

$$\|Tu\| \le \|u\|, \quad u\in K\cap\partial\Omega_1. \tag{3.4}$$

On the other hand, since $1 < A_2 f_\infty$ and $1 < B_2 g_\infty$, there exists $\varepsilon_2 > 0$ such that

$$A_2(f_\infty - \varepsilon_2) \ge 1, \qquad B_2(g_\infty - \varepsilon_2) \ge 1. \tag{3.5}$$

By the definition of f_∞, g_∞, we may choose $\sigma_2' > \sigma_1$ so that

$$\begin{aligned}
&f(t,x,y) \ge (f_\infty - \varepsilon_2)(x+y), \\
&g(t,x,y) \ge (g_\infty - \varepsilon_2)(x+y), \quad t\in[\theta,1-\theta], (x+y)\in[\sigma_2',+\infty).
\end{aligned} \tag{3.6}$$

Let $\sigma_2 = \max\{2\sigma_1, \sigma_2'/d_0\}$ and set $\Omega_2 = \{u\in E \mid \|u\| < \sigma_2\}$. Then $u\in K\cap\partial\Omega_2$ implies that $\sigma_2' \le d_0\|u\| \le u'(t)$, $t\in[\theta,1-\theta]$. So, for $s\in[\theta,1-\theta]$, in view of Lemmas 2.2 and 2.3, we have

$$\begin{aligned}
&\int_0^1 \left(G_S(s,\tau) + \frac{\partial G_S(s,\tau)}{\partial s}\right) g\big(\tau,u(\tau),u'(\tau)\big)\,d\tau \\
&\quad \ge \int_\theta^{1-\theta} \big(c_0\Phi_1(\tau) + d_0\Phi_2(\tau)\big)\big(g_\infty - \varepsilon_2\big)\big(u(\tau)+u'(\tau)\big)\,d\tau \\
&\quad \ge \|u\|(g_\infty - \varepsilon_2)d_0 \int_\theta^{1-\theta} \big(c_0\Phi_1(\tau) + d_0\Phi_2(\tau)\big)\,d\tau \ge \sigma_2.
\end{aligned} \tag{3.7}$$

Then, for $t\in[\theta,1-\theta]$, by (3.5), (3.6), (3.7), Lemmas 2.2 and 2.3, we have

$$\begin{aligned}
(Tu)'(t) &= \int_0^1 \frac{\partial G_S(t,s)}{\partial t} f\left(s, \int_0^1 G_S(s,\tau) g\big(\tau,u(\tau),u'(\tau)\big)\,d\tau,\right. \\
&\qquad \left.\int_0^1 \frac{\partial G_S(s,\tau)}{\partial s} g\big(\tau,u(\tau),u'(\tau)\big)\,d\tau\right) ds \\
&\ge d_0 \int_\theta^{1-\theta} \Phi_2(s)(f_\infty - \varepsilon_2)\left(\int_0^1 \left(G_S(s,\tau) + \frac{\partial G_S(s,\tau)}{\partial s}\right) g\big(\tau,u(\tau),u'(\tau)\big)\,d\tau\right) ds \\
&\ge d_0 \int_\theta^{1-\theta} \Phi_2(s)\,ds(f_\infty - \varepsilon_2)\sigma_2 \\
&\ge \|u\|, \quad t\in[0,1].
\end{aligned}$$

Therefore,

$$\|Tu\| \ge \|u\|, \quad u\in K\cap\partial\Omega_2. \tag{3.8}$$

Therefore, it follows from the first part of Theorem 1.1 that T has a fixed point $u_1 \in K\cap(\overline{\Omega}_2\backslash\Omega_1)$. Consequently, BVP (1.6) has a positive solution $(u,v)\in K\times K$, here

$$\begin{cases}
u(t) = u_1(t), \\
v(t) = \int_0^1 G_S(t,s) g(s,u_1(s),u_1'(s))\,ds.
\end{cases}$$

□

Theorem 3.2 *Assume that* $A_1 f^\infty < 1 < A_2 f_0$ *and* $B_1 g^\infty < 1 < B_2 g_0$. *Then BVP* (1.6) *has at least one positive solution.*

Proof The proof is similar to Theorem 3.1 and therefore omitted. □

Theorem 3.3 *Assume that* $A_2 f_0 > 1$, $A_2 f_\infty > 1$, $B_2 g_0 > 1$, $B_2 g_\infty > 1$, $B_2 g^0 < 2$ *and there is a* $\mu > 0$ *such that*

$$\max\{g(t,x,y), t \in [0,1], (x+y) \in [0,\mu]\} < \frac{2\mu}{B_1}; \tag{3.9}$$

$$\max\{f(t,x,y), t \in [0,1], (x+y) \in [0,\mu]\} < \frac{\mu}{2A_1}. \tag{3.10}$$

Then BVP (1.6) *has at least two positive solutions.*

Proof Firstly, in view of $A_2 f_0 > 1$ and $B_2 g_0 > 1$, there exists $\varepsilon > 0$ such that

$$A_2(f_0 - \varepsilon) \ge 1, \qquad B_2(g_0 - \varepsilon) \ge 1. \tag{3.11}$$

By the definition of f_0, g_0, we may choose $\hat{\sigma}_1 > 0$ so that

$$\begin{aligned} &f(t,x,y) \ge (f_0 - \varepsilon)(x+y), \\ &g(t,x,y) \ge (g_0 - \varepsilon)(x+y), \quad t \in [0,1], (x+y) \in [0,\hat{\sigma}_1]. \end{aligned} \tag{3.12}$$

Moreover, from $B_2 g^0 < 2$, take ρ_1 satisfying $0 < \rho_1 < \frac{B_2}{B_1}\hat{\sigma}_1 < \mu$ such that

$$g(t,x,y) \le \frac{2\rho_1}{B_2}, \quad \forall t \in [0,1], x+y \in [0,\rho_1].$$

Set $\Omega_1 = \{u \in E \mid \|u\| < \rho_1/2\}$. It follows from (3.11), (3.12), Lemmas 2.2 and 2.3 that for any $u \in K \cap \partial\Omega_1$,

$$\begin{aligned} \int_0^1 \left(G_S(s,\tau) + \frac{\partial G_S(s,\tau)}{\partial s}\right) g\bigl(\tau, u(\tau), u'(\tau)\bigr)\,d\tau &\le \frac{2\rho_1}{B_2}\int_0^1 \bigl(\Phi_1(\tau) + \Phi_2(\tau)\bigr)\,d\tau \\ &= \frac{B_1}{B_2}\rho_1 < \hat{\sigma}_1, \quad s \in [0,1]. \end{aligned} \tag{3.13}$$

Then, for $t \in [\theta, 1-\theta]$, by (3.11), (3.12), (3.13), Lemmas 2.2 and 2.3, we have

$$\begin{aligned} (Tu)'(t) &= \int_0^1 \frac{\partial G_S(t,s)}{\partial t} f\left(s, \int_0^1 G_S(s,\tau) g\bigl(\tau, u(\tau), u'(\tau)\bigr)\,d\tau,\right. \\ &\qquad \left.\int_0^1 \frac{\partial G_S(s,\tau)}{\partial s} g\bigl(\tau, u(\tau), u'(\tau)\bigr)\,d\tau\right) ds \\ &\ge d_0 \int_\theta^{1-\theta} \Phi_2(s)(f_0 - \varepsilon)\left(\int_0^1 \left(G_S(s,\tau) + \frac{\partial G_S(s,\tau)}{\partial s}\right) g\bigl(\tau, u(\tau), u'(\tau)\bigr)\,d\tau\right) ds \\ &\ge d_0 \int_\theta^{1-\theta} \Phi_2(s)(f_0 - \varepsilon)\left(\int_\theta^{1-\theta} \left(G_S(s,\tau) + \frac{\partial G_S(s,\tau)}{\partial s}\right) g\bigl(\tau, u(\tau), u'(\tau)\bigr)\,d\tau\right) ds \end{aligned}$$

$$\begin{aligned}
&\geq d_0 \int_\theta^{1-\theta} \Phi_2(s)(f_0-\varepsilon) \\
&\quad \times \left(\int_\theta^{1-\theta} \big(c_0\Phi_1(\tau)+d_0\Phi_2(\tau)\big)(g_0-\varepsilon)\big(u(\tau)+u'(\tau)\big)\,d\tau \right) ds \\
&\geq d_0 \int_\theta^{1-\theta} \Phi_2(s)\,ds(f_0-\varepsilon)\left(\int_\theta^{1-\theta} \big(c_0\Phi_1(\tau)+d_0\Phi_2(\tau)\big)\,d\tau \right)(g_0-\varepsilon)d_0\|u\| \\
&\geq \|u\|, \quad t\in[0,1].
\end{aligned}$$

Thus,

$$\|Tu\| \geq \|u\|, \quad u \in K \cap \partial\Omega_1. \tag{3.14}$$

Secondly, similar to the proof of (3.8), we may choose $\sigma_2 > \mu$ and set $\Omega_2 = \{u \in E \mid \|u\| < \sigma_2\}$, and easily get

$$\|Tu\| \geq \|u\|, \quad u \in K \cap \partial\Omega_2. \tag{3.15}$$

Let $\Omega_3 = \{u \in E \mid \|u\| < \mu/2\}$. Then, for any $u \in K \cap \partial\Omega_3$, it follows by (3.9) and (3.10) that

$$\begin{aligned}
\int_0^1 \left(G_S(s,\tau) + \frac{\partial G_S(s,\tau)}{\partial s} \right) g\big(\tau,u(\tau),u'(\tau)\big)\,d\tau &\leq \frac{2\mu}{B_1}\int_0^1 \big(\Phi_1(\tau)+\Phi_2(\tau)\big)\,d\tau \\
&\leq \mu, \quad s\in[0,1]
\end{aligned} \tag{3.16}$$

and

$$\begin{aligned}
(Tu)'(t) &= \int_0^1 \frac{\partial G_S(t,s)}{\partial t} f\left(s, \int_0^1 G_S(s,\tau)g\big(\tau,u(\tau),u'(\tau)\big)\,d\tau, \right. \\
&\quad \left. \int_0^1 \frac{\partial G_S(s,\tau)}{\partial s} g\big(\tau,u(\tau),u'(\tau)\big)\,d\tau \right) ds \\
&< \int_0^1 \Phi_2(s)\,ds \frac{\mu}{2A_1} = \frac{\mu}{2} = \|u\|, \quad t\in[0,1].
\end{aligned}$$

Thus,

$$\|Tu\| < \|u\|, \quad u \in K \cap \partial\Omega_3, \tag{3.17}$$

which together with (3.13), (3.14) shows that T has at least two fixed points in $u_1 \in K \cap (\overline{\Omega}_2 \setminus \Omega_3)$ and $u_2 \in K \cap (\overline{\Omega}_3 \setminus \Omega_1)$. □

Similarly, we can get the following theorem.

Theorem 3.4 *Assume that* $A_1 f^0 < 1$, $A_1 f^\infty < \frac{1}{2}$, $B_1 g^0 < 1$, $B_2 g_\infty > d_0$ *and there exists* $\eta > 0$ *such that*

$$\max\big\{g(t,x,y), t\in[\theta,1-\theta], (x+y)\in[d_0\eta,+\infty)\big\} > \frac{d_0^2\eta}{B_2}; \tag{3.18}$$

$$\max\big\{f(t,x,y), t\in[\theta,1-\theta], (x+y)\in[d_0\eta,+\infty)\big\} > \frac{\eta}{A_2}. \tag{3.19}$$

Then BVP (1.6) *has at least two positive solutions.*

Theorem 3.5 *If* $A_1 f(t,x,y) < (x+y)$ *and* $B_1 g(t,x,y) < (x+y)$ *for* $t \in [0,1]$ *and* $(x+y) \in [0,+\infty)$, *then BVP* (1.6) *has no monotone positive solution.*

Proof Suppose on the contrary that u is a monotone positive solution of BVP (1.6). Then $u(t) \geq 0$ and $u'(t) \geq 0$ for $t \in [0,1]$, and

$$\begin{aligned} u'(t) &= \int_0^1 \frac{\partial G_S(t,s)}{\partial t} f\biggl(s, \int_0^1 G_S(s,\tau) g\bigl(\tau, u(\tau), u'(\tau)\bigr)\,d\tau, \\ &\quad \int_0^1 \frac{\partial G_S(s,\tau)}{\partial s} g\bigl(\tau, u(\tau), u'(\tau)\bigr)\,d\tau\biggr)\,ds \\ &< \frac{1}{A_1}\int_0^1 \Phi_2(s)\biggl(\int_0^1 G_S(s,\tau) g\bigl(\tau, u(\tau), u'(\tau)\bigr)\,d\tau \\ &\quad + \int_0^1 \frac{\partial G_S(s,\tau)}{\partial s} g\bigl(\tau, u(\tau), u'(\tau)\bigr)\,d\tau\biggr)\,ds \\ &< \frac{1}{A_1}\frac{1}{B_1}\int_0^1 \Phi_2(s)\,ds \int_0^1 \bigl(\Phi_1(s)+\Phi_2(s)\bigr)\,ds \int_0^1 \bigl(u(\tau)+u'(\tau)\bigr)\,d\tau \\ &< \|u\|, \end{aligned}$$

which shows that $\|u\| < \|u\|$. This is a contradiction. □

Similarly, we can prove the following theorem.

Theorem 3.6 *If* $A_2 f(t,x,y) > (x+y)$ *and* $B_2 g(t,x,y) > (x+y)$ *for* $t \in [\theta, 1-\theta]$ *and* $(x+y) \in [0,+\infty)$, *then BVP* (1.6) *has no monotone positive solution.*

4 Example

In this section, we give an example to illustrate our main results.

Consider the BVP:

$$\begin{cases} -u'''(t) = f(t, v(t), v'(t)), \quad t \in (0,1), \\ -v'''(t) = g(t, u(t), u'(t)), \quad t \in (0,1), \\ u(0) = 0, \qquad u'(0) = \alpha[u] = \int_0^1 (1-s)u(s)\,ds, \qquad u'(1) = \beta[u] = \int_0^1 s u(s)\,ds, \\ v(0) = 0, \qquad v'(0) = \alpha[v] = \int_0^1 (1-s)v(s)\,ds, \qquad v'(1) = \beta[v] = \int_0^1 s v(s)\,ds. \end{cases} \tag{4.1}$$

For this BCs, the corresponding $\gamma(t) = \frac{2t-t^2}{2}$ and $\delta(t) = \frac{t^2}{2}$. A simple calculation shows that

$$\begin{aligned} &\alpha[\gamma] = \frac{1}{8}, \qquad \alpha[\delta] = \frac{1}{24}, \qquad \beta[\gamma] = \frac{5}{24}, \qquad \beta[\delta] = \frac{1}{8}, \\ &D = \bigl(1-\alpha[\gamma]\bigr)\bigl(1-\beta[\delta]\bigr) - \alpha[\delta]\beta[\gamma] = \frac{109}{144}, \\ &\mathcal{K}_A(s) := \int_0^1 G(t,s)(1-t)\,dt = \frac{s}{8} - \frac{s^2}{4} + \frac{s^3}{6} - \frac{s^4}{24}, \\ &\mathcal{K}_B(s) := \int_0^1 G(t,s)t\,dt = \frac{5s}{24} - \frac{s^2}{4} + \frac{s^4}{24}, \\ &\Phi_1(s) = \frac{265s}{218} - \frac{145s^2}{109} + \frac{13s^3}{109} - \frac{s^4}{218}, \qquad \Phi_2(s) = \frac{156s}{109} - \frac{181s^2}{109} + \frac{26s^3}{109} - \frac{s^4}{109}. \end{aligned}$$

Let $\theta = 1/4$, then $A_1 = 719/3{,}270 \approx 0.2199$, $A_2 = 126{,}517/3{,}348{,}480 \approx 0.0378$, $B_1 = 2{,}767/3{,}270 \approx 0.8462$, $B_2 = 4{,}438{,}339/428{,}605{,}440 \approx 0.0104$.

Example 4.1 Let

$$f\bigl(t, v(t), v'(t)\bigr) = \frac{1}{1+t}\left[\frac{v(t)+v'(t)}{e^{v(t)+v'(t)}} + \frac{1{,}000(v(t)+v'(t))^2}{1+v(t)+v'(t)}\right],$$
$$g\bigl(t, u(t), u'(t)\bigr) = \frac{1}{10(1+t)}\left[\frac{u(t)+u'(t)}{e^{u(t)+u'(t)}} + \frac{2{,}000(u(t)+u'(t))^2}{1+u(t)+u'(t)}\right].$$

It is easy to compute that $f^0 = 1$, $f_\infty = \frac{4{,}000}{7}$, $g^0 = \frac{1}{10}$, $g_\infty = \frac{800}{7}$, which show that $A_1 f^0 < 1 < A_2 f_\infty$ and $B_1 g^0 < 1 < B_2 g_\infty$. So, it follows from Theorem 3.1 that BVP (4.1) has at least one positive solution.

Example 4.2 Let

$$f\bigl(t, v(t), v'(t)\bigr) = \frac{1}{1+t}\left[\frac{50(v(t)+v'(t))}{e^{v(t)+v'(t)}} + \frac{50(v(t)+v'(t))^2}{1{,}000+v(t)+v'(t)}\right],$$
$$g\bigl(t, u(t), u'(t)\bigr) = \left[1+\left(t-\frac{3}{4}\right)^2\right]\left[\frac{100(u(t)+u'(t))}{e^{200(u(t)+u'(t))}} + \frac{100(u(t)+u'(t))^2}{1{,}000+u(t)+u'(t)}\right].$$

It is easy to compute that $f_0 = \frac{200}{7}$, $f_\infty = \frac{200}{7}$, $g_0 = 100$, $g_\infty = 100$ and $g^0 = \frac{625}{4}$, which show that $A_2 f_0 > 1$, $A_2 f_\infty > 1$, $B_2 g_0 > 1$, $B_2 g_\infty > 1$ and $B_2 g^0 < 2$.

Choose $\mu = 14$,

$$\max\bigl\{g(t,x,y), t \in [0,1], (x+y) \in [0,14]\bigr\} = \frac{50}{e} + \frac{4{,}900}{507} < 29 < \frac{2\mu}{B_1};$$
$$\max\bigl\{f(t,x,y), t \in [0,1], (x+y) \in [0,14]\bigr\} = \frac{25}{16}\left(\frac{1}{2e} + \frac{9{,}800}{507}\right) < 21 < \frac{\mu}{2A_1}.$$

So, it follows from Theorem 3.3 that BVP (4.1) has at least two positive solutions.

Competing interests
The author declares that she has no competing interests.

Author's contributions
The author read and approved the final manuscript.

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