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On impulsive partial differential equations with Caputo-Hadamard fractional derivatives

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Abstract

In this paper, the mixed Caputo-Hadamard fractional derivative is introduced based on the Caputo-type modification of Hadamard fractional derivatives in the existing paper, and impulsive partial differential equations with Caputo-Hadamard fractional derivatives are studied. The formula of a general solution for these impulsive fractional partial differential equations is found by considering some limiting cases (impulses tending to zero), and its validity is shown by an example.

MSC: 34A08; 34A37

Keywords: impulsive fractional partial differential equations; fractional partial differential equations; impulse; general solution

1 Introduction

The fractional calculus was developed within the frame of the Hadamard fractional derivative in [1–6], and for the general theory of Hadamard fractional calculus we refer the interested reader to [7]. Moreover, some progress was achieved in controllability, some new definitions, some new methods of numerical solution etc. for fractional differential equations [8–13].

Recently, Jarad *et al.* presented the definition of Caputo-Hadamard fractional derivative in [14], and developed the fundamental theorem of fractional calculus in the Caputo-Hadamard setting in [14, 15].

Furthermore, Vityuk and Golushkov were concerned with the existence and uniqueness of solution for a kind of fractional partial differential equations in [16]. Next, Abbas and Benchohra first considered fractional partial differential equations with impulses in [17], and the authors gave some results as regards the existence and uniqueness of solution for these impulsive systems in [17–22].

Now the equivalent integral equations were found for several fractional-order systems with impulses in [23–28], and the obtained results show that there is a general solution for their impulsive fractional-order systems.

Motivated by the above-mentioned work, we will give the definition of a mixed Caputo-Hadamard fractional derivative and seek the equivalent integral equations for a kind of impulsive partial differential equations with Caputo-Hadamard fractional derivatives to find the essential result that there exists a general solution for impulsive fractional differential equations in this paper. We have

$$\begin{cases} ({}_{\text{C-H}}D^{q}_{(a+,b+)}u)(x,y) = f(x,y,u(x,y)), & (x,y) \in J \text{ and } x \neq x_i \ (i = 1,2,\ldots,m), \\ u(x_i^+,y) = u(x_i^-,y) + I_i(u(x_i^-,y)), & i = 1,2,\ldots,m, \\ u(x,b) = \phi(x), \qquad u(a,y) = \psi(y), & x \in [a,A], y \in [b,B], \end{cases} \tag{1}$$

where $J = [a,A] \times [b,B]$ $(a,b > 0)$, $q = (q_1,q_2)$ (here $q_1,q_2 \in \mathbb{C}$ and $(\Re(q_1),\Re(q_2)) \in (0,1] \times (0,1]$), ${}_{\text{C-H}}D^{q}_{(a+,b+)}$ denotes the Caputo-Hadamard fractional derivative of order q. We have the impulsive points $a = x_0 < x_1 < \cdots < x_m < x_{m+1} = A$. $u(x_i^+,y) = \lim_{\varepsilon\to 0^+} u(x_i + \varepsilon,y)$ and $u(x_i^-,y) = \lim_{\varepsilon\to 0^-} u(x_i + \varepsilon,y)$ represent the right and left limits of $u(x,y)$ at $x = x_i$ $(i = 1,2,\ldots,m)$, respectively. $f : J \times \mathbb{C}^n \to \mathbb{C}^n$ and $I_i : \mathbb{C}^n \to \mathbb{C}^n$ $(i = 1,2,\ldots,m)$ are given functions. $\phi : [a,A] \to \mathbb{C}^n$, $\psi : [b,B] \to \mathbb{C}^n$ are given continuous functions with $\phi(a) = \psi(b)$.

Consider a limiting case in system (1):

$$\begin{aligned} &\lim_{I_i(u(x_i^-,y))\to 0 \text{ for all } i\in\{1,2,\ldots,m\}} \{\text{system (1)}\} \\ &\quad\to \begin{cases} ({}_{\text{C-H}}D^{q}_{(a+,b+)}u)(x,y) = f(x,y,u(x,y)), & (x,y) \in J, \\ u(x,b) = \phi(x), \qquad u(a,y) = \psi(y), & x \in [a,A], y \in [b,B]. \end{cases} \end{aligned} \tag{2}$$

Therefore,

$$\begin{aligned} &\lim_{I_i(u(x_i^-,y))\to 0 \text{ for all } i\in\{1,2,\ldots,m\}} \{\text{the solution of system (1)}\} \\ &\quad= \{\text{the solution of system (2)}\}. \end{aligned} \tag{3}$$

Next, some preliminaries are given in Section 2, and the equivalent integral equation will be provided for a fractional partial differential system with impulses in Section 3. Finally, an example is presented to illuminate the main result in Section 4.

2 Preliminaries

In this section, we shall present the definition of Caputo-Hadamard fractional partial derivatives according to definition of left-sided Caputo-Hadamard fractional derivatives suggested by Jarad *et al.* in [14, 15], and we draw a conclusion.

Definition 2.1 Let $a_1 \in [a,A]$, $z^+ = (a_1+,b+)$, $J_z = [a_1,A] \times [b,B]$, $q = (q_1,q_2)$ (here $q_1,q_2 \in \mathbb{C}$ and $(\Re(q_1),\Re(q_2)) \in (0,1] \times (0,1]$). For the function w, the expression

$$\big({}_{\text{H}}\mathcal{J}^{q}_{z^+}w\big)(x,y) = \frac{1}{\Gamma(q_1)\Gamma(q_2)} \int_{a_1}^{x}\int_{b}^{y} \left(\ln\frac{x}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1} w(s,t)\frac{dt}{t}\frac{ds}{s},$$

where Γ is the gamma function, is called the left-sided mixed Hadamard integral of order q.

Definition 2.2 Let $q = (q_1,q_2)$ (here $q_1,q_2 \in \mathbb{C}$ and $(\Re(q_1),\Re(q_2)) \in (0,1] \times (0,1]$). For $w \in L^1(J_z,\mathbb{C}^n)$ the mixed Caputo-Hadamard fractional derivative of order q can be defined by the expression

$$\begin{aligned}
&\left({}_{\text{C-H}}D^{q}_{z^+}w\right)(x,y)\\
&\quad=\frac{1}{\Gamma(1-q_1)\Gamma(1-q_2)}\int_{a_1}^{x}\int_{b}^{y}\left(\ln\frac{x}{s}\right)^{-q_1}\left(\ln\frac{y}{t}\right)^{-q_2}\delta_s\delta_t w(s,t)\frac{dt}{t}\frac{ds}{s}\\
&\quad=\left({}_{\mathrm{H}}\mathcal{J}^{1-q}_{z^+}\delta_x\delta_y w\right)(x,y),
\end{aligned}$$

where we have the partial differential operator $\delta_x = x\frac{\partial}{\partial x}$.

Lemma 2.3 *Let $h \in C(J_z,\mathbb{C}^n)$, $q = (q_1,q_2)$ (here $q_1,q_2 \in \mathbb{C}$ and $(\Re(q_1),\Re(q_2)) \in (0,1] \times (0,1]$). A function $u \in C(J_z,\mathbb{C}^n)$ is a solution of the differential equation*

$$\left({}_{\text{C-H}}D^{q}_{z^+}u\right)(x,y) = h(x,y), \quad (x,y) \in J_z, \tag{4}$$

if and only if

$$\begin{aligned}
u(x,y) &= u(x,b) + u\left(a_1^+,y\right) - u\left(a_1^+,b\right) + \left(I^{q}_{z^+}h\right)(x,y)\\
&= u(x,b) + u\left(a_1^+,y\right) - u\left(a_1^+,b\right)\\
&\quad+\frac{1}{\Gamma(q_1)\Gamma(q_2)}\int_{a_1}^{x}\int_{b}^{y}\left(\ln\frac{x}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1}h(s,t)\frac{dt}{t}\frac{ds}{s},\\
&\quad\textit{for } (x,y) \in J_z.
\end{aligned}\tag{5}$$

Proof Let $u(x,y)$ is a solution of the equation $({}_{\text{C-H}}D^{q}_{z^+}u)(x,y) = h(x,y)$, $(x,y) \in J_z$. Due to

$$\left({}_{\text{C-H}}D^{q}_{z^+}u\right)(x,y) = \left({}_{\mathrm{H}}\mathcal{J}^{1-q}_{z^+}\delta_x\delta_y u\right)(x,y)$$

we have

$${}_{\mathrm{H}}\mathcal{J}^{q}_{z^+}\left({}_{\mathrm{H}}\mathcal{J}^{1-q}_{z^+}\delta_x\delta_y u\right)(x,y) = \left({}_{\mathrm{H}}\mathcal{J}^{q}_{z^+}h\right)(x,y), \quad (x,y) \in J_z.$$

On the other hand,

$$\begin{aligned}
&{}_{\mathrm{H}}\mathcal{J}^{q}_{z^+}\left({}_{\mathrm{H}}\mathcal{J}^{1-q}_{z^+}\delta_x\delta_y u\right)(x,y)\\
&\quad= {}_{\mathrm{H}}\mathcal{J}^{1}_{z^+}(\delta_x\delta_y u)(x,y)\\
&\quad= u(x,y) - u(x,b) - u\left(a_1^+,y\right) + u\left(a_1^+,b\right), \quad \text{for } (x,y) \in J_z.
\end{aligned}$$

Therefore,

$$u(x,y) = u(x,b) + u\left(a_1^+,y\right) - u\left(a_1^+,b\right) + \left({}_{\mathrm{H}}\mathcal{J}^{q}_{z^+}h\right)(x,y), \quad \text{for } (x,y) \in J_z.$$

Moreover, equation (5) satisfies (4) by Definition 2.2. The proof is completed. □

3 Main results

For convenience, let $\sum_{i=1}^{0} z_i = 0$, $\Xi(x,y) = \phi(x) + \psi(y) - \phi(a)$, and $f = f(s,t,u(s,t))$. Define

$$\begin{aligned}\bar{u}(x,y) &= u(x,b) + u\big(x_k^+,y\big) - u\big(x_k^+,b\big)\\ &\quad + \frac{1}{\Gamma(q_1)\Gamma(q_2)}\int_{x_k}^{x}\int_b^y \left(\ln\frac{x}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1} f\frac{dt}{t}\frac{ds}{s},\\ &\quad \text{for } (x,y)\in(x_k,x_{k+1}]\times[b,B], \text{and } k\in\{1,2,\ldots,m\}, \end{aligned} \tag{6}$$

with $u(x_k^+,y) = u(x_k^-,y) + I_k(u(x_k^-,y))$.

By Lemma 2.3, it is sure that $\bar{u}(x,y)$ satisfies the fractional derivative condition and impulsive conditions in system (1). But $\bar{u}(x,y)$ is not a solution of (1) because *it does not satisfy* (3). Therefore, $\bar{u}(x,y)$ will be considered an approximate solution to seek the exact solution of system (1).

Theorem 3.1 *Let $q = (q_1,q_2)$, here $q_1,q_2 \in \mathbb{C}$ and $(\Re(q_1),\Re(q_2)) \in (0,1]\times(0,1]$. $I_i(u(x_i^-,y))$ $(i = 1,2,\ldots,m)$ are differentiable functions on y. System* (1) *is equivalent to the integral equation*

$$\begin{aligned}u(x,y) = {}& \Xi(x,y) + \frac{1}{\Gamma(q_1)\Gamma(q_2)}\int_a^x\int_b^y\left(\ln\frac{x}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1} f\frac{dt}{t}\frac{ds}{s}\\ &+\sum_{i=1}^{k}\big[I_i\big(u\big(x_i^-,y\big)\big) - I_i\big(u\big(x_i^-,b\big)\big)\big]\\ &+\frac{\sigma(y)}{\Gamma(q_1)\Gamma(q_2)}\sum_{i=1}^{k} I_i\big(u\big(x_i^-,y\big)\big)\bigg[\int_a^{x_i}\int_b^y\left(\ln\frac{x_i}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1} f\frac{dt}{t}\frac{ds}{s}\\ &+\int_{x_i}^{x}\int_b^y\left(\ln\frac{x}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1} f\frac{dt}{t}\frac{ds}{s}\\ &-\int_a^x\int_b^y\left(\ln\frac{x}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1} f\frac{dt}{t}\frac{ds}{s}\bigg],\\ &\textit{for } (x,y)\in(x_k,x_{k+1}]\times[b,B]\ \big(\textit{here } k\in\{0,1,2,\ldots,m\}\big),\end{aligned}\tag{7}$$

provided that the integral in (7) *exists, where $\sigma(y)$ is an arbitrary differentiable function on y.*

Proof As regards necessity; letting $I_i(u(x_i^-,y)) \to 0$ for all $i \in \{1,2,\ldots,m\}$ in equation (7), we obtain

$$\begin{aligned}&\lim_{I_i(u(x_i^-,y))\to 0 \text{ for all } i\in\{1,2,\ldots,m\}} u(x,y)\\ &\quad= \Xi(x,y) + \frac{1}{\Gamma(q_1)\Gamma(q_2)}\int_a^x\int_b^y\left(\ln\frac{x}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1} f\frac{dt}{t}\frac{ds}{s},\\ &\qquad \text{for } (x,y)\in(x_k,x_{k+1}]\times[b,B], k\in\{0,1,2,\ldots,m\}.\end{aligned}$$

Therefore, by Lemma 2.3, equation (7) (under conditions $I_i(u(x_i^-,y)) \to 0$ for all $i \in \{1,2,\ldots,m\}$) is the solution of system (2), that is, equation (7) satisfies condition (3).

Next, for $\forall x_i$ ($i \in \{1,2,\ldots,m\}$) in equation (7), we get

$$\begin{aligned} u(x_i^+,y) - u(x_i^-,y) &= \Xi(x_i^+,y) + I_i(u(x_i^-,y)) - I_i(u(x_i^-,b)) - \Xi(x_i^-,y) \\ &= I_i(u(x_i^-,y)) - I_i(u(x_i^-,b)) + \phi(x_i^+) - \phi(x_i^-) \\ &= I_i(u(x_i^-,y)). \end{aligned}$$

Therefore, equation (7) satisfies the impulsive conditions in system (1).

Finally, taking fractional derivatives of both sides of equation (7) as $(x,y) \in (x_k,x_{k+1}] \times [b,B]$ (here $k = 0,1,2,\ldots,m$), we obtain

$$\begin{aligned} &\left({}_{\text{C-H}}D^q_{(a+,b+)}u\right)(x,y) \\ &\quad= \left({}_{\text{H}}\mathcal{J}^{1-q}_{(a+,b+)}\delta_x\delta_y u\right)(x,y) \\ &\quad= {}_{\text{H}}\mathcal{J}^{1-q}_{(a+,b+)}\delta_x\delta_y \Bigg\{ \Xi(x,y) + \sum_{i=1}^{k}\left[I_i(u(x_i^-,y)) - I_i(u(x_i^-,0))\right] \\ &\qquad+ \frac{1}{\Gamma(q_1)\Gamma(q_2)}\int_a^x\int_b^y\left(\ln\frac{x}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1} f\frac{dt}{t}\frac{ds}{s} \\ &\qquad+ \frac{\sigma(y)}{\Gamma(q_1)\Gamma(q_2)}\sum_{i=1}^{k} I_i(u(x_i^-,y))\left[\int_a^{x_i}\int_b^y\left(\ln\frac{x_i}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1} f\frac{dt}{t}\frac{ds}{s}\right. \\ &\qquad+ \int_{x_i}^x\int_b^y\left(\ln\frac{x}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1} f\frac{dt}{t}\frac{ds}{s} \\ &\qquad\left.- \int_a^x\int_b^y\left(\ln\frac{x}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1} f\frac{dt}{t}\frac{ds}{s}\right]\Bigg\} \\ &\quad= \Bigg\{ f(x,y,u(x,y))|_{(x,y)\in[a,x_{k+1}]\times[b,B]} + \frac{1}{\Gamma(q_1)\Gamma(q_2)} \\ &\qquad\times \sum_{i=1}^{k} {}_{\text{H}}\mathcal{J}^{1-q}_{(a+,b+)}\delta_x\delta_y\left[\int_{x_i}^x\left(\ln\frac{x}{s}\right)^{q_1-1}\left(\int_b^y\sigma(y)I_i(u(x_i^-,y))\left(\ln\frac{y}{t}\right)^{q_2-1} f\frac{dt}{t}\right)\frac{ds}{s}\right. \\ &\qquad\left.- \int_a^x\left(\ln\frac{x}{s}\right)^{q_1-1}\left(\int_b^y\sigma(y)I_i(u(x_i^-,y))\left(\ln\frac{y}{t}\right)^{q_2-1} f\frac{dt}{t}\right)\frac{ds}{s}\right]\Bigg\}_{(x,y)\in(x_k,x_{k+1}]\times[b,B]}. \end{aligned}$$

We have

$$\begin{aligned} &{}_{\text{H}}\mathcal{J}^{1-q}_{(a+,b+)}\delta_x\delta_y\left[\int_{x_i}^x\left(\ln\frac{x}{s}\right)^{q_1-1}\left(\int_b^y\sigma(y)I_i(u(x_i^-,y))\left(\ln\frac{y}{t}\right)^{q_2-1} f\frac{dt}{t}\right)\frac{ds}{s}\right. \\ &\quad\left.- \int_a^x\left(\ln\frac{x}{s}\right)^{q_1-1}\left(\int_b^y\sigma(y)I_i(u(x_i^-,y))\left(\ln\frac{y}{t}\right)^{q_2-1} f\frac{dt}{t}\right)\frac{ds}{s}\right] = 0. \end{aligned} \tag{8}$$

Also, we will give the proof of equation (8) in the Appendix. Thus

$$\left({}_{\text{C-H}}D^q_{(a+,b+)}u\right)(x,y) = f(x,y,u(x,y))|_{(x,y)\in(x_k,x_{k+1}]\times[b,B]}.$$

So, equation (7) satisfies all conditions of (1).

As regards sufficiency: we will prove that the solution of system (1) satisfies equation (7) by mathematical induction. By Lemma 2.3, the solution of system (1) satisfies

$$u(x,y) = \Xi(x,y) + \frac{1}{\Gamma(q_1)\Gamma(q_2)}\int_a^x\int_b^y\left(\ln\frac{x}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1} f\frac{dt}{t}\frac{ds}{s},\quad \text{for } (x,y)\in[a,x_1]\times[b,B]. \tag{9}$$

Using (9), the approximate solution (as $(x,y)\in(x_1,x_2]\times[b,B]$) of system (1) is given by

$$\begin{aligned}\bar{u}(x,y) &= u(x,b)+u\left(x_1^+,y\right)-u\left(x_1^+,b\right)\\ &\quad+\frac{1}{\Gamma(q_1)\Gamma(q_2)}\int_{x_1}^x\int_b^y\left(\ln\frac{x}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1} f\frac{dt}{t}\frac{ds}{s}\\ &=\phi(x)+\phi\left(x_1^-\right)+\psi(y)-\phi(a)+I_1\left(u\left(x_1^-,y\right)\right)\\ &\quad+\frac{1}{\Gamma(q_1)\Gamma(q_2)}\int_a^{x_1}\int_b^y\left(\ln\frac{x_1}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1} f\frac{dt}{t}\frac{ds}{s}\\ &\quad-\phi\left(x_1^-\right)-\psi(b)+\phi(a)-I_1\left(u\left(x_1^-,b\right)\right)\\ &\quad+\frac{1}{\Gamma(q_1)\Gamma(q_2)}\int_{x_1}^x\int_b^y\left(\ln\frac{x}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1} f\frac{dt}{t}\frac{ds}{s}\\ &=\Xi(x,y)+I_1\left(u\left(x_1^-,y\right)\right)-I_1\left(u\left(x_1^-,b\right)\right)\\ &\quad+\frac{1}{\Gamma(q_1)\Gamma(q_2)}\left[\int_a^{x_1}\int_b^y\left(\ln\frac{x_1}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1} f\frac{dt}{t}\frac{ds}{s}\right.\\ &\quad\left.+\int_{x_1}^x\int_b^y\left(\ln\frac{x}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1} f\frac{dt}{t}\frac{ds}{s}\right],\\ &\quad\text{for } (x,y)\in(x_1,x_2]\times[b,B].\end{aligned}\tag{10}$$

Let $e_1(x,y)=u(x,y)-\bar{u}(x,y)$ for $(x,y)\in(x_1,x_2]\times[b,B]$, here $u(x,y)$ denotes the exact solution of system (1). Moreover, by equation (9), the exact solution $u(x,y)$ of system (1) satisfies

$$\lim_{I_1(u(x_1^-,y))\to 0} u(x,y) = \Xi(x,y) + \frac{1}{\Gamma(q_1)\Gamma(q_2)}\int_a^x\int_b^y\left(\ln\frac{x}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1} f\frac{dt}{t}\frac{ds}{s},\quad \text{for } (x,y)\in(x_1,x_2]\times[b,B]. \tag{11}$$

Thus,

$$\begin{aligned}&\lim_{I_1(u(x_1^-,y))\to 0} e_1(x,y)\\ &\quad=\lim_{I_1(u(x_1^-,y))\to 0}\left\{u(x,y)-\bar{u}(x,y)\right\}\\ &\quad=\frac{1}{\Gamma(q_1)\Gamma(q_2)}\left[\int_a^x\int_b^y\left(\ln\frac{x}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1} f\frac{dt}{t}\frac{ds}{s}\right.\\ &\quad\quad-\int_a^{x_1}\int_b^y\left(\ln\frac{x_1}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1} f\frac{dt}{t}\frac{ds}{s}\\ &\quad\quad\left.-\int_{x_1}^x\int_b^y\left(\ln\frac{x}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1} f\frac{dt}{t}\frac{ds}{s}\right].\end{aligned}\tag{12}$$

Equation (12) means that $e_1(x,y)$ is connected with $\lim_{I_1(u(x_1^-,y))\to 0} e_1(x,y)$ and $I_1(u(x_1^-,y))$. Therefore, we suppose

$$\begin{aligned} e_1(x,y) &= \kappa\bigl(I_1\bigl(u\bigl(x_1^-,y\bigr)\bigr)\bigr)\lim_{I_1(u(x_1^-,y))\to 0} e_1(x,y) \\ &= \frac{\kappa(I_1(u(x_1^-,y)))}{\Gamma(q_1)\Gamma(q_2)}\biggl[\int_a^x\int_b^y\left(\ln\frac{x}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1} f\frac{dt}{t}\frac{ds}{s} \\ &\quad - \int_a^{x_1}\int_b^y\left(\ln\frac{x_1}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1} f\frac{dt}{t}\frac{ds}{s} \\ &\quad - \int_{x_1}^x\int_b^y\left(\ln\frac{x}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1} f\frac{dt}{t}\frac{ds}{s}\biggr], \end{aligned} \tag{13}$$

where κ is an undetermined function with $\kappa(0) = 1$. Thus,

$$\begin{aligned} u(x,y) &= \bar{u}(x,y) + e_1(x,y) \\ &= \Xi(x,y) + I_1\bigl(u\bigl(x_1^-,y\bigr)\bigr) - I_1\bigl(u\bigl(x_1^-,b\bigr)\bigr) \\ &\quad + \frac{1}{\Gamma(q_1)\Gamma(q_2)}\int_a^x\int_b^y\left(\ln\frac{x}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1} f\frac{dt}{t}\frac{ds}{s} \\ &\quad + \frac{1-\kappa(I_1(u(x_1^-,y)))}{\Gamma(q_1)\Gamma(q_2)}\biggl[\int_a^{x_1}\int_b^y\left(\ln\frac{x_1}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1} f\frac{dt}{t}\frac{ds}{s} \\ &\quad + \int_{x_1}^x\int_b^y\left(\ln\frac{x}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1} f\frac{dt}{t}\frac{ds}{s} \\ &\quad - \int_a^x\int_b^y\left(\ln\frac{x}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1} f\frac{dt}{t}\frac{ds}{s}\biggr], \\ &\quad \text{for } (x,y)\in(x_1,x_2]\times[b,B]. \end{aligned} \tag{14}$$

Next, using equation (14), the approximate solution (as $(x,y)\in(x_2,x_3]\times[b,B]$) of system (1) is provided by

$$\begin{aligned} \bar{u}(x,y) &= u(x,b) + u\bigl(x_2^+,y\bigr) - u\bigl(x_2^+,b\bigr) \\ &\quad + \frac{1}{\Gamma(q_1)\Gamma(q_2)}\int_{x_2}^x\int_b^y\left(\ln\frac{x}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1} f\frac{dt}{t}\frac{ds}{s} \\ &= \Xi(x,y) + I_1\bigl(u\bigl(x_1^-,y\bigr)\bigr) - I_1\bigl(u\bigl(x_1^-,b\bigr)\bigr) + I_2\bigl(u\bigl(x_2^-,y\bigr)\bigr) - I_2\bigl(u\bigl(x_2^-,b\bigr)\bigr) \\ &\quad + \frac{1}{\Gamma(q_1)\Gamma(q_2)}\biggl[\int_a^{x_2}\int_b^y\left(\ln\frac{x_2}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1} f\frac{dt}{t}\frac{ds}{s} \\ &\quad + \int_{x_2}^x\int_b^y\left(\ln\frac{x}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1} f\frac{dt}{t}\frac{ds}{s}\biggr] \\ &\quad + \frac{1-\kappa(I_1(u(x_1^-,y)))}{\Gamma(q_1)\Gamma(q_2)}\biggl[\int_a^{x_1}\int_b^y\left(\ln\frac{x_1}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1} f\frac{dt}{t}\frac{ds}{s} \\ &\quad + \int_{x_1}^{x_2}\int_b^y\left(\ln\frac{x_2}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1} f\frac{dt}{t}\frac{ds}{s} \\ &\quad - \int_a^{x_2}\int_b^y\left(\ln\frac{x_2}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1} f\frac{dt}{t}\frac{ds}{s}\biggr], \\ &\quad \text{for } (x,y)\in(x_2,x_3]\times[b,B]. \end{aligned} \tag{15}$$

Let $e_2(x,y) = u(x,y) - \bar{u}(x,y)$ for $(x,y) \in (x_2,x_3] \times [b,B]$. Moreover, by equation (14), the exact solution of (1) satisfies

$$\begin{aligned}
&\lim_{\substack{I_1(u(x_1^-,y))\to 0,\\ I_2(u(x_2^-,y))\to 0}} u(x,y) = \Xi(x,y) + \frac{1}{\Gamma(q_1)\Gamma(q_2)} \int_a^x \int_b^y \left(\ln \frac{x}{s}\right)^{q_1-1} \left(\ln \frac{y}{t}\right)^{q_2-1} f \frac{dt}{t} \frac{ds}{s}, \\
&\quad \text{for } (x,y) \in (x_2,x_3] \times [b,B], \\
&\lim_{I_2(u(x_2^-,y))\to 0} u(x,y) \\
&\quad = \Xi(x,y) + I_1\bigl(u\bigl(x_1^-,y\bigr)\bigr) - I_1\bigl(u\bigl(x_1^-,b\bigr)\bigr) \\
&\qquad + \frac{1}{\Gamma(q_1)\Gamma(q_2)} \int_a^x \int_b^y \left(\ln \frac{x}{s}\right)^{q_1-1} \left(\ln \frac{y}{t}\right)^{q_2-1} f \frac{dt}{t} \frac{ds}{s} \\
&\qquad + \frac{1-\kappa(I_1(u(x_1^-,y)))}{\Gamma(q_1)\Gamma(q_2)} \left[\int_a^{x_1} \int_b^y \left(\ln \frac{x_1}{s}\right)^{q_1-1} \left(\ln \frac{y}{t}\right)^{q_2-1} f \frac{dt}{t} \frac{ds}{s} \right. \\
&\qquad \left. + \int_{x_1}^x \int_b^y \left(\ln \frac{x}{s}\right)^{q_1-1} \left(\ln \frac{y}{t}\right)^{q_2-1} f \frac{dt}{t} \frac{ds}{s} - \int_a^x \int_b^y \left(\ln \frac{x}{s}\right)^{q_1-1} \left(\ln \frac{y}{t}\right)^{q_2-1} f \frac{dt}{t} \frac{ds}{s}\right], \\
&\qquad \text{for } (x,y) \in (x_2,x_3] \times [b,B], \\
&\lim_{I_1(u(x_1^-,y))\to 0} u(x,y) \\
&\quad = \Xi(x,y) + I_2\bigl(u\bigl(x_2^-,y\bigr)\bigr) - I_2\bigl(u\bigl(x_2^-,b\bigr)\bigr) \\
&\qquad + \frac{1}{\Gamma(q_1)\Gamma(q_2)} \int_a^x \int_b^y \left(\ln \frac{x}{s}\right)^{q_1-1} \left(\ln \frac{y}{t}\right)^{q_2-1} f \frac{dt}{t} \frac{ds}{s} \\
&\qquad + \frac{1-\kappa(I_2(u(x_2^-,y)))}{\Gamma(q_1)\Gamma(q_2)} \left[\int_a^{x_2} \int_b^y \left(\ln \frac{x_2}{s}\right)^{q_1-1} \left(\ln \frac{y}{t}\right)^{q_2-1} f \frac{dt}{t} \frac{ds}{s} \right. \\
&\qquad \left. + \int_{x_2}^x \int_b^y \left(\ln \frac{x}{s}\right)^{q_1-1} \left(\ln \frac{y}{t}\right)^{q_2-1} f \frac{dt}{t} \frac{ds}{s} - \int_a^x \int_b^y \left(\ln \frac{x}{s}\right)^{q_1-1} \left(\ln \frac{y}{t}\right)^{q_2-1} f \frac{dt}{t} \frac{ds}{s}\right], \\
&\qquad \text{for } (x,y) \in (x_2,x_3] \times [b,B].
\end{aligned}$$

Thus,

$$\begin{aligned}
&\lim_{\substack{I_1(u(x_1^-,y))\to 0,\\ I_2(u(x_2^-,y))\to 0}} e_2(x,y) \\
&\quad = \lim_{\substack{I_1(u(x_1^-,y))\to 0,\\ I_2(u(x_2^-,y))\to 0}} \bigl\{u(x,y) - \bar{u}(x,y)\bigr\} \\
&\quad = \frac{1}{\Gamma(q_1)\Gamma(q_2)} \left[\int_a^x \int_b^y \left(\ln \frac{x}{s}\right)^{q_1-1} \left(\ln \frac{y}{t}\right)^{q_2-1} f \frac{dt}{t} \frac{ds}{s} \right. \\
&\qquad - \int_a^{x_2} \int_b^y \left(\ln \frac{x_2}{s}\right)^{q_1-1} \left(\ln \frac{y}{t}\right)^{q_2-1} f \frac{dt}{t} \frac{ds}{s} \\
&\qquad \left. - \int_{x_2}^x \int_b^y \left(\ln \frac{x}{s}\right)^{q_1-1} \left(\ln \frac{y}{t}\right)^{q_2-1} f \frac{dt}{t} \frac{ds}{s}\right], \qquad (16) \\
&\lim_{I_2(u(x_2^-,y))\to 0} e_2(x,y)
\end{aligned}$$

$$\begin{aligned}
&= \lim_{I_2(u(x_2^-,y))\to 0}\{u(x,y)-\bar{u}(x,y)\} \\
&= \frac{-\kappa(I_1(u(x_1^-,y)))}{\Gamma(q_1)\Gamma(q_2)}\biggl[\int_a^{x_2}\int_b^y\left(\ln\frac{x_2}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1}f\frac{dt}{t}\frac{ds}{s} \\
&\quad+\int_{x_2}^x\int_b^y\left(\ln\frac{x}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1}f\frac{dt}{t}\frac{ds}{s}-\int_a^x\int_b^y\left(\ln\frac{x}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1}f\frac{dt}{t}\frac{ds}{s}\biggr] \\
&\quad+\frac{1-\kappa(I_1(u(x_1^-,y)))}{\Gamma(q_1)\Gamma(q_2)}\biggl[\int_{x_1}^x\int_b^y\left(\ln\frac{x}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1}f\frac{dt}{t}\frac{ds}{s} \\
&\quad-\int_{x_1}^{x_2}\int_b^y\left(\ln\frac{x_2}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1}f\frac{dt}{t}\frac{ds}{s} \\
&\quad-\int_{x_2}^x\int_b^y\left(\ln\frac{x}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1}f\frac{dt}{t}\frac{ds}{s}\biggr],
\end{aligned}\tag{17}$$

$$\begin{aligned}
&\lim_{I_1(u(x_1^-,y))\to 0} e_2(x,y) \\
&= \lim_{I_1(u(x_1^-,y))\to 0}\{u(x,y)-\bar{u}(x,y)\} \\
&= \frac{-\kappa(I_2(u(x_2^-,y)))}{\Gamma(q_1)\Gamma(q_2)}\biggl[\int_a^{x_2}\int_b^y\left(\ln\frac{x_2}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1}f\frac{dt}{t}\frac{ds}{s} \\
&\quad+\int_{x_2}^x\int_b^y\left(\ln\frac{x}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1}f\frac{dt}{t}\frac{ds}{s} \\
&\quad-\int_a^x\int_b^y\left(\ln\frac{x}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1}f\frac{dt}{t}\frac{ds}{s}\biggr].
\end{aligned}\tag{18}$$

By (16)-(18), we get

$$\begin{aligned}
e_2(x,y) &= \frac{1-\kappa(I_2(u(x_2^-,y)))-\kappa(I_1(u(x_1^-,y)))}{\Gamma(q_1)\Gamma(q_2)}\biggl[\int_a^{x_2}\int_b^y\left(\ln\frac{x_2}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1}f\frac{dt}{t}\frac{ds}{s} \\
&\quad+\int_{x_2}^x\int_b^y\left(\ln\frac{x}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1}f\frac{dt}{t}\frac{ds}{s} \\
&\quad-\int_a^x\int_b^y\left(\ln\frac{x}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1}f\frac{dt}{t}\frac{ds}{s}\biggr] \\
&\quad+\frac{1-\kappa(I_1(u(x_1^-,y)))}{\Gamma(q_1)\Gamma(q_2)}\biggl[\int_{x_1}^x\int_b^y\left(\ln\frac{x}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1}f\frac{dt}{t}\frac{ds}{s} \\
&\quad-\int_{x_1}^{x_2}\int_b^y\left(\ln\frac{x_2}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1}f\frac{dt}{t}\frac{ds}{s} \\
&\quad-\int_{x_2}^x\int_b^y\left(\ln\frac{x}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1}f\frac{dt}{t}\frac{ds}{s}\biggr].
\end{aligned}\tag{19}$$

Therefore, by (15) and (19), we have

$$\begin{aligned}
u(x,y) &= \bar{u}(x,y)+e_2(x,y) \\
&= \Xi(x,y)+I_1\bigl(u\bigl(x_1^-,y\bigr)\bigr)-I_1\bigl(u\bigl(x_1^-,b\bigr)\bigr)+I_2\bigl(u\bigl(x_2^-,y\bigr)\bigr)-I_2\bigl(u\bigl(x_2^-,b\bigr)\bigr) \\
&\quad+\frac{1}{\Gamma(q_1)\Gamma(q_2)}\int_a^x\int_b^y\left(\ln\frac{x}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1}f\frac{dt}{t}\frac{ds}{s}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1-\kappa(I_1(u(x_1^-,y)))}{\Gamma(q_1)\Gamma(q_2)}\biggl[\int_a^{x_1}\int_b^y\left(\ln\frac{x_1}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1}f\frac{dt}{t}\frac{ds}{s}\\
&+\int_{x_1}^x\int_b^y\left(\ln\frac{x}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1}f\frac{dt}{t}\frac{ds}{s}\\
&-\int_a^x\int_b^y\left(\ln\frac{x}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1}f\frac{dt}{t}\frac{ds}{s}\biggr]\\
&+\frac{1-\kappa(I_2(u(x_2^-,y)))}{\Gamma(q_1)\Gamma(q_2)}\biggl[\int_a^{x_2}\int_b^y\left(\ln\frac{x_2}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1}f\frac{dt}{t}\frac{ds}{s}\\
&+\int_{x_2}^x\int_b^y\left(\ln\frac{x}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1}f\frac{dt}{t}\frac{ds}{s}\\
&-\int_a^x\int_b^y\left(\ln\frac{x}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1}f\frac{dt}{t}\frac{ds}{s}\biggr],\\
&\text{for } (x,y)\in(x_2,x_3]\times[b,B].
\end{aligned}
\tag{20}
$$

On the other hand, for system (1), we have

$$
\lim_{x_2\to x_1}\begin{cases}({}_{\text{C-H}}D^q_{(a+,b+)}u)(x,y)=f(x,y,u(x,y)), & (x,y)\in J \text{ and } x\neq x_1,x_2,\\ u(x_i^+,y)=u(x_i^-,y)+I_i(u(x_i^-,y)), & i=1,2,\\ u(x,b)=\phi(x), \quad u(a,y)=\psi(y), & x\in[a,A], y\in[b,B]\end{cases}
\tag{21}
$$

$$
=\begin{cases}({}_{\text{C-H}}D^q_{(a+,b+)}u)(x,y)=f(x,y,u(x,y)), & (x,y)\in J \text{ and } x\neq x_1,\\ u(x_1^+,y)=u(x_1^-,y)+I_1(u(x_1^-,y))+I_2(u(x_1^-,y)), \\ u(x,b)=\phi(x), \quad u(a,y)=\psi(y), & x\in[a,A], y\in[b,B].\end{cases}
\tag{22}
$$

Using (20) and (14) to (21) and (22), respectively, we get

$$
\begin{aligned}
&1-\kappa\bigl[I_1\bigl(u(x_1^-,y)\bigr)+I_2\bigl(u(x_1^-,y)\bigr)\bigr]=1-\kappa\bigl(I_1\bigl(u(x_1^-,y)\bigr)\bigr)+1-\kappa\bigl(I_2\bigl(u(x_1^-,y)\bigr)\bigr),\\
&\quad\text{for } \forall I_1\bigl(u(x_1^-,y)\bigr) \text{ and } I_2\bigl(u(x_1^-,y)\bigr).
\end{aligned}
\tag{23}
$$

Therefore, $1-\kappa(I_i(u(x_i^-,y)))=\sigma(y)I_i(u(x_i^-,y))$, here $\sigma(y)$ is a differentiable function on y. Thus, (14) and (20) can be rewritten into

$$
\begin{aligned}
u(x,y)={}&\Xi(x,y)+I_1\bigl(u(x_1^-,y)\bigr)-I_1\bigl(u(x_1^-,b)\bigr)\\
&+\frac{1}{\Gamma(q_1)\Gamma(q_2)}\int_a^x\int_b^y\left(\ln\frac{x}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1}f\frac{dt}{t}\frac{ds}{s}\\
&+\frac{\sigma(y)I_1(u(x_1^-,y))}{\Gamma(q_1)\Gamma(q_2)}\biggl[\int_a^{x_1}\int_b^y\left(\ln\frac{x_1}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1}f\frac{dt}{t}\frac{ds}{s}\\
&+\int_{x_1}^x\int_b^y\left(\ln\frac{x}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1}f\frac{dt}{t}\frac{ds}{s}\\
&-\int_a^x\int_b^y\left(\ln\frac{x}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1}f\frac{dt}{t}\frac{ds}{s}\biggr],\\
&\text{for } (x,y)\in(x_1,x_2]\times[b,B],
\end{aligned}
\tag{24}
$$

$$
u(x,y)=\Xi(x,y)+I_1\bigl(u(x_1^-,y)\bigr)-I_1\bigl(u(x_1^-,b)\bigr)+I_2\bigl(u(x_2^-,y)\bigr)-I_2\bigl(u(x_2^-,b)\bigr)
$$

$$
\begin{aligned}
&+ \frac{1}{\Gamma(q_1)\Gamma(q_2)} \int_a^x \int_b^y \left(\ln \frac{x}{s}\right)^{q_1-1} \left(\ln \frac{y}{t}\right)^{q_2-1} f \frac{dt}{t} \frac{ds}{s} \\
&+ \frac{\sigma(y) I_1(u(x_1^-, y))}{\Gamma(q_1)\Gamma(q_2)} \biggl[\int_a^{x_1} \int_b^y \left(\ln \frac{x_1}{s}\right)^{q_1-1} \left(\ln \frac{y}{t}\right)^{q_2-1} f \frac{dt}{t} \frac{ds}{s} \\
&+ \int_{x_1}^x \int_b^y \left(\ln \frac{x}{s}\right)^{q_1-1} \left(\ln \frac{y}{t}\right)^{q_2-1} f \frac{dt}{t} \frac{ds}{s} \\
&- \int_a^x \int_b^y \left(\ln \frac{x}{s}\right)^{q_1-1} \left(\ln \frac{y}{t}\right)^{q_2-1} f \frac{dt}{t} \frac{ds}{s} \biggr] \\
&+ \frac{\sigma(y) I_2(u(x_2^-, y))}{\Gamma(q_1)\Gamma(q_2)} \biggl[\int_a^{x_2} \int_b^y \left(\ln \frac{x_2}{s}\right)^{q_1-1} \left(\ln \frac{y}{t}\right)^{q_2-1} f \frac{dt}{t} \frac{ds}{s} \\
&+ \int_{x_2}^x \int_b^y \left(\ln \frac{x}{s}\right)^{q_1-1} \left(\ln \frac{y}{t}\right)^{q_2-1} f \frac{dt}{t} \frac{ds}{s} \\
&- \int_a^x \int_b^y \left(\ln \frac{x}{s}\right)^{q_1-1} \left(\ln \frac{y}{t}\right)^{q_2-1} f \frac{dt}{t} \frac{ds}{s} \biggr], \\
&\text{for } (x,y) \in (x_2, x_3] \times [b, B].
\end{aligned}
\tag{25}
$$

For $(x,y) \in (x_n, x_{n+1}] \times [b,B]$, suppose

$$
\begin{aligned}
u(x,y) = {} & \Xi(x,y) + \frac{1}{\Gamma(q_1)\Gamma(q_2)} \int_a^x \int_b^y \left(\ln \frac{x}{s}\right)^{q_1-1} \left(\ln \frac{y}{t}\right)^{q_2-1} f \frac{dt}{t} \frac{ds}{s} \\
&+ \sum_{i=1}^{n} \bigl[I_i\bigl(u(x_i^-, y)\bigr) - I_i\bigl(u(x_i^-, b)\bigr) \bigr] \\
&+ \frac{\sigma(y)}{\Gamma(q_1)\Gamma(q_2)} \sum_{i=1}^{n} I_i\bigl(u(x_i^-, y)\bigr) \biggl[\int_a^{x_i} \int_b^y \left(\ln \frac{x_i}{s}\right)^{q_1-1} \left(\ln \frac{y}{t}\right)^{q_2-1} f \frac{dt}{t} \frac{ds}{s} \\
&+ \int_{x_i}^x \int_b^y \left(\ln \frac{x}{s}\right)^{q_1-1} \left(\ln \frac{y}{t}\right)^{q_2-1} f \frac{dt}{t} \frac{ds}{s} \\
&- \int_a^x \int_b^y \left(\ln \frac{x}{s}\right)^{q_1-1} \left(\ln \frac{y}{t}\right)^{q_2-1} f \frac{dt}{t} \frac{ds}{s} \biggr], \\
&\text{for } (x,y) \in (x_n, x_{n+1}] \times [b, B].
\end{aligned}
\tag{26}
$$

Using (26), the approximate solution (when $(x,y) \in (x_{n+1}, x_{n+2}] \times [b,B]$) of (1) can be given by

$$
\begin{aligned}
\bar{u}(x,y) = {} & u(x,b) + u\bigl(x_{n+1}^+, y\bigr) - u\bigl(x_{n+1}^+, b\bigr) \\
&+ \frac{1}{\Gamma(q_1)\Gamma(q_2)} \int_{x_{n+1}}^x \int_b^y \left(\ln \frac{x}{s}\right)^{q_1-1} \left(\ln \frac{y}{t}\right)^{q_2-1} f \frac{dt}{t} \frac{ds}{s}, \\
= {} & \Xi(x,y) + \sum_{i=1}^{n+1} \bigl[I_i\bigl(u(x_i^-, y)\bigr) - I_i\bigl(u(x_i^-, b)\bigr) \bigr] \\
&+ \frac{1}{\Gamma(q_1)\Gamma(q_2)} \biggl[\int_a^{x_{n+1}} \int_b^y \left(\ln \frac{x_{n+1}}{s}\right)^{q_1-1} \left(\ln \frac{y}{t}\right)^{q_2-1} f \frac{dt}{t} \frac{ds}{s} \\
&+ \int_{x_{n+1}}^x \int_b^y \left(\ln \frac{x}{s}\right)^{q_1-1} \left(\ln \frac{y}{t}\right)^{q_2-1} f \frac{dt}{t} \frac{ds}{s} \biggr]
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{\sigma(y)}{\Gamma(q_1)\Gamma(q_2)} \sum_{i=1}^{n} I_i\big(u\big(x_i^-, y\big)\big)\bigg[\int_a^{x_i}\int_b^y \left(\ln\frac{x_i}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1} f \frac{dt}{t}\frac{ds}{s} \\
&+ \int_{x_i}^{x_{n+1}}\int_b^y \left(\ln\frac{x_{n+1}}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1} f \frac{dt}{t}\frac{ds}{s} \\
&- \int_a^{x_{n+1}}\int_b^y \left(\ln\frac{x_{n+1}}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1} f \frac{dt}{t}\frac{ds}{s}\bigg], \\
&\text{for } (x,y) \in (x_{n+1}, x_{n+2}] \times [b,B].
\end{aligned} \tag{27}
$$

Let $e_{n+1}(x,y) = u(x,y) - \bar{u}(x,y)$ for $(x,y) \in (x_{n+1}, x_{n+2}] \times [b,B]$, here $u(x,y)$ denotes the exact solution of system (1). Moreover, by equation (26), the exact solution satisfies

$$
\begin{aligned}
&\lim_{\substack{I_i(u(x_i^-,y))\to 0,\\ \text{for all } i\in\{1,2,\ldots,n+1\}}} u(x,y) = \Xi(x,y) + \frac{1}{\Gamma(q_1)\Gamma(q_2)}\int_a^x\int_b^y \left(\ln\frac{x}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1} f \frac{dt}{t}\frac{ds}{s}, \\
&\text{for } (x,y) \in (x_{n+1}, x_{n+2}] \times [b,B],
\end{aligned} \tag{28}
$$

$$
\begin{aligned}
&\lim_{\substack{I_j(u(x_j^-,y))\to 0,\\ \text{here } j\in\{1,2,\ldots,n+1\}}} u(x,y) \\
&= \Xi(x,y) + \sum_{\substack{1\le i\le n+1,\\ \text{and } i\ne j}} \big[I_i\big(u\big(x_i^-, y\big)\big) - I_i\big(u\big(x_i^-, b\big)\big)\big] \\
&\quad + \frac{1}{\Gamma(q_1)\Gamma(q_2)}\int_a^x\int_b^y \left(\ln\frac{x}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1} f \frac{dt}{t}\frac{ds}{s} \\
&\quad + \frac{\sigma(y)}{\Gamma(q_1)\Gamma(q_2)} \sum_{\substack{1\le i\le n+1,\\ \text{and } i\ne j}} I_i\big(u\big(x_i^-, y\big)\big)\bigg[\int_a^{x_i}\int_b^y \left(\ln\frac{x_i}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1} f \frac{dt}{t}\frac{ds}{s} \\
&\quad + \int_{x_i}^{x}\int_b^y \left(\ln\frac{x}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1} f \frac{dt}{t}\frac{ds}{s} \\
&\quad - \int_a^{x}\int_b^y \left(\ln\frac{x}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1} f \frac{dt}{t}\frac{ds}{s}\bigg], \\
&\quad \text{for } (x,y) \in (x_{n+1}, x_{n+2}] \times [b,B].
\end{aligned} \tag{29}
$$

Thus,

$$
\begin{aligned}
&\lim_{\substack{I_i(u(x_i^-,y))\to 0,\\ \text{for all } i\in\{1,2,\ldots,n+1\}}} e_{n+1}(x,y) \\
&= \lim_{\substack{I_i(u(x_i^-,y))\to 0,\\ \text{for all } i\in\{1,2,\ldots,n+1\}}} \big\{u(x,y) - \bar{u}(x,y)\big\} \\
&= \frac{1}{\Gamma(q_1)\Gamma(q_2)}\bigg[\int_a^x\int_b^y \left(\ln\frac{x}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1} f \frac{dt}{t}\frac{ds}{s} \\
&\quad - \int_a^{x_{n+1}}\int_b^y \left(\ln\frac{x_{n+1}}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1} f \frac{dt}{t}\frac{ds}{s} \\
&\quad - \int_{x_{n+1}}^{x}\int_b^y \left(\ln\frac{x}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1} f \frac{dt}{t}\frac{ds}{s}\bigg],
\end{aligned} \tag{30}
$$

$$\begin{aligned}
&\lim_{\substack{I_j(u(x_j^-,y))\to 0,\\ \text{here } j\in\{1,2,\ldots,n+1\}}} e_{n+1}(x,y)\\
&\quad= \lim_{\substack{I_j(u(x_j^-,y))\to 0,\\ \text{here } j\in\{1,2,\ldots,n+1\}}} \bigl\{u(x,y)-\bar{u}(x,y)\bigr\}\\
&\quad= \frac{1-\sigma(y)\sum_{\substack{1\le i\le n+1,\\ \text{and } i\neq j}} I_i(u(x_i^-,y))}{\Gamma(q_1)\Gamma(q_2)}\biggl[\int_a^x\int_b^y\left(\ln\frac{x}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1} f\frac{dt}{t}\frac{ds}{s}\\
&\qquad-\int_a^{x_{n+1}}\int_b^y\left(\ln\frac{x_{n+1}}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1} f\frac{dt}{t}\frac{ds}{s}\\
&\qquad-\int_{x_{n+1}}^x\int_b^y\left(\ln\frac{x}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1} f\frac{dt}{t}\frac{ds}{s}\biggr]\\
&\qquad+\frac{\sigma(y)}{\Gamma(q_1)\Gamma(q_2)}\sum_{\substack{1\le i\le n+1,\\ \text{and } i\neq j}} I_i\bigl(u\bigl(x_i^-,y\bigr)\bigr)\biggl[\int_{x_i}^x\int_b^y\left(\ln\frac{x}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1} f\frac{dt}{t}\frac{ds}{s}\\
&\qquad-\int_{x_i}^{x_{n+1}}\int_b^y\left(\ln\frac{x_{n+1}}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1} f\frac{dt}{t}\frac{ds}{s}\\
&\qquad-\int_{x_{n+1}}^x\int_b^y\left(\ln\frac{x}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1} f\frac{dt}{t}\frac{ds}{s}\biggr].
\end{aligned}\tag{31}$$

By (30) and (31), we obtain

$$\begin{aligned}
e_{n+1}(x,y) &= \frac{1-\sigma(y)\sum_{1\le i\le n+1} I_i(u(x_i^-,y))}{\Gamma(q_1)\Gamma(q_2)}\biggl[\int_a^x\int_b^y\left(\ln\frac{x}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1} f\frac{dt}{t}\frac{ds}{s}\\
&\quad-\int_a^{x_{n+1}}\int_b^y\left(\ln\frac{x_{n+1}}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1} f\frac{dt}{t}\frac{ds}{s}\\
&\quad-\int_{x_{n+1}}^x\int_b^y\left(\ln\frac{x}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1} f\frac{dt}{t}\frac{ds}{s}\biggr]\\
&\quad+\frac{\sigma(y)}{\Gamma(q_1)\Gamma(q_2)}\sum_{i=1}^{n+1} I_i\bigl(u\bigl(x_i^-,y\bigr)\bigr)\biggl[\int_{x_i}^x\int_b^y\left(\ln\frac{x}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1} f\frac{dt}{t}\frac{ds}{s}\\
&\quad-\int_{x_i}^{x_{n+1}}\int_b^y\left(\ln\frac{x_{n+1}}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1} f\frac{dt}{t}\frac{ds}{s}\\
&\quad-\int_{x_{n+1}}^x\int_b^y\left(\ln\frac{x}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1} f\frac{dt}{t}\frac{ds}{s}\biggr].
\end{aligned}\tag{32}$$

Therefore, by (27) and (32), we get

$$\begin{aligned}
u(x,y) &= \bar{u}(x,y)+e_{n+1}(x,y)\\
&= \Xi(x,y)+\sum_{i=1}^{n+1}\bigl[I_i\bigl(u\bigl(x_i^-,y\bigr)\bigr)-I_i\bigl(u\bigl(x_i^-,b\bigr)\bigr)\bigr]\\
&\quad+\frac{1}{\Gamma(q_1)\Gamma(q_2)}\int_a^x\int_b^y\left(\ln\frac{x}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1} f\frac{dt}{t}\frac{ds}{s}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\sigma(y)}{\Gamma(q_1)\Gamma(q_2)}\sum_{i=1}^{n+1} I_i\big(u\big(x_i^-,y\big)\big)\biggl[\int_a^{x_i}\int_b^y \left(\ln\frac{x_i}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1} f\frac{dt}{t}\frac{ds}{s}\\
&+\int_{x_i}^x\int_b^y \left(\ln\frac{x}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1} f\frac{dt}{t}\frac{ds}{s}\\
&-\int_a^x\int_b^y \left(\ln\frac{x}{s}\right)^{q_1-1}\left(\ln\frac{y}{t}\right)^{q_2-1} f\frac{dt}{t}\frac{ds}{s}\biggr],\\
&\text{for } (x,y)\in(x_{n+1},x_{n+2}]\times[b,B].
\end{aligned}
\tag{33}
$$

Therefore, the solution of system (1) satisfies equation (7). Thus, by necessity and sufficiency, system (1) is equivalent to equation (7). The proof is completed. □

4 Examples

In this section, we will give an example to reveal that there exists a general solution for impulsive fractional partial differential equations.

Example 4.1 Let us consider the impulsive fractional system

$$
\begin{cases}
({}_{\text{C-H}}D^{q}_{(1^+,1^+)}u)(x,y)=\ln x\ln y, & (x,y)\in[1,3]\times[1,3] \text{ and } x\neq 2\\
u(2^+,y)=u(2^-,y)+ly,\\
u(x,1)=u(1,y)\equiv 0, & x\in[1,3], y\in[1,3],
\end{cases}
\tag{34}
$$

where $q=(\frac{1}{2}+j,\frac{1}{2}+j)$ (here j denotes the imaginary unit) and l is a constant. By Theorem 3.1, the general solution of (34) is given by

$$
\begin{aligned}
u(x,y)&=\frac{1}{\Gamma(\frac{1}{2}+j)\Gamma(\frac{1}{2}+j)}\int_1^x\int_1^y\left(\ln\frac{x}{s}\right)^{\frac{1}{2}+j-1}\left(\ln\frac{y}{t}\right)^{\frac{1}{2}+j-1}\ln t\ln s\frac{dt}{t}\frac{ds}{s}\\
&=\frac{1}{\Gamma(\frac{5}{2}+j)\Gamma(\frac{5}{2}+j)}(\ln x)^{\frac{3}{2}+j}(\ln y)^{\frac{3}{2}+j},\quad \text{for } (x,y)\in(1,2]\times(1,3],
\end{aligned}
\tag{35}
$$

$$
\begin{aligned}
u(x,y)&=ly+\frac{1}{\Gamma(\frac{1}{2}+j)\Gamma(\frac{1}{2}+j)}\int_1^x\int_1^y\left(\ln\frac{x}{s}\right)^{\frac{1}{2}+j-1}\left(\ln\frac{y}{t}\right)^{\frac{1}{2}+j-1}\ln t\ln s\frac{dt}{t}\frac{ds}{s}\\
&\quad+\frac{\sigma(y)ly}{\Gamma(\frac{1}{2}+j)\Gamma(\frac{1}{2}+j)}\biggl[\int_1^2\int_1^y\left(\ln\frac{2}{s}\right)^{\frac{1}{2}+j-1}\left(\ln\frac{y}{t}\right)^{\frac{1}{2}+j-1}\ln t\ln s\frac{dt}{t}\frac{ds}{s}\\
&\quad+\int_2^x\int_1^y\left(\ln\frac{x}{s}\right)^{\frac{1}{2}+j-1}\left(\ln\frac{y}{t}\right)^{\frac{1}{2}+j-1}\ln t\ln s\frac{dt}{t}\frac{ds}{s}\\
&\quad-\int_1^x\int_1^y\left(\ln\frac{x}{s}\right)^{\frac{1}{2}+j-1}\left(\ln\frac{y}{t}\right)^{\frac{1}{2}+j-1}\ln t\ln s\frac{dt}{t}\frac{ds}{s}\biggr]\\
&=ly+\frac{1}{\Gamma(\frac{5}{2}+j)\Gamma(\frac{5}{2}+j)}(\ln x)^{\frac{3}{2}+j}\Big|_{x>1}(\ln y)^{\frac{3}{2}+j}\Big|_{y>1}\\
&\quad+\frac{\sigma(y)ly}{\Gamma(\frac{5}{2}+j)\Gamma(\frac{5}{2}+j)}\biggl[(\ln 2)^{\frac{3}{2}+j}+\biggl[\ln x+\left(\frac{1}{2}+j\right)\ln 2\biggr]\left(\ln\frac{x}{2}\right)^{\frac{1}{2}+j}\Big|_{x>2}\\
&\quad-(\ln x)^{\frac{3}{2}+j}\Big|_{x>1}\biggr](\ln y)^{\frac{3}{2}+j}\Big|_{y>1},\quad \text{for } (x,y)\in(2,3]\times(1,3],
\end{aligned}
\tag{36}
$$

where $\sigma(y)$ is a differentiable function on y in equation (36). Next, we will verify that the general solution (35)-(36) satisfies all conditions in system (34). By the Appendix, we have

$$\begin{aligned}
\text{(i)}\quad & {}_{\text{C-H}}D^q_{(1+,1+)}\left\{\frac{\sigma(y)y}{\Gamma(\frac{1}{2}+j)\Gamma(\frac{1}{2}+j)}\left[\int_2^x\int_1^y\left(\ln\frac{x}{s}\right)^{\frac{1}{2}+j-1}\left(\ln\frac{y}{t}\right)^{\frac{1}{2}+j-1}\ln t\ln s\frac{dt}{t}\frac{ds}{s}\right.\right.\\
&\left.\left.-\int_1^x\int_1^y\left(\ln\frac{x}{s}\right)^{\frac{1}{2}+j-1}\left(\ln\frac{y}{t}\right)^{\frac{1}{2}+j-1}\ln t\ln s\frac{dt}{t}\frac{ds}{s}\right]\right\}\\
&=\frac{1}{\Gamma(\frac{1}{2}+j)\Gamma(\frac{1}{2}+j)}\\
&\quad\times {}_{\text{H}}\mathcal{J}^{1-q}_{(1+,1+)}\delta_x\delta_y\left\{\left[\int_2^x\int_1^y\left(\ln\frac{x}{s}\right)^{\frac{1}{2}+j-1}\left(\ln\frac{y}{t}\right)^{\frac{1}{2}+j-1}\ln t\ln s\frac{dt}{t}\frac{ds}{s}\right.\right.\\
&\quad\left.\left.-\int_1^x\int_1^y\left(\ln\frac{x}{s}\right)^{\frac{1}{2}+j-1}\left(\ln\frac{y}{t}\right)^{\frac{1}{2}+j-1}\ln t\ln s\frac{dt}{t}\frac{ds}{s}\right]\sigma(y)y\right\}\\
&\equiv 0.
\end{aligned}$$

Taking fractional derivatives of the two sides of equations (35)-(36), we have

$$\begin{aligned}
&\left({}_{\text{C-H}}D^q_{(1+,1+)}u\right)(x,y)\\
&\quad={}_{\text{C-H}}D^q_{(1+,1+)}\left(\frac{1}{\Gamma(\frac{5}{2}+j)\Gamma(\frac{5}{2}+j)}(\ln x)^{\frac{3}{2}+j}\Big|_{x>1}(\ln y)^{\frac{3}{2}+j}\Big|_{y>1}\right)\\
&\quad=\ln x\ln y,\quad\text{for }(x,y)\in(1,2]\times(1,3],\\
&\left({}_{\text{C-H}}D^q_{(1+,1+)}u\right)(x,y)\\
&\quad={}_{\text{C-H}}D^q_{(1+,1+)}\left\{ly+\frac{1}{\Gamma(\frac{1}{2}+j)\Gamma(\frac{1}{2}+j)}\int_1^x\int_1^y\left(\ln\frac{x}{s}\right)^{\frac{1}{2}+j-1}\left(\ln\frac{y}{t}\right)^{\frac{1}{2}+j-1}\ln t\ln s\frac{dt}{t}\frac{ds}{s}\right.\\
&\qquad+\frac{\sigma(y)ly}{\Gamma(\frac{1}{2}+j)\Gamma(\frac{1}{2}+j)}\left[\int_1^2\int_1^y\left(\ln\frac{2}{s}\right)^{\frac{1}{2}+j-1}\left(\ln\frac{y}{t}\right)^{\frac{1}{2}+j-1}\ln t\ln s\frac{dt}{t}\frac{ds}{s}\right.\\
&\qquad+\int_2^x\int_1^y\left(\ln\frac{x}{s}\right)^{\frac{1}{2}+j-1}\left(\ln\frac{y}{t}\right)^{\frac{1}{2}+j-1}\ln t\ln s\frac{dt}{t}\frac{ds}{s}\\
&\qquad\left.\left.-\int_1^x\int_1^y\left(\ln\frac{x}{s}\right)^{\frac{1}{2}+j-1}\left(\ln\frac{y}{t}\right)^{\frac{1}{2}+j-1}\ln t\ln s\frac{dt}{t}\frac{ds}{s}\right]\right\}\quad\text{(using (i))}\\
&\quad=\ln x\ln y,\quad\text{for }(x,y)\in(2,3]\times(1,3].
\end{aligned}$$

Therefore, equations (35)-(36) satisfy the fractional derivative condition in system (34).

Next, by equations (35)-(36), we have

$$\begin{aligned}
&\left[u(2^+,y)-u(2^-,y)\right]_{y\in(1,3]}\\
&\quad=ly+\frac{1}{\Gamma(\frac{1}{2}+j)\Gamma(\frac{1}{2}+j)}\int_1^2\int_1^y\left(\ln\frac{2}{s}\right)^{\frac{1}{2}+j-1}\left(\ln\frac{y}{t}\right)^{\frac{1}{2}+j-1}\ln t\ln s\frac{dt}{t}\frac{ds}{s}\\
&\qquad-\frac{1}{\Gamma(\frac{1}{2}+j)\Gamma(\frac{1}{2}+j)}\int_1^2\int_1^y\left(\ln\frac{2}{s}\right)^{\frac{1}{2}+j-1}\left(\ln\frac{y}{t}\right)^{\frac{1}{2}+j-1}\ln t\ln s\frac{dt}{t}\frac{ds}{s}
\end{aligned}$$

$$\text{(using (35) and (36))}$$
$$= ly|_{y\in(1,3]}.$$

Therefore, equations (35)-(36) satisfy the impulsive conditions in system (34).

Finally, for system (34), we have

$$\lim_{l\to 0}\begin{cases} ({}_{\text{C-H}}D^{q}_{(1+,1+)}u)(x,y) = \ln x \ln y, & (x,y)\in[1,3]\times[1,3] \text{ and } x\neq 2, \\ u(2^+,y) = u(2^-,y) + ly, \\ u(x,1) = u(1,y) \equiv 0, & x\in[1,3], y\in[1,3] \end{cases}$$
$$= \begin{cases} ({}_{\text{C-H}}D^{q}_{(1+,1+)}u)(x,y) = \ln x \ln y, & (x,y)\in[1,3]\times[1,3], \\ u(x,1) = u(1,y) \equiv 0, & x\in[1,3], y\in[1,3]. \end{cases} \tag{37}$$

On the other hand, by equations (35)-(36), we have

$$\lim_{l\to 0}\{\text{equations (35)-(36)}\}$$
$$\Rightarrow \quad \lim_{l\to 0} u(x,y)$$
$$= \lim_{l\to 0}\begin{cases} \frac{1}{\Gamma(\frac{5}{2}+j)\Gamma(\frac{5}{2}+j)}(\ln x)^{\frac{3}{2}+j}(\ln y)^{\frac{3}{2}+j}, & \text{for } (x,y)\in(1,2]\times(1,3], \\ ly + \frac{1}{\Gamma(\frac{5}{2}+j)\Gamma(\frac{5}{2}+j)}(\ln x)^{\frac{3}{2}+j}|_{x>1}(\ln y)^{\frac{3}{2}+j}|_{y>1} & \\ \quad + \frac{\sigma(y)ly}{\Gamma(\frac{5}{2}+j)\Gamma(\frac{5}{2}+j)}[(\ln 2)^{\frac{3}{2}+j} & \\ \quad + [\ln x + (\frac{1}{2}+j)\ln 2](\ln\frac{x}{2})^{\frac{1}{2}+j}|_{x>2} & \\ \quad - (\ln x)^{\frac{3}{2}+j}|_{x>1}](\ln y)^{\frac{3}{2}+j}|_{y>1}, & \text{for } (x,y)\in(2,3]\times(1,3] \end{cases} \tag{38}$$
$$\Rightarrow \quad u(x,y) = \begin{cases} \frac{1}{\Gamma(\frac{5}{2}+j)\Gamma(\frac{5}{2}+j)}(\ln x)^{\frac{3}{2}+j}(\ln y)^{\frac{3}{2}+j}, & \text{for } (x,y)\in(1,2]\times(1,3], \\ \frac{1}{\Gamma(\frac{5}{2}+j)\Gamma(\frac{5}{2}+j)}(\ln x)^{\frac{3}{2}+j}|_{x>1}(\ln y)^{\frac{3}{2}+j}|_{y>1}, & \text{for } (x,y)\in(2,3]\times(1,3]. \end{cases}$$

By Lemma 2.3, equation (38) is equivalent to system (37). Therefore, equations (35)-(36) satisfy the corresponding condition (3) of system (34). Thus, equations (35)-(36) satisfy all conditions of system (34), that is, equations (35)-(36) is the general solution of system (34).

Appendix

In the section, we will prove the following conclusion.

$$\frac{1}{\Gamma(q_1)\Gamma(q_2)}{}_{\text{H}}\mathcal{J}^{1-q}_{(a+,b+)}\delta_x\delta_y\biggl[\int_{x_i}^{x}\left(\ln\frac{x}{s}\right)^{q_1-1}\left(\int_b^y \sigma(y)I_i\bigl(u\bigl(x_i^-,y\bigr)\bigr)\left(\ln\frac{y}{t}\right)^{q_2-1}h(s,t)\frac{dt}{t}\right)\frac{ds}{s}$$
$$-\int_a^x\left(\ln\frac{x}{s}\right)^{q_1-1}\left(\int_b^y \sigma(y)I_i\bigl(u\bigl(x_i^-,y\bigr)\bigr)\left(\ln\frac{y}{t}\right)^{q_2-1}h(s,t)\frac{dt}{t}\right)\frac{ds}{s}\biggr] = 0. \tag{A.1}$$

Proof Let $\sigma(y)I_i(u(x_i^-,y)) = \vartheta(y)$. For the sake of convenience, we divide the calculation into several steps.

Step 1. Compute

$$\frac{1}{\Gamma(q_1)\Gamma(q_2)}{}_{\text{H}}\mathcal{J}^{1-q}_{(a+,b+)}\delta_x\delta_y\biggl[\int_{x_i}^{x}\left(\ln\frac{x}{s}\right)^{q_1-1}\left(\int_b^y \vartheta(y)\left(\ln\frac{y}{t}\right)^{q_2-1}h(s,t)\frac{dt}{t}\right)\frac{ds}{s}\biggr].$$

First of all, $\int_{x_i}^{x}(\ln\frac{x}{s})^{q_1-1}(\int_b^y\vartheta(y)(\ln\frac{y}{t})^{q_2-1}h(s,t)\frac{dt}{t})\frac{ds}{s}$ is transformed as

$$\begin{aligned}
&\int_{x_i}^{x}\left(\ln\frac{x}{s}\right)^{q_1-1}\left(\int_b^y\vartheta(y)\left(\ln\frac{y}{t}\right)^{q_2-1}h(s,t)\frac{dt}{t}\right)\frac{ds}{s}\\
&=\frac{1}{q_1q_2}\int_{x_i}^{x}\left(\int_b^y\vartheta(y)h(s,t)d\left(\ln\frac{y}{t}\right)^{q_2}\right)d\left(\ln\frac{x}{s}\right)^{q_1}\\
&=\frac{1}{q_1q_2}\left[\left(\ln\frac{x}{s}\right)^{q_1}\int_b^y\vartheta(y)h(s,t)\,d\left(\ln\frac{y}{t}\right)^{q_2}\Bigg|_{x_i}^{x}\right.\\
&\quad\left.-\int_{x_i}^{x}\left(\ln\frac{x}{s}\right)^{q_1}\partial_s\left(\int_b^y\vartheta(y)h(s,t)\,d\left(\ln\frac{y}{t}\right)^{q_2}\right)\right]\\
&=\frac{1}{q_1q_2}\left[-\left(\ln\frac{x}{x_i}\right)^{q_1}\int_b^y\vartheta(y)h(x_i,t)\,d\left(\ln\frac{y}{t}\right)^{q_2}\right.\\
&\quad\left.-\int_{x_i}^{x}\left(\ln\frac{x}{s}\right)^{q_1}\left(\int_b^y\vartheta(y)\frac{\partial h(s,t)}{\partial s}\,d\left(\ln\frac{y}{t}\right)^{q_2}\right)ds\right]\\
&=\frac{1}{q_1q_2}\left\{-\left(\ln\frac{x}{x_i}\right)^{q_1}\left(\ln\frac{y}{t}\right)^{q_2}\vartheta(y)h(x_i,t)\Bigg|_b^y\right.\\
&\quad+\left(\ln\frac{x}{x_i}\right)^{q_1}\int_b^y\left(\ln\frac{y}{t}\right)^{q_2}\vartheta(y)\frac{\partial h(x_i,t)}{\partial t}\,dt\\
&\quad\left.-\int_{x_i}^{x}\left(\ln\frac{x}{s}\right)^{q_1}\left[-\left(\ln\frac{y}{b}\right)^{q_2}\vartheta(y)\frac{\partial h(s,b)}{\partial s}-\int_b^y\left(\ln\frac{y}{t}\right)^{q_2}\vartheta(y)\frac{\partial^2 h(s,t)}{\partial s\,\partial t}\,dt\right]ds\right\}\\
&=\frac{1}{q_1q_2}\left\{\left(\ln\frac{x}{x_i}\right)^{q_1}\left(\ln\frac{y}{b}\right)^{q_2}\vartheta(y)h(x_i,b)+\left(\ln\frac{x}{x_i}\right)^{q_1}\int_b^y\left(\ln\frac{y}{t}\right)^{q_2}\vartheta(y)\frac{\partial h(x_i,t)}{\partial t}\,dt\right.\\
&\quad+\left(\ln\frac{y}{b}\right)^{q_2}\vartheta(y)\int_{x_i}^{x}\left(\ln\frac{x}{s}\right)^{q_1}\frac{\partial h(s,b)}{\partial s}\,ds\\
&\quad\left.+\int_{x_i}^{x}\left(\ln\frac{x}{s}\right)^{q_1}\left[\int_b^y\left(\ln\frac{y}{t}\right)^{q_2}\vartheta(y)\frac{\partial^2 h(s,t)}{\partial s\,\partial t}\,dt\right]ds\right\}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{1}{\Gamma(q_1)\Gamma(q_2)}{}_{\mathrm{H}}\mathcal{J}^{1-q}_{(a+,b+)}\delta_x\delta_y\left[\int_{x_i}^{x}\left(\ln\frac{x}{s}\right)^{q_1-1}\left(\int_b^y\vartheta(y)\left(\ln\frac{y}{t}\right)^{q_2-1}h(s,t)\frac{dt}{t}\right)\frac{ds}{s}\right]\\
&=\frac{{}_{\mathrm{H}}\mathcal{J}^{1-q}_{(a+,b+)}\delta_x\delta_y}{\Gamma(q_1)\Gamma(q_2)q_1q_2}\left\{\left(\ln\frac{x}{x_i}\right)^{q_1}\left(\ln\frac{y}{b}\right)^{q_2}\vartheta(y)h(x_i,b)\right.\\
&\quad+\left(\ln\frac{x}{x_i}\right)^{q_1}\int_b^y\left(\ln\frac{y}{t}\right)^{q_2}\vartheta(y)\frac{\partial h(x_i,t)}{\partial t}\,dt\\
&\quad+\left(\ln\frac{y}{b}\right)^{q_2}\vartheta(y)\int_{x_i}^{x}\left(\ln\frac{x}{s}\right)^{q_1}\frac{\partial h(s,b)}{\partial s}\,ds\\
&\quad\left.+\int_{x_i}^{x}\left(\ln\frac{x}{s}\right)^{q_1}\left[\int_b^y\left(\ln\frac{y}{t}\right)^{q_2}\vartheta(y)\frac{\partial^2 h(s,t)}{\partial s\,\partial t}\,dt\right]ds\right\}_{x\ge x_i}\\
&=\frac{1}{\Gamma(q_1)\Gamma(q_2)}{}_{\mathrm{H}}\mathcal{J}^{1-q}_{(a+,b+)}\left\{\left(\ln\frac{x}{x_i}\right)^{q_1-1}\left(\ln\frac{y}{b}\right)^{q_2-1}h(x_i,b)\left(\vartheta(y)+\frac{1}{q_2}y\vartheta'(y)\ln\frac{y}{b}\right)\right.\\
&\quad+\left(\ln\frac{x}{x_i}\right)^{q_1-1}\int_b^y\left(\ln\frac{y}{t}\right)^{q_2-1}\frac{\partial h(x_i,t)}{\partial t}\left(\vartheta(y)+\frac{1}{q_2}y\vartheta'(y)\ln\frac{y}{t}\right)dt
\end{aligned}$$

$$+\left(\ln\frac{y}{b}\right)^{q_2-1}\left(\vartheta(y)+\frac{1}{q_2}y\vartheta'(y)\ln\frac{y}{b}\right)\int_{x_i}^{x}\left(\ln\frac{x}{s}\right)^{q_1-1}\frac{\partial h(s,b)}{\partial s}\,ds+\int_{x_i}^{x}\left(\ln\frac{x}{s}\right)^{q_1-1}$$
$$\times\left[\int_b^y\left(\ln\frac{y}{t}\right)^{q_2-1}\left(\vartheta(y)+\frac{1}{q_2}y\vartheta'(y)\ln\frac{y}{t}\right)\frac{\partial^2 h(s,t)}{\partial s\,\partial t}\,dt\right]ds\Bigg\}_{x\geq x_i}. \tag{A.2}$$

By Definition 2.1, we have

$$\begin{aligned}
&\frac{1}{\Gamma(q_1)\Gamma(q_2)}{}_{\mathrm{H}}\mathcal{J}^{1-q}_{(a+,b+)}\left\{\left(\ln\frac{x}{x_i}\right)^{q_1-1}\left(\ln\frac{y}{b}\right)^{q_2-1}h(x_i,b)\left(\vartheta(y)+\frac{1}{q_2}y\vartheta'(y)\ln\frac{y}{b}\right)\right\}_{x\geq x_i}\\
&\quad=\frac{1}{\Gamma(q_1)\Gamma(q_2)}\frac{1}{\Gamma(1-q_1)\Gamma(1-q_2)}\\
&\qquad\times\int_{x_i}^{x}\int_b^y\left(\ln\frac{x}{\xi}\right)^{-q_1}\left(\ln\frac{y}{\eta}\right)^{-q_2}\left(\ln\frac{\xi}{x_i}\right)^{q_1-1}\left(\ln\frac{\eta}{b}\right)^{q_2-1}h(x_i,b)\\
&\qquad\times\left(\vartheta(\eta)+\frac{1}{q_2}\eta\vartheta'(\eta)\ln\frac{\eta}{b}\right)\frac{d\eta}{\eta}\frac{d\xi}{\xi}\\
&\quad=\frac{h(x_i,b)}{\Gamma(q_1)\Gamma(q_2)}\frac{1}{\Gamma(1-q_1)\Gamma(1-q_2)}\int_b^y\left(\ln\frac{y}{\eta}\right)^{-q_2}\left(\ln\frac{\eta}{b}\right)^{q_2-1}\left(\vartheta(\eta)+\frac{1}{q_2}\eta\vartheta'(\eta)\ln\frac{\eta}{b}\right)\\
&\qquad\times\left\{\int_{x_i}^{x}\big(\ln x-\ln x_i-(\ln\xi-\ln x_i)\big)^{1-q_1-1}(\ln\xi-\ln x_i)^{q_1-1}\,d(\ln\xi-\ln x_i)\right\}\frac{d\eta}{\eta}\\
&\qquad\left(\text{let }\frac{\xi}{x_i}=\varphi\right)\\
&\quad=\frac{h(x_i,b)}{\Gamma(q_1)\Gamma(1-q_2)}\int_b^y\left(\ln\frac{y}{\eta}\right)^{-q_2}\left(\ln\frac{\eta}{b}\right)^{q_2-1}\left(\vartheta(\eta)+\frac{1}{q_2}\eta\vartheta'(\eta)\ln\frac{\eta}{b}\right)\frac{d\eta}{\eta};
\end{aligned}\tag{A.3}$$

$$\begin{aligned}
&\frac{1}{\Gamma(q_1)\Gamma(q_2)}{}_{\mathrm{H}}\mathcal{J}^{1-q}_{(a+,b+)}\left\{\left(\ln\frac{x}{x_i}\right)^{q_1-1}\int_b^y\left(\ln\frac{y}{t}\right)^{q_2-1}\right.\\
&\qquad\left.\times\frac{\partial h(x_i,t)}{\partial t}\left(\vartheta(y)+\frac{1}{q_2}y\vartheta'(y)\ln\frac{y}{t}\right)dt\right\}_{x\geq x_i}\\
&\quad=\frac{1}{\Gamma(q_1)\Gamma(q_2)}\frac{1}{\Gamma(1-q_1)\Gamma(1-q_2)}\int_{x_i}^{x}\int_b^y\left(\ln\frac{x}{\xi}\right)^{-q_1}\left(\ln\frac{y}{\eta}\right)^{-q_2}\left\{\left(\ln\frac{\xi}{x_i}\right)^{q_1-1}\right.\\
&\qquad\left.\times\int_b^\eta\left(\ln\frac{\eta}{t}\right)^{q_2-1}\frac{\partial h(x_i,t)}{\partial t}\left(\vartheta(\eta)+\frac{1}{q_2}\eta\vartheta'(\eta)\ln\frac{\eta}{t}\right)dt\right\}\frac{d\eta}{\eta}\frac{d\xi}{\xi}\\
&\quad=\frac{1}{\Gamma(q_1)\Gamma(q_2)}\frac{1}{\Gamma(1-q_1)\Gamma(1-q_2)}\int_{x_i}^{x}\left(\ln\frac{x}{x_i}-\ln\frac{\xi}{x_i}\right)^{-q_1}\left(\ln\frac{\xi}{x_i}\right)^{q_1-1}d\left(\ln\frac{\xi}{x_i}\right)\\
&\qquad\times\int_b^y\left(\ln\frac{y}{\eta}\right)^{-q_2}\left[\int_b^\eta\left(\ln\frac{\eta}{t}\right)^{q_2-1}\frac{\partial h(x_i,t)}{\partial t}\left(\vartheta(\eta)+\frac{1}{q_2}\eta\vartheta'(\eta)\ln\frac{\eta}{t}\right)dt\right]\frac{d\eta}{\eta}\\
&\qquad\left(\text{let }\frac{\xi}{x_i}=\varphi\right)\\
&\quad=\frac{1}{\Gamma(q_2)\Gamma(1-q_2)}\int_b^y\left(\ln\frac{y}{\eta}\right)^{-q_2}\\
&\qquad\times\left[\int_b^\eta\left(\ln\frac{\eta}{t}\right)^{q_2-1}\frac{\partial h(x_i,t)}{\partial t}\left(\vartheta(\eta)+\frac{1}{q_2}\eta\vartheta'(\eta)\ln\frac{\eta}{t}\right)dt\right]\frac{d\eta}{\eta}\\
&\quad=\frac{1}{\Gamma(q_2)\Gamma(1-q_2)}\int_b^y\int_t^y\left(\ln\frac{y}{\eta}\right)^{-q_2}\left(\ln\frac{\eta}{t}\right)^{q_2-1}
\end{aligned}$$

$$\times \frac{\partial h(x_i,t)}{\partial t}\left(\vartheta(\eta)+\frac{1}{q_2}\eta\vartheta'(\eta)\ln\frac{\eta}{t}\right)\frac{d\eta}{\eta}\,dt; \tag{A.4}$$

$$\begin{aligned}
&\frac{1}{\Gamma(q_1)\Gamma(q_2)}{}_{\mathrm{H}}\mathcal{J}^{1-q}_{(a+,b+)}\left\{\left(\ln\frac{y}{b}\right)^{q_2-1}\left(\vartheta(y)+\frac{1}{q_2}y\vartheta'(y)\ln\frac{y}{b}\right)\right.\\
&\quad\left.\times\int_{x_i}^{x}\left(\ln\frac{x}{s}\right)^{q_1-1}\frac{\partial h(s,b)}{\partial s}\,ds\right\}_{x\geq x_i}\\
&=\frac{1}{\Gamma(q_1)\Gamma(q_2)}\frac{1}{\Gamma(1-q_1)\Gamma(1-q_2)}\int_{x_i}^{x}\int_b^y\left(\ln\frac{x}{\xi}\right)^{-q_1}\left(\ln\frac{y}{\eta}\right)^{-q_2}\left\{\left(\ln\frac{\eta}{b}\right)^{q_2-1}\right.\\
&\quad\left.\times\left(\vartheta(\eta)+\frac{1}{q_2}\eta\vartheta'(\eta)\ln\frac{\eta}{b}\right)\int_{x_i}^{\xi}\left(\ln\frac{\xi}{s}\right)^{q_1-1}\frac{\partial h(s,b)}{\partial s}\,ds\right\}\frac{d\eta}{\eta}\frac{d\xi}{\xi}\\
&=\frac{1}{\Gamma(q_1)\Gamma(q_2)}\frac{1}{\Gamma(1-q_1)\Gamma(1-q_2)}\int_{x_i}^{x}\left(\ln\frac{x}{\xi}\right)^{-q_1}\left\{\int_{x_i}^{\xi}\left(\ln\frac{\xi}{s}\right)^{q_1-1}\frac{\partial h(s,b)}{\partial s}\right.\\
&\quad\left.\times\int_b^y\left(\ln\frac{y}{\eta}\right)^{-q_2}\left(\ln\frac{\eta}{b}\right)^{q_2-1}\left(\vartheta(\eta)+\frac{1}{q_2}\eta\vartheta'(\eta)\ln\frac{\eta}{b}\right)\frac{d\eta}{\eta}\right\}ds\frac{d\xi}{\xi}\\
&=\frac{1}{\Gamma(q_1)\Gamma(q_2)}\frac{1}{\Gamma(1-q_1)\Gamma(1-q_2)}\int_{x_i}^{x}\frac{\partial h(s,b)}{\partial s}\int_s^x\left(\ln\frac{x}{\xi}\right)^{-q_1}\left(\ln\frac{\xi}{s}\right)^{q_1-1}\frac{d\xi}{\xi}\,ds\\
&\quad\times\int_b^y\left(\ln\frac{y}{\eta}\right)^{-q_2}\left(\ln\frac{\eta}{b}\right)^{q_2-1}\left(\vartheta(\eta)+\frac{1}{q_2}\eta\vartheta'(\eta)\ln\frac{\eta}{b}\right)\frac{d\eta}{\eta}\\
&=\frac{1}{\Gamma(q_2)\Gamma(1-q_2)}\int_{x_i}^{x}\frac{\partial h(s,b)}{\partial s}\,ds\\
&\quad\times\int_b^y\left(\ln\frac{y}{\eta}\right)^{-q_2}\left(\ln\frac{\eta}{b}\right)^{q_2-1}\left(\vartheta(\eta)+\frac{1}{q_2}\eta\vartheta'(\eta)\ln\frac{\eta}{b}\right)\frac{d\eta}{\eta}\\
&=\frac{h(x,b)-h(x_i,b)}{\Gamma(q_2)\Gamma(1-q_2)}\int_b^y\left(\ln\frac{y}{\eta}\right)^{-q_2}\left(\ln\frac{\eta}{b}\right)^{q_2-1}\left(\vartheta(\eta)+\frac{1}{q_2}\eta\vartheta'(\eta)\ln\frac{\eta}{b}\right)\frac{d\eta}{\eta};
\end{aligned}\tag{A.5}$$

$$\begin{aligned}
&\frac{1}{\Gamma(q_1)\Gamma(q_2)}{}_{\mathrm{H}}\mathcal{J}^{1-q}_{(a+,b+)}\left\{\int_{x_i}^{x}\left(\ln\frac{x}{s}\right)^{q_1-1}\right.\\
&\quad\left.\times\left[\int_b^y\left(\ln\frac{y}{t}\right)^{q_2-1}\left(\vartheta(y)+\frac{1}{q_2}y\vartheta'(y)\ln\frac{y}{t}\right)\frac{\partial^2 h(s,t)}{\partial s\,\partial t}\,dt\right]ds\right\}_{x\geq x_i}\\
&=\frac{1}{\Gamma(q_1)\Gamma(q_2)}\frac{1}{\Gamma(1-q_1)\Gamma(1-q_2)}\int_{x_i}^{x}\int_b^y\left(\ln\frac{x}{\xi}\right)^{-q_1}\left(\ln\frac{y}{\eta}\right)^{-q_2}\int_{x_i}^{\xi}\left(\ln\frac{\xi}{s}\right)^{q_1-1}\\
&\quad\times\left[\int_b^\eta\left(\ln\frac{\eta}{t}\right)^{q_2-1}\left(\vartheta(\eta)+\frac{1}{q_2}\eta\vartheta'(\eta)\ln\frac{\eta}{t}\right)\frac{\partial^2 h(s,t)}{\partial s\,\partial t}\,dt\right]ds\frac{d\eta}{\eta}\frac{d\xi}{\xi}\\
&=\frac{1}{\Gamma(q_1)\Gamma(q_2)}\frac{1}{\Gamma(1-q_1)\Gamma(1-q_2)}\int_{x_i}^{x}\int_{x_i}^{\xi}\left(\ln\frac{\xi}{s}\right)^{q_1-1}\left(\ln\frac{x}{\xi}\right)^{-q_1}\\
&\quad\times\int_b^y\left(\ln\frac{y}{\eta}\right)^{-q_2}\left[\int_b^\eta\left(\ln\frac{\eta}{t}\right)^{q_2-1}\left(\vartheta(\eta)+\frac{1}{q_2}\eta\vartheta'(\eta)\ln\frac{\eta}{t}\right)\frac{\partial^2 h(s,t)}{\partial s\,\partial t}\,dt\right]\frac{d\eta}{\eta}\,ds\frac{d\xi}{\xi}\\
&=\frac{1}{\Gamma(q_1)\Gamma(q_2)}\frac{1}{\Gamma(1-q_1)\Gamma(1-q_2)}\int_{x_i}^{x}\int_s^x\left(\ln\frac{\xi}{s}\right)^{q_1-1}\left(\ln\frac{x}{\xi}\right)^{-q_1}\\
&\quad\times\int_b^y\frac{\partial^2 h(s,t)}{\partial s\,\partial t}\left[\int_t^y\left(\ln\frac{y}{\eta}\right)^{-q_2}\left(\ln\frac{\eta}{t}\right)^{q_2-1}\left(\vartheta(\eta)+\frac{1}{q_2}\eta\vartheta'(\eta)\ln\frac{\eta}{t}\right)\frac{d\eta}{\eta}\right]dt\frac{d\xi}{\xi}\,ds
\end{aligned}$$

$$= \frac{1}{\Gamma(q_2)\Gamma(1-q_2)} \int_{x_i}^{x} \int_{b}^{y} \frac{\partial^2 h(s,t)}{\partial s\, \partial t}$$

$$\times \left[\int_t^y \left(\ln \frac{y}{\eta}\right)^{-q_2} \left(\ln \frac{\eta}{t}\right)^{q_2-1} \left(\vartheta(\eta) + \frac{1}{q_2} \eta \vartheta'(\eta) \ln \frac{\eta}{t}\right) \frac{d\eta}{\eta} \right] dt\, ds$$

$$= \frac{1}{\Gamma(q_2)\Gamma(1-q_2)} \int_b^y \int_t^y \left(\ln \frac{y}{\eta}\right)^{-q_2} \left(\ln \frac{\eta}{t}\right)^{q_2-1}$$

$$\left(\vartheta(\eta) + \frac{1}{q_2} \eta \vartheta'(\eta) \ln \frac{\eta}{t}\right) \frac{\partial h(x,t) - \partial h(x_i,t)}{\partial t} \frac{d\eta}{\eta}\, dt. \tag{A.6}$$

Substitute (A.3)-(A.6) into equation (A.2), we have

$$\frac{1}{\Gamma(q_1)\Gamma(q_2)} {}_{\mathrm{H}}\mathcal{J}^{1-q}_{(a+,b+)} \delta_x \delta_y \left[\int_{x_i}^{x} \left(\ln \frac{x}{s}\right)^{q_1-1} \left(\int_b^y \vartheta(y) \left(\ln \frac{y}{t}\right)^{q_2-1} h(s,t) \frac{dt}{t} \right) \frac{ds}{s} \right]$$

$$= \frac{h(x,b)}{\Gamma(q_2)\Gamma(1-q_2)} \int_b^y \left(\ln \frac{y}{\eta}\right)^{-q_2} \left(\ln \frac{\eta}{b}\right)^{q_2-1} \left(\vartheta(\eta) + \frac{1}{q_2} \eta \vartheta'(\eta) \ln \frac{\eta}{b}\right) \frac{d\eta}{\eta}$$

$$+ \frac{1}{\Gamma(q_2)\Gamma(1-q_2)} \int_b^y \int_t^y \left(\ln \frac{y}{\eta}\right)^{-q_2} \left(\ln \frac{\eta}{t}\right)^{q_2-1}$$

$$\times \left(\vartheta(\eta) + \frac{1}{q_2} \eta \vartheta'(\eta) \ln \frac{\eta}{t}\right) \frac{\partial h(x,t)}{\partial t} \frac{d\eta}{\eta}\, dt. \tag{A.7}$$

Step 2. Compute

$$\frac{1}{\Gamma(q_1)\Gamma(q_2)} {}_{\mathrm{H}}\mathcal{J}^{1-q}_{(a+,b+)} \delta_x \delta_y \left[\int_a^x \left(\ln \frac{x}{s}\right)^{q_1-1} \right.$$

$$\left. \times \left(\int_b^y \sigma(y) I_i\big(u\big(x_i^-, y\big)\big) \left(\ln \frac{y}{t}\right)^{q_2-1} h(s,t) \frac{dt}{t} \right) \frac{ds}{s} \right].$$

Letting $x_i = a$ in (A.7), we obtain

$$\frac{1}{\Gamma(q_1)\Gamma(q_2)} {}_{\mathrm{H}}\mathcal{J}^{1-q}_{(a+,b+)} \delta_x \delta_y \int_a^x \left(\ln \frac{x}{s}\right)^{q_1-1} \left(\int_b^y \sigma(y) I_i\big(u\big(x_i^-, y\big)\big) \left(\ln \frac{y}{t}\right)^{q_2-1} h(s,t) \frac{dt}{t} \right) \frac{ds}{s}$$

$$= \frac{h(x,b)}{\Gamma(q_2)\Gamma(1-q_2)} \int_b^y \left(\ln \frac{y}{\eta}\right)^{-q_2} \left(\ln \frac{\eta}{b}\right)^{q_2-1} \left(\vartheta(\eta) + \frac{1}{q_2} \eta \vartheta'(\eta) \ln \frac{\eta}{b}\right) \frac{d\eta}{\eta}$$

$$+ \frac{1}{\Gamma(q_2)\Gamma(1-q_2)} \int_b^y \int_t^y \left(\ln \frac{y}{\eta}\right)^{-q_2} \left(\ln \frac{\eta}{t}\right)^{q_2-1}$$

$$\times \left(\vartheta(\eta) + \frac{1}{q_2} \eta \vartheta'(\eta) \ln \frac{\eta}{t}\right) \frac{\partial h(x,t)}{\partial t} \frac{d\eta}{\eta}\, dt. \tag{A.8}$$

Therefore, by (A.7) and (A.8), we get

$$\frac{1}{\Gamma(q_1)\Gamma(q_2)} {}_{\mathrm{H}}\mathcal{J}^{1-q}_{(a+,b+)} \delta_x \delta_y \left[\int_{x_i}^{x} \left(\ln \frac{x}{s}\right)^{q_1-1} \left(\int_b^y \sigma(y) I_i\big(u\big(x_i^-, y\big)\big) \left(\ln \frac{y}{t}\right)^{q_2-1} h(s,t) \frac{dt}{t} \right) \frac{ds}{s} \right.$$

$$\left. - \int_a^x \left(\ln \frac{x}{s}\right)^{q_1-1} \left(\int_b^y \sigma(y) I_i\big(u\big(x_i^-, y\big)\big) \left(\ln \frac{y}{t}\right)^{q_2-1} h(s,t) \frac{dt}{t} \right) \frac{ds}{s} \right] = 0.$$

The proof is now completed. □

Competing interests
The author declares that he has no competing interests.

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