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Strong convergence theorems for Bregman W-mappings with applications to convex feasibility problems in Banach spaces

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Abstract

In this paper we introduce new modified Mann iterative processes for computing fixed points of an infinite family of Bregman W-mappings in reflexive Banach spaces. Let W_n be the Bregman W-mapping generated by $S_n, S_{n-1}, \ldots, S_1$ and $\beta_{n,n}, \beta_{n,n-1}, \ldots, \beta_{n,1}$. We first express the set of fixed points of W_n as the intersection of fixed points of $\{S_i\}_{i=1}^n$. As a consequence, we show that W_n is a Bregman weak relatively nonexpansive mapping if S_i is a Bregman weak relatively nonexpansive mapping for each $i = 1, 2, \ldots, n$. When specialized to the fixed point set of a Bregman nonexpansive type mapping T, the required sufficient condition $\widetilde{F}(T) = F(T)$ is less restrictive than the usual condition $\hat{F}(T) = F(T)$ which is based on the demiclosedness principle. We then prove some strong convergence theorems for these mappings. Some application of our results to convex feasibility problem is also presented. Our results improve and generalize many known results in the current literature.

MSC: 47H10; 37C25

Keywords: Bregman function; uniformly convex function; Bregman W-mapping generated by $S_n, S_{n-1}, \ldots, S_1$ and $\beta_{n,n}, \beta_{n,n-1}, \ldots, \beta_{n,1}$; uniformly smooth function; fixed point; strong convergence

1 Introduction

Throughout this paper, we denote the set of real numbers and the set of positive integers by $\mathbb{R}$ and $\mathbb{N}$, respectively. Let E be a Banach space with the norm $\|\cdot\|$ and the dual space E^*. For any $x \in E$, we denote the value of $x^* \in E^*$ at x by $\langle x, x^* \rangle$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in E, we denote the strong convergence of $\{x_n\}_{n\in\mathbb{N}}$ to $x \in E$ as $n \to \infty$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. The modulus δ of convexity of E is denoted by

$$\delta(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \epsilon\right\}$$

for every ϵ with $0 \leq \epsilon \leq 2$. A Banach space E is said to be *uniformly convex* if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. Let $S_E = \{x \in E : \|x\| = 1\}$. The norm of E is said to be *Gâteaux differentiable* if for each $x, y \in S_E$, the limit

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{1.1}$$

exists. In this case, E is called *smooth*. If the limit (1.1) is attained uniformly for all $x, y \in S_E$, then E is called *uniformly smooth*. The Banach space E is said to be *strictly convex* if $\|\frac{x+y}{2}\| < 1$ whenever $x, y \in S_E$ and $x \neq y$. It is well known that E is uniformly convex if and only if E^* is uniformly smooth. It is also known that if E is reflexive, then E is strictly convex if and only if E^* is smooth; for more details, see [1, 2].

Let C be a nonempty subset of E. Let $T : C \to E$ be a mapping. We denote the set of fixed points of T by $F(T)$, *i.e.*, $F(T) = \{x \in C : Tx = x\}$. A mapping $T : C \to E$ is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A mapping $T : C \to E$ is said to be *quasi-nonexpansive* if $F(T) \neq \emptyset$ and $\|Tx - y\| \leq \|x - y\|$ for all $x \in C$ and $y \in F(T)$. The mapping T is called *closed*, if for any sequence $\{x_n\}_{n\in\mathbb{N}} \subset C$ with $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} Tx_n = y_0$, we have $Tx_0 = y_0$. The concept of nonexpansivity plays an important role in the study of Mann-type iteration [3] for finding fixed points of a mapping $T : C \to C$. Recall that the Mann-type iteration is given by the following formula:

$$x_{n+1} = \gamma_n Tx_n + (1 - \gamma_n)x_n, \quad x_1 \in C. \tag{1.2}$$

Here, $\{\gamma_n\}_{n\in\mathbb{N}}$ is a sequence of real numbers in $[0, 1]$ satisfying some appropriate conditions. The construction of fixed points of nonexpansive mappings via Mann's algorithm [3] has been extensively investigated recently in the current literature (see, for example, [4] and the references therein). In [4], Reich proved that the sequence $\{x_n\}_{n\in\mathbb{N}}$ generated by Mann's algorithm (1.2) converges weakly to a fixed point of T. However, the convergence of the sequence $\{x_n\}_{n\in\mathbb{N}}$ generated by Mann's algorithm (1.2) is in general not strong (see a counterexample in [5]; see also [6, 7]). Some attempts to modify the Mann iteration method (1.2) so that strong convergence is guaranteed have recently been made. Bauschke and Combettes [8] proposed another modification of the Mann iteration process for a single nonexpansive mapping T in a Hilbert space H. Then they proved that if the sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ is bounded above from one, then the sequence $\{x_n\}_{n\in\mathbb{N}}$ generated by (1.2) converges strongly to a fixed point of T, see also Nakajo and Takahashi [9].

Let E be a smooth, strictly convex and reflexive Banach space and let J be the normalized duality mapping of E. Let C be a nonempty, closed and convex subset of E. The generalized projection Π_C from E onto C is defined and denoted by

$$\Pi_C(x) = \operatorname*{arg\,min}_{y\in C} \phi(y, x), \tag{1.3}$$

where $\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$. For more details, see [10].

Let C be a nonempty, closed and convex subset of a smooth Banach space E, let T be a mapping from C into itself. A point $p \in C$ is said to be an *asymptotic fixed point* [11] of T if there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in C which converges weakly to p and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. We denote the set of all asymptotic fixed points of T by $\hat{F}(T)$. A point $p \in C$ is called a *strong asymptotic fixed point* of T if there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in C which converges strongly to p and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. We denote the set of all strong asymptotic fixed points of T by $\tilde{F}(T)$.

Following Matsushita and Takahashi [12], a mapping $T : C \to C$ is said to be *relatively nonexpansive* if the following conditions are satisfied:

(1) $F(T)$ is nonempty;

(2) $\phi(u, Tx) \leq \phi(u,x)$, $\forall u \in F(T)$, $x \in C$;
(3) $\hat{F}(T) = F(T)$.

In 2005, Matsushita and Takahashi [12] proved the following strong convergence theorem for relatively nonexpansive mappings in a Banach space.

Theorem 1.1 *Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty, closed and convex subset of E, let T be a relatively nonexpansive mapping from C into itself, and let $\{\alpha_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers such that $0 \leq \alpha_n < 1$ and $\limsup_{n\to\infty} \alpha_n < 1$. Suppose that $\{x_n\}_{n\in\mathbb{N}}$ is given by*

$$\begin{cases} x_0 = x \in C, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \\ H_n = \{z \in C_n : \phi(z,y_n) \leq \phi(z,x_n)\}, \\ W_n = \{z \in C : \langle x_n - z, Jx - Jx_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{H_n\cap W_n} x. \end{cases} \tag{1.4}$$

If $F(T)$ is nonempty, then $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to $\Pi_{F(T)}x$.

1.1 Some facts about gradients

For any convex function $g : E \to (-\infty, +\infty]$, we denote the domain of g by $\operatorname{dom} g = \{x \in E : g(x) < \infty\}$. For any $x \in \operatorname{int}\operatorname{dom} g$ and any $y \in E$, the *right-hand derivative* of g at x in the direction y is defined by

$$g^o(x,y) = \lim_{t\downarrow 0} \frac{g(x+ty) - g(x)}{t}. \tag{1.5}$$

The function g is said to be *Gâteaux differentiable* at x if $\lim_{t\to 0} \frac{g(x+ty)-g(x)}{t}$ exists for any y. In this case $g^o(x,y)$ coincides with $\nabla g(x)$, the value of the *gradient* ∇g of g at x. The function g is said to be *Gâteaux differentiable* if it is Gâteaux differentiable everywhere. The function g is said to be *Fréchet differentiable* at x if this limit is attained uniformly in $\|y\| = 1$. The function g is said to be *Fréchet differentiable* if it is Fréchet differentiable everywhere. It is well known that if a continuous convex function $g : E \to \mathbb{R}$ is Gâteaux differentiable, then ∇g is norm-to-weak* continuous (see, for example, [13]). Also, it is known that if g is Fréchet differentiable, then ∇g is norm-to-norm continuous (see [13]). The mapping ∇g is said to be *weakly sequentially continuous* if $x_n \rightharpoonup x$ as $n \to \infty$ implies that $\nabla g(x_n) \rightharpoonup^* \nabla g(x)$ as $n \to \infty$ (for more details, see [13] or [14]). The function g is said to be *strongly coercive* if

$$\lim_{\|x_n\|\to\infty} \frac{g(x_n)}{\|x_n\|} = \infty.$$

It is also said to be *bounded on bounded subsets of E* if $g(U)$ is bounded for each bounded subset U of E. Finally, g is said to be *uniformly Fréchet differentiable* on a subset X of E if the limit (1.5) is attained uniformly for all $x \in X$ and $\|y\| = 1$.

Let $A : E \to 2^{E^*}$ be a set-valued mapping. We define the domain and range of A by $\operatorname{dom} A = \{x \in E : Ax \neq \emptyset\}$ and $\operatorname{ran} A = \bigcup_{x\in E} Ax$, respectively. The graph of A is denoted by $G(A) = \{(x, x^*) \in E \times E^* : x^* \in Ax\}$. The mapping $A \subset E \times E^*$ is said to be *monotone* [15] if $\langle x - y, x^* - y^*\rangle \geq 0$ whenever $(x, x^*), (y, y^*) \in A$. It is also said to be *maximal monotone* [16] if its graph is not contained in the graph of any other monotone operator on E. If $A \subset E \times E^*$ is maximal monotone, then we can show that the set $A^{-1}0 = \{z \in E : 0 \in Az\}$ is closed and convex.

1.2 Some facts about Legendre functions

Let E be a reflexive Banach space. For any proper, lower semicontinuous and convex function $g : E \to (-\infty, +\infty]$, the *conjugate function* g^* of g is defined by

$$g^*(x^*) = \sup_{x\in E}\{\langle x, x^*\rangle - g(x)\}$$

for all $x^* \in E^*$. It is well known that $g(x) + g^*(x^*) \geq \langle x, x^*\rangle$ for all $(x, x^*) \in E \times E^*$. It is also known that $(x, x^*) \in \partial g$ is equivalent to

$$g(x) + g^*(x^*) = \langle x, x^*\rangle. \tag{1.6}$$

Here, ∂g is the subdifferential of g [17, 18]. We also know that if $g : E \to (-\infty, +\infty]$ is a proper, lower semicontinuous and convex function, then $g^* : E^* \to (-\infty, +\infty]$ is a proper, weak* lower semicontinuous and convex function; see [2] for more details on convex analysis.

Let $g : E \to (-\infty, +\infty]$ be a mapping. The function g is said to be:

(i) *Essentially smooth* if ∂g is both locally bounded and single-valued on its domain.

(ii) *Essentially strictly convex* if $(\partial g)^{-1}$ is locally bounded on its domain and g is strictly convex on every convex subset of $\operatorname{dom} \partial g$.

(iii) *Legendre* if it is both essentially smooth and essentially strictly convex (for more details, we refer to [19]).

If E is a reflexive Banach space and $g : E \to (-\infty, +\infty]$ is a Legendre function, then in view of [20]

$$\nabla g^* = (\nabla g)^{-1}, \qquad \operatorname{ran} \nabla g = \operatorname{dom} g^* = \operatorname{int} \operatorname{dom} g^*, \quad \text{and} \quad \operatorname{ran} \nabla g = \operatorname{int} \operatorname{dom} g.$$

Examples of Legendre functions are given in [21, 22]. The most notable example of a Legendre function is $\frac{1}{s}\|\cdot\|^s$ $(1 < s < \infty)$, where the Banach space E is smooth and strictly convex and, in particular, a Hilbert space.

1.3 Some facts about Bregman distances

Let E be a Banach space and let E^* be the dual space of E. Let $g : E \to \mathbb{R}$ be a convex and Gâteaux differentiable function. Then the *Bregman distance* [23, 24] corresponding to g is the function $D_g : E \times E \to \mathbb{R}$ defined by

$$D_g(x, y) = g(x) - g(y) - \langle x - y, \nabla g(y)\rangle, \quad \forall x, y \in E. \tag{1.7}$$

It is clear that $D_g(x,y) \geq 0$ for all $x, y \in E$. It is well known [25] that for $x \in E$ and $x_0 \in C$, $D_g(x_0, x) = \min_{y \in C} D_g(y,x)$ if and only if

$$\bigl\langle y - x_0, \nabla g(x) - \nabla g(x_0)\bigr\rangle \leq 0, \quad \forall y \in C. \tag{1.8}$$

In that case when E is a smooth Banach space, setting $g(x) = \|x\|^2$ for all $x \in E$, we obtain that $\nabla g(x) = 2Jx$ for all $x \in E$ and hence $D_g(x,y) = \phi(x,y)$ for all $x, y \in E$.

A *Bregman projection* [13, 23] of $x \in \operatorname{int}(\operatorname{dom} g)$ onto the nonempty, closed and convex set $C \subset \operatorname{dom} g$ is the unique vector $\operatorname{proj}_C^g(x) := x_0 \in C$ satisfying

$$D_g(x_0, x) = \min_{y \in C} D_g(y, x).$$

It is well known that proj_C^g has the following property:

$$D_g\bigl(y, \operatorname{proj}_C^g x\bigr) + D_g\bigl(\operatorname{proj}_C^g x, x\bigr) \leq D_g(y,x) \tag{1.9}$$

for all $y \in C$ and $x \in E$ (see [13] for more details).

1.4 Some facts about uniformly convex functions

Let E be a Banach space and let $B_r := \{z \in E : \|z\| \leq r\}$ for all $r > 0$. Then a function $g : E \to \mathbb{R}$ is said to be *uniformly convex on bounded subsets of E* [26] if $\rho_r(t) > 0$ for all $r, t > 0$, where $\rho_r : [0, +\infty) \to [0, \infty]$ is defined by

$$\rho_r(t) = \inf_{x,y \in B_r, \|x-y\| = t, \alpha \in (0,1)} \frac{\alpha g(x) + (1-\alpha) g(y) - g(\alpha x + (1-\alpha) y)}{\alpha(1-\alpha)} \tag{1.10}$$

for all $t \geq 0$. The function ρ_r is called the gauge of uniform convexity of g. The function g is also said to be *uniformly smooth on bounded subsets of E* [26] if $\lim_{t \downarrow 0} \frac{\sigma_r(t)}{t} = 0$ for all $r > 0$, where $\sigma_r : [0, +\infty) \to [0, \infty]$ is defined by

$$\sigma_r(t) = \sup_{x \in B_r, y \in S_E, \alpha \in (0,1)} \frac{\alpha g(x + (1-\alpha) t y) + (1-\alpha) g(x - \alpha t y) - g(x)}{\alpha(1-\alpha)}$$

for all $t \geq 0$. The function g is said to be *uniformly convex* if the function $\delta_g : [0, +\infty) \to [0, +\infty]$, defined by

$$\delta_g(t) := \sup\left\{\frac{1}{2} g(x) + \frac{1}{2} g(y) - g\left(\frac{x+y}{2}\right) : \|y - x\| = t\right\},$$

satisfies that $\lim_{t \downarrow 0} \frac{\sigma_r(t)}{t} = 0$.

1.5 Some facts about resolvents

Let E be a reflexive Banach space with the dual space E^* and let $g : E \to (-\infty, +\infty]$ be a proper, lower semicontinuous and convex function. Let A be a maximal monotone operator from E to E^*. For any $r > 0$, let the mapping $\operatorname{Res}_{rA}^g : E \to \operatorname{dom} A$ be defined by

$$\operatorname{Res}_{rA}^g = (\nabla g + rA)^{-1} \nabla g.$$

The mapping $\operatorname{Res}_{rA}^g$ is called the *g-resolvent* of A (see [27]). It is well known that $A^{-1}(0) = F(\operatorname{Res}_{rA}^g)$ for each $r > 0$ (for more details, see, for example, [1]).

Examples and some important properties of such operators are discussed in [28].

1.6 Some facts about Bregman quasi-nonexpansive mappings

Let C be a nonempty, closed and convex subset of a reflexive Banach space E. Let $g : E \to (-\infty, +\infty]$ be a proper, lower semicontinuous and convex function. Recall that a mapping $T : C \to C$ is said to be *Bregman quasi-nonexpansive* if $F(T) \neq \emptyset$ and

$$D_g(p, Tx) \leq D_g(p, x), \quad \forall x \in C, p \in F(T).$$

A mapping $T : C \to C$ is said to be *Bregman relatively nonexpansive* if the following conditions are satisfied:

(1) $F(T)$ is nonempty;
(2) $D_g(p, Tv) \leq D_g(p, v)$, $\forall p \in F(T)$, $v \in C$;
(3) $\hat{F}(T) = F(T)$.

A mapping $T : C \to C$ is said to be *Bregman weak relatively nonexpansive* if the following conditions are satisfied:

(1) $F(T)$ is nonempty;
(2) $D_g(p, Tv) \leq D_g(p, v)$, $\forall p \in F(T)$, $v \in C$;
(3) $\tilde{F}(T) = F(T)$.

It is clear that any Bregman relatively nonexpansive mapping is a Bregman quasi-nonexpansive mapping. It is also obvious that every Bregman relatively nonexpansive mapping is a Bregman weak relatively nonexpansive mapping, but the converse in not true in general; see, for example, [29]. Indeed, for any mapping $T : C \to C$, we have $F(T) \subset \tilde{F}(T) \subset \hat{F}(T)$. If T is Bregman relatively nonexpansive, then $F(T) = \tilde{F}(T) = \hat{F}(T)$.

The concept of W-mapping was first introduced by Atsushiba and Takahashi [30] in 1999 and ever since has been extensively investigated for a finite family of mappings (see [31] and the references therein). Now, we are in a position to introduce the concept of Bregman W-mapping in a Banach space. Let C be a nonempty, closed and convex subset of a reflexive Banach space E. Let $\{S_n\}_{n\in\mathbb{N}}$ be an infinite family of Bregman weak relatively nonexpansive mappings of C into itself, and let $\{\beta_{n,k} : k, n \in \mathbb{N}, 1 \leq k \leq n\}$ be a sequence of real numbers such that $0 \leq \beta_{i,j} \leq 1$ for every $i, j \in \mathbb{N}$ with $i \geq j$. Then, for any $n \in \mathbb{N}$, we define a mapping W_n of C into itself as follows:

$$\begin{aligned}
&U_{n,n+1}x = x, \\
&U_{n,n}x = \operatorname{proj}_C^g\bigl(\nabla g^*\bigl[\beta_{n,n}\nabla g(S_n U_{n,n+1}x) + (1 - \beta_{n,n})\nabla g(x)\bigr]\bigr), \\
&U_{n,n-1}x = \operatorname{proj}_C^g\bigl(\nabla g^*\bigl[\beta_{n,n-1}\nabla g(S_{n-1} U_{n,n}x) + (1 - \beta_{n,n-1})\nabla g(x)\bigr]\bigr), \\
&\vdots \\
&U_{n,k}x = \operatorname{proj}_C^g\bigl(\nabla g^*\bigl[\beta_{n,k}\nabla g(S_k U_{n,k+1}x) + (1 - \beta_{n,k})\nabla g(x)\bigr]\bigr), \\
&\vdots
\end{aligned}$$

$$U_{n,2}x = \operatorname{proj}_C^g\big(\nabla g^*\big[\beta_{n,2}\nabla g(S_2U_{n,3}x) + (1-\beta_{n,2})\nabla g(x)\big]\big),$$
$$W_nx = U_{n,1}x = \nabla g^*\big[\beta_{n,1}\nabla g(S_1U_{n,2}x) + (1-\beta_{n,1})\nabla g(x)\big]$$

for all $x \in C$, where proj_C^g is the Bregman projection from E onto C. Such a mapping W_n is called the Bregman W-mapping generated by $S_n, S_{n-1}, \ldots, S_1$ and $\beta_{n,n}, \beta_{n,n-1}, \ldots, \beta_{n,1}$.

The theory of fixed points with respect to Bregman distances has been studied in the last ten years and much intensively in the last four years. For some recent articles on the existence of fixed points for Bregman nonexpansive type mappings, we refer the readers to [20–22, 27, 28]. But it is worth mentioning that, in all the above results for Bregman nonexpansive type mappings, the assumption $\hat{F}(T) = F(T)$ is imposed on the map T. So, the following question arises naturally in a Banach space setting.

Question 1.1 Is it possible to obtain strong convergence of modified Mann-type schemes to a common fixed point of an infinite family of Bregman W-mappings $\{S_j\}_{j\in\mathbb{N}}$ without imposing the assumption $\hat{F}(S_j) = F(S_j)$ on S_j?

In this paper we introduce new modified Mann iterative processes for computing fixed points of an infinite family of Bregman W-mappings in reflexive Banach spaces. Let W_n be the Bregman W-mapping generated by $S_n, S_{n-1}, \ldots, S_1$ and $\beta_{n,n}, \beta_{n,n-1}, \ldots, \beta_{n,1}$. We first express the set of fixed points of W_n as the intersection of fixed points of $\{S_i\}_{i=1}^n$. As a consequence, we show that W_n is a Bregman weak relatively nonexpansive mapping if S_i is a Bregman weak relatively nonexpansive mapping for each $i = 1, 2, \ldots, n$. We then prove some strong convergence theorems for these mappings. Some application of our results to convex feasibility problem is also presented. No assumption $\hat{F}(T) = F(T)$ is imposed on the mapping T. Consequently, the above question is answered in the affirmative in a reflexive Banach space setting. Our results improve and generalize many known results in the current literature; see, for example, [8, 9, 12, 30–33].

2 Preliminaries

In this section, we begin by recalling some preliminaries and lemmas which will be used in the sequel.

The following definition is slightly different from that in Butnariu and Iusem [13].

Definition 2.1 ([14]) Let E be a Banach space. The function $g : E \to \mathbb{R}$ is said to be a Bregman function if the following conditions are satisfied:

(1) g is continuous, strictly convex and Gâteaux differentiable;
(2) the set $\{y \in E : D_g(x, y) \leq r\}$ is bounded for all $x \in E$ and $r > 0$.

The following lemma follows from Butnariu and Iusem [13] and Zălinescu [26].

Lemma 2.1 *Let E be a reflexive Banach space and $g : E \to \mathbb{R}$ be a strongly coercive Bregman function. Then*

(1) $\nabla g : E \to E^*$ *is one-to-one, onto and norm-to-weak* continuous*;
(2) $\langle x - y, \nabla g(x) - \nabla g(y)\rangle = 0$ *if and only if* $x = y$;
(3) $\{x \in E : D_g(x, y) \leq r\}$ *is bounded for all* $y \in E$ *and* $r > 0$;
(4) $\operatorname{dom} g^* = E^*, g^*$ *is Gâteaux differentiable and* $\nabla g^* = (\nabla g)^{-1}$.

We know the following two results from [26].

Theorem 2.1 *Let E be a reflexive Banach space and $g : E \to \mathbb{R}$ be a convex function which is bounded on bounded subsets of E. Then the following assertions are equivalent*:

(1) *g is strongly coercive and uniformly convex on bounded subsets of E*;
(2) $\operatorname{dom} g^* = E^*$, *$g^*$ is bounded on bounded subsets and uniformly smooth on bounded subsets of E^**;
(3) $\operatorname{dom} g^* = E^*$, *$g^*$ is Fréchet differentiable and ∇g^* is uniformly norm-to-norm continuous on bounded subsets of E^*.*

Theorem 2.2 *Let E be a reflexive Banach space and $g : E \to \mathbb{R}$ be a continuous convex function which is strongly coercive. Then the following assertions are equivalent*:

(1) *g is bounded on bounded subsets and uniformly smooth on bounded subsets of E*;
(2) *g^* is Fréchet differentiable and ∇g^* is uniformly norm-to-norm continuous on bounded subsets of E^**;
(3) $\operatorname{dom} g^* = E^*$, *$g^*$ is strongly coercive and uniformly convex on bounded subsets of E^*.*

Let E be a Banach space and let $g : E \to \mathbb{R}$ be a convex and Gâteaux differentiable function. Then the Bregman distance [34] (see also [23, 24]) satisfies the *three point identity* that is

$$D_g(x,z) = D_g(x,y) + D_g(y,z) + \langle x - y, \nabla g(y) - \nabla g(z)\rangle, \quad \forall x,y,z \in E. \tag{2.1}$$

In particular, it can be easily seen that

$$D_g(x,y) = -D_g(y,x) + \langle y - x, \nabla g(y) - \nabla g(x)\rangle, \quad \forall x,y \in E. \tag{2.2}$$

The following result was proved in [29].

Lemma 2.2 *Let E be a Banach space and $g : E \to \mathbb{R}$ be a Gâteaux differentiable function which is uniformly convex on bounded subsets of E. Let $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be bounded sequences in E. Then*

$$\lim_{n\to\infty} D_g(x_n, y_n) = 0 \quad \Longleftrightarrow \quad \lim_{n\to\infty} \|x_n - y_n\| = 0.$$

The following result was first proved in [19] (see also [14]).

Lemma 2.3 *Let E be a reflexive Banach space, $g : E \to \mathbb{R}$ be a strongly coercive Bregman function and V be the function defined by*

$$V_g(x, x^*) = g(x) - \langle x, x^*\rangle + g^*(x^*), \quad x \in E, x^* \in E^*.$$

Then the following assertions hold:

(1) *$D_g(x, \nabla g^*(x^*)) = V_g(x, x^*)$ for all $x \in E$ and $x^* \in E^*$.*
(2) *$V_g(x, x^*) + \langle \nabla g^*(x^*) - x, y^*\rangle \le V_g(x, x^* + y^*)$ for all $x \in E$ and $x^*, y^* \in E^*$.*

The following result was proved in [29].

Lemma 2.4 *Let E be a Banach space, $r > 0$ be a constant, ρ_r be the gauge of uniform convexity of g and $g : E \to \mathbb{R}$ be a convex function which is uniformly convex on bounded subsets of E. Then*

(i) *For any $x, y \in B_r$ and $\alpha \in (0,1)$,*

$$g\big(\alpha x + (1-\alpha)y\big) \leq \alpha g(x) + (1-\alpha)g(y) - \alpha(1-\alpha)\rho_r\big(\|x-y\|\big).$$

(ii) *For any $x, y \in B_r$,*

$$\rho_r\big(\|x-y\|\big) \leq D_g(x,y).$$

(iii) *If, in addition, g is bounded on bounded subsets and uniformly convex on bounded subsets of E then, for any $x \in E$, $y^*, z^* \in B_r$ and $\alpha \in (0,1)$,*

$$V_g\big(x, \alpha y^* + (1-\alpha)z^*\big) \leq \alpha V_g\big(x, y^*\big) + (1-\alpha)V_g\big(x, z^*\big) - \alpha(1-\alpha)\rho_r^*\big(\big\|y^* - z^*\big\|\big).$$

The following result was proved in [29].

Lemma 2.5 *Let E be a Banach space, $r > 0$ be a constant and $g : E \to \mathbb{R}$ be a convex function which is uniformly convex on bounded subsets of E. Then*

$$g\left(\sum_{k=0}^{n} \alpha_k x_k\right) \leq \sum_{k=0}^{n} \alpha_k g(x_k) - \alpha_i \alpha_j \rho_r\big(\|x_i - x_j\|\big)$$

for all $i, j \in \{0, 1, 2, \ldots, n\}$, $x_k \in B_r$, $\alpha_k \in (0,1)$ and $k = 0, 1, 2, \ldots, n$ with $\sum_{k=0}^{n} \alpha_k = 1$, where ρ_r is the gauge of uniform convexity of g.

Now we prove the following important result.

Proposition 2.1 *Let E be a reflexive Banach space and $g : E \to \mathbb{R}$ be a convex, continuous, strongly coercive and Gâteaux differentiable function which is bounded on bounded subsets and uniformly convex on bounded subsets of E. Let C be a nonempty, closed and convex subset of E. Let $S_1, S_2, \ldots, S_n$ be Bregman weak relatively nonexpansive mappings of C into itself such that $\bigcap_{i=1}^{n} F(S_i) \neq \emptyset$, and let $\{\beta_{n,k} : k, n \in \mathbb{N}, 1 \leq k \leq n\}$ be a sequence of real numbers such that $0 < \beta_{n,1} \leq 1$ and $0 < \beta_{n,i} < 1$ for every $i = 2, 3, \ldots, n$. Let W_n be the Bregman W-mapping generated by $S_n, S_{n-1}, \ldots, S_1$ and $\beta_{n,n}, \beta_{n,n-1}, \ldots, \beta_{n,1}$. Then the following assertions hold*:

(i) $F(W_n) = \bigcap_{i=1}^{n} F(S_i)$;

(ii) *for every $k = 1, 2, \ldots, n$, $x \in C$ and $z \in F(W_n)$, $D_g(z, U_{n,k}x) \leq D_g(z,x)$ and $D_g(z, S_k U_{n,k+1}x) \leq D_g(z,x)$;*

(iii) *for every $n \in \mathbb{N}$, W_n is a Bregman weak relatively nonexpansive mapping.*

Proof (i) It is clear that $\bigcap_{i=1}^{n} F(S_i) \subset F(W_n)$. For the converse inclusion, take any $w \in \bigcap_{i=1}^{n} F(S_i)$ and $z \in F(W_n)$.

Let $r_1 = \sup\{\|\nabla g(z)\|, \|\nabla g(S_k z)\|, \|\nabla g(S_k U_{n,k+1} z)\| : k = 1, 2, \ldots, n\}$ and $\rho_{r_1}^* : E^* \to \mathbb{R}$ be the gauge of uniform convexity of the conjugate function g^*. In view of (1.9) and Lemma 2.4, we obtain

$$\begin{aligned}
D_g(w,z) &= D_g(w, W_n z) \\
&= D_g\bigl(w, \nabla g^*\bigl[\beta_{n,1}\nabla g(S_1U_{n,2}z) + (1-\beta_{n,1})\nabla g(z)\bigr]\bigr) \\
&= g(w) - \bigl\langle w, \beta_{n,1}\beta_{n,1}\nabla g(S_1U_{n,2}z) + (1-\beta_{n,1})\nabla g(z)\bigr\rangle \\
&\quad + g^*\bigl(\beta_{n,1}\nabla g(S_1U_{n,2}z) + (1-\beta_{n,1})\nabla g(z)\bigr) \\
&\le \beta_{n,1}g(w) + (1-\beta_{n,1})g(w) + \beta_{n,1}g^*\bigl(\nabla g(S_1U_{n,2}z)\bigr) + (1-\beta_{n,1})g^*\bigl(\nabla g(z)\bigr) \\
&\quad - \beta_{n,1}(1-\beta_{n,1})\rho^*_{r_1}\bigl(\bigl\|\nabla g(S_1U_{n,2}z) - \nabla g(z)\bigr\|\bigr) \\
&= \beta_{n,1}V_g\bigl(w, \nabla g(S_1U_{n,2}z)\bigr) + (1-\beta_{n,1})V_g\bigl(w, \nabla g(z)\bigr) \\
&\quad - \beta_{n,1}(1-\beta_{n,1})\rho^*_{r_1}\bigl(\bigl\|\nabla g(S_1U_{n,2}z) - \nabla g(z)\bigr\|\bigr) \\
&= \beta_{n,1}D_g(w, S_1U_{n,2}z) + (1-\beta_{n,1})D_g(w,z) \\
&\quad - \beta_{n,1}(1-\beta_{n,1})\rho^*_{r_1}\bigl(\bigl\|\nabla g(S_1U_{n,2}z) - \nabla g(z)\bigr\|\bigr) \\
&\le \beta_{n,1}D_g(w, U_{n,2}z) + (1-\beta_{n,1})D_g(w,z) \\
&\quad - \beta_{n,1}(1-\beta_{n,1})\rho^*_{r_1}\bigl(\bigl\|\nabla g(S_1U_{n,2}z) - \nabla g(z)\bigr\|\bigr) \\
&= \beta_{n,1}\bigl[\beta_{n,2}D_g(w, U_{n,3}z) + (1-\beta_{n,2})D_g(w,z) \\
&\quad - \beta_{n,2}(1-\beta_{n,2})\rho^*_{r_1}\bigl(\bigl\|\nabla g(S_2U_{n,3}z) - \nabla g(z)\bigr\|\bigr)\bigr] + (1-\beta_{n,1})D_g(w,z) \\
&\quad - \beta_{n,1}(1-\beta_{n,1})\rho^*_{r_1}\bigl(\bigl\|\nabla g(S_1U_{n,2}z) - \nabla g(z)\bigr\|\bigr) \\
&\le \cdots \\
&\le D_g(w,z) - \beta_{n,1}(1-\beta_{n,1})\rho^*_{r_1}\bigl(\bigl\|\nabla g(S_1U_{n,2}z) - \nabla g(z)\bigr\|\bigr) \\
&\quad - \beta_{n,1}\beta_{n,2}(1-\beta_{n,2})\rho^*_{r_1}\bigl(\bigl\|\nabla g(S_2U_{n,3}z) - \nabla g(z)\bigr\|\bigr) - \cdots \\
&\quad - \beta_{n,1}\beta_{n,2}\cdots\beta_{n,n}(1-\beta_{n,n})\rho^*_{r_1}\bigl(\bigl\|\nabla g(S_nz) - \nabla g(z)\bigr\|\bigr).
\end{aligned}$$

This implies that

$$\rho^*_{r_1}\bigl(\bigl\|\nabla g(S_2U_{n,3}z) - \nabla g(z)\bigr\|\bigr) = \cdots = \rho^*_{r_1}\bigl(\bigl\|\nabla g(S_nz) - \nabla g(z)\bigr\|\bigr) = 0$$

and hence, from the properties of $\rho^*_{r_1}$, we conclude that

$$S_kz = z, \qquad U_{n,k}z = z \quad (k = 2, 3, \ldots, n).$$

If $\beta_{n,1} < 1$, then we get from $\|\nabla g(S_1U_{n,2}z) - \nabla g(z)\| = 0$ that $S_1z = z$. And if $\beta_{n,1} = 1$, then we obtain from $z = W_nz = S_1U_{n,2}z$ that $S_1z = z$. Thus we have $z \in \bigcap_{i=1}^n F(S_i)$. This shows that $F(W_n) \subset \bigcap_{i=1}^n F(S_i)$.

(ii) Let $k = 1, 2, \ldots, n, x \in C$ and $z \in F(W_n)$. By a similar way as in the proof of (i), we arrive at

$$\begin{aligned}
D_g(z, U_{n,k}x) &\le \beta_{n,k}D_g(z, S_kU_{n,k+1}x) + (1-\beta_{n,k})D_g(z,x) \\
&\le \beta_{n,k}D_g(z, U_{n,k+1}x) + (1-\beta_{n,k})D_g(z,x)
\end{aligned}$$

$$\begin{aligned}
&\leq \beta_{n,k}\big[\beta_{n,k+1}D_g(z, U_{n,k+1}x) + (1-\beta_{n,k+1})D_g(z,x)\big] + (1-\beta_{n,k})D_g(z,x) \\
&\leq \cdots \leq D_g(z,x).
\end{aligned}$$

This implies that

$$D_g(z, S_k U_{n,k+1}x) \leq D_g(z,x).$$

(iii) Since we have already proved that $F(W_n) = \bigcap_{i=1}^n F(S_i)$, then the fact that W_n is a Bregman weak relatively nonexpansive mapping is a consequence of each S_i being Bregman weak relatively nonexpansive. Indeed, let $\{z_m\}_{m\in\mathbb{N}}$ be a sequence in C such that $z_m \to z \in C$ and $\|z_m - W_n z_m\| \to 0$ as $m \to \infty$. We will show that $z \in F(W_n)$. To this end, let $w \in F(W_n)$. In view of Lemma 2.2, we get that

$$\lim_{m\to\infty} D_g(W_n z_m, z_m) = 0.$$

On the other hand, we have from (2.1) that

$$\begin{aligned}
D_g(w, z_m) - D_g(w, W_n z_m) &= D_g(w, W_n z_m) + D_g(W_n z_m, z_m) \\
&\quad + \big\langle w - W_n z_m, \nabla g(W_n z_m) - \nabla g(z_m)\big\rangle - D_g(w, W_n z_m) \\
&= D_g(W_n z_m, z_m) + \big\langle w - W_n z_m, \nabla g(W_n z_m) - \nabla g(z_m)\big\rangle.
\end{aligned}$$

This, together with (2.2), implies that

$$\lim_{m\to\infty} \big|D_g(w, z_m) - D_g(w, W_n z_m)\big| = 0.$$

Let $r_2 = \sup\{\|\nabla g(z_m)\|, \|\nabla g(S_k z_m)\|, \|\nabla g(S_k U_{n,k+1} z_m)\| : m \in \mathbb{N}, k = 1,2,\ldots,n\}$ and $\rho_{r_2}^* : E^* \to \mathbb{R}$ be the gauge of uniform convexity of the conjugate function g^*. By the same arguments as in (ii), we conclude that

$$\begin{aligned}
D_g(w, W_n z_m) &= D_g\big(w, \nabla g^*\big[\beta_{n,1}\nabla g(S_1 U_{n,2} z_m) + (1-\beta_{n,1})\nabla g(z_m)\big]\big) \\
&= g(w) - \big\langle w, \beta_{n,1}\beta_{n,1}\nabla g(S_1 U_{n,2} z_m) + (1-\beta_{n,1})\nabla g(z_m)\big\rangle \\
&\quad + g^*\big(\beta_{n,1}\nabla g(S_1 U_{n,2} z_m) + (1-\beta_{n,1})\nabla g(z_m)\big) \\
&\leq \beta_{n,1} g(w) + (1-\beta_{n,1})g(w) + \beta_{n,1} g^*\big(\nabla g(S_2 U_{n,3} z_m)\big) + (1-\beta_{n,1})g^*\big(\nabla g(z_m)\big) \\
&\quad - \beta_{n,1}(1-\beta_{n,1})\rho_{r_2}^*\big(\big\|\nabla g(S_1 U_{n,2} z_m) - \nabla g(z_m)\big\|\big) \\
&= \beta_{n,1} V_g\big(w, \nabla g(S_1 U_{n,2} z_m)\big) + (1-\beta_{n,1})V_g\big(w, \nabla g(z_m)\big) \\
&\quad - \beta_{n,1}(1-\beta_{n,1})\rho_{r_2}\big(\big\|\nabla g(S_1 U_{n,2} z_m) - \nabla g(z_m)\big\|\big) \\
&= \beta_{n,1} D_g(w, S_1 U_{n,2} z_m) + (1-\beta_{n,1})D_g(u, z_m) \\
&\quad - \beta_{n,1}(1-\beta_{n,1})\rho_{r_2}^*\big(\big\|\nabla g(S_1 U_{n,2} z_m) - \nabla g(z_m)\big\|\big) \\
&\leq \beta_{n,1} D_g(w, U_{n,2} z_m) + (1-\beta_{n,1})D_g(w, z_m) \\
&\quad - \beta_{n,1}(1-\beta_{n,1})\rho_{r_2}^*\big(\big\|\nabla g(S_1 U_{n,2} z_m) - \nabla g(z_m)\big\|\big)
\end{aligned}$$

$$\begin{aligned}
&= \beta_{n,1}\big[\beta_{n,2}D_g(w, U_{n,3}z_m) + (1-\beta_{n,2})D_g(w,z_m) \\
&\quad - \beta_{n,2}(1-\beta_{n,2})\rho^*_{r_2}\big(\big\|\nabla g(S_2U_{n,3}z_m) - \nabla g(z_m)\big\|\big)\big] + (1-\beta_{n,1})D_g(w,z_m) \\
&\quad - \beta_{n,1}(1-\beta_{n,1})\rho^*_{r_2}\big(\big\|\nabla g(S_1U_{n,2}z_m) - \nabla g(z_m)\big\|\big) \\
&\le \cdots \\
&\le D_g(w,z_m) - \beta_{n,1}(1-\beta_{n,1})\rho^*_{r_2}\big(\big\|\nabla g(S_1U_{n,2}z_m) - \nabla g(z_m)\big\|\big) \\
&\quad - \beta_{n,1}\beta_{n,2}(1-\beta_{n,2})\rho^*_{r_2}\big(\big\|\nabla g(S_2U_{n,3}z_m) - \nabla g(z_m)\big\|\big) - \cdots \\
&\quad - \beta_{n,1}\beta_{n,2}\cdots\beta_{n,n}(1-\beta_{n,n})\rho^*_{r_2}\big(\big\|\nabla g(S_nz_m) - \nabla g(z_m)\big\|\big).
\end{aligned}$$

This implies that

$$\begin{aligned}
&\lim_{m\to\infty}\rho^*_{r_2}\big(\big\|\nabla g(S_1U_{n,2}z_m) - \nabla g(z_m)\big\|\big) = \cdots \\
&\qquad = \lim_{m\to\infty}\rho^*_{r_2}\big(\big\|\nabla g(S_nz_m) - \nabla g(z_m)\big\|\big) = 0.
\end{aligned}$$

Therefore, from the property of $\rho^*_{r_2}$ we deduce that

$$\lim_{m\to\infty}\big\|\nabla g(S_1z_m) - \nabla g(S_kz_m)\big\| = 0, \quad \forall k \in \{2,\ldots,n\}$$

and hence

$$S_kz = z, \qquad U_{n,k}z = z \quad (k = 2,3,\ldots,n).$$

If $\beta_{n,1} < 1$, then we get from $\|\nabla g(S_1U_{n,2}z) - \nabla g(z)\| = 0$ that $S_1z = z$. And if $\beta_{n,1} = 1$, then we obtain from $z = W_nz = S_1U_{n,2}z$ that $S_1z = z$. Thus we have $z \in \bigcap_{i=1}^n F(S_i)$ and hence W_n is a Bregman weak relatively nonexpansive mapping for every $n \in \mathbb{N}$. This completes the proof. □

Next we prove the following convex combination of Bregman weak relatively nonexpansive mappings in a Banach space.

Proposition 2.2 *Let E be a reflexive Banach space and $g : E \to \mathbb{R}$ be a convex, continuous, strongly coercive and Gâteaux differentiable function which is bounded on bounded subsets and uniformly convex on bounded subsets of E. Let C be a nonempty, closed and convex subset of E. Let $\{S_n\}_{n\in\mathbb{N}}$ be a family of Bregman weak relatively nonexpansive mappings of C into itself such that $F := \bigcap_{n=1}^\infty F(S_n) \neq \emptyset$, and let $T_nx = \nabla g^*(\sum_{j=1}^n \beta_{n,j}\nabla g(S_jx))$ for every $n \in \mathbb{N}$ and $x \in C$, where $0 \le \beta_{n,j} \le 1$ ($n \in \mathbb{N}$, $j = 1,2,\ldots,n$) with $\sum_{j=1}^n \beta_{n,j} = 1$ for all $n \in \mathbb{N}$ and $\liminf_{n\to\infty}\beta_{n,j} > 0$ for each $j \in \mathbb{N}$. Then the following assertions hold*:

(i) $\bigcap_{n=1}^\infty F(T_n) = F$;

(ii) *for every $n \in \mathbb{N}$, $x \in C$ and $z \in F$, $D_g(z, T_nx) \le D_g(z,x)$;*

(iii) *for every $n \in \mathbb{N}$, T_n is a Bregman weak relatively nonexpansive mapping.*

Proof (i) It is clear that $F \subset \bigcap_{n=1}^\infty F(T_n) \neq \emptyset$. For the converse inclusion, take $w \in F$ and $z \in \bigcap_{n=1}^\infty F(T_n)$. Let $n \in \mathbb{N}$ be large enough and $l, m \in \mathbb{N}$ with $1 \le l \le m \le n$. Let $r_3 = \sup\{\|\nabla g(z)\|, \|\nabla g(S_kz)\|, \|\nabla g(S_kU_{n,k+1}z_m)\| : m \in \mathbb{N}, k = 1,2,\ldots,n\}$ and $\rho^*_{r_3} : E^* \to \mathbb{R}$ be the gauge of uniform convexity of the conjugate function g^*. In view of Lemma 2.5, we obtain

$$\begin{aligned}
D_g(w,z) &= D_g(w,T_nz) \\
&= D_g\left(w,\nabla g^*\left(\sum_{j=1}^{n}\beta_{n,j}\nabla g(S_j)(z)\right)\right) \\
&= D_g\left(w,\nabla g^*\left[\sum_{j=1}^{n}\beta_{n,j}\nabla g(S_jz)\right]\right) \\
&= V_g\left(w,\sum_{j=1}^{n}\beta_{n,j}\nabla g(S_jz)\right) \\
&= g(w)-\left\langle w,\sum_{j=1}^{n}\beta_{n,j}\nabla g(S_jz)\right\rangle \\
&\quad + g^*\left((\beta_{n,l}+\beta_{n,m})\frac{(\beta_{n,l}\nabla g(S_lz))+\beta_{n,m}\nabla g(S_mz)}{\beta_{n,l}+\beta_{n,m}}\right. \\
&\quad \left.\times\big(1-(\beta_{n,l}+\beta_{n,m})\big)\frac{\sum_{j=1,2,\ldots,n,j\neq l,m}^{n}\beta_{n,j}\nabla g(S_jz)}{1-(\beta_{n,l}+\beta_{n,m})}\right) \\
&\le g(w)-\sum_{j=1}^{n}\beta_{n,j}\langle w,\nabla g(S_jz)\rangle \\
&\quad + (\beta_{n,l}+\beta_{n,m})\left[\frac{\beta_{n,l}}{(\beta_{n,l}+\beta_{n,m})}g^*\big(\nabla g(S_lz)\big)+\frac{\beta_{n,l}}{(\beta_{n,l}+\beta_{n,m})}g^*\big(\nabla g(S_mz)\big)\right. \\
&\quad \left.-\frac{\beta_{n,l}}{(\beta_{n,l}+\beta_{n,m})}\frac{\beta_{n,m}}{(\beta_{n,l}+\beta_{n,m})}\rho_{r_3}^*\big(\|\nabla g(S_lz)-\nabla g(S_mz)\|\big)\right] \\
&\quad + \sum_{j=1,2,\ldots,n,j\neq l,m}^{n}\beta_{n,j}g^*\big(\nabla g(S_jz)\big) \\
&= \sum_{j=1,2,\ldots,n}^{n}\beta_{n,j}\big[g(w)-\langle w,\nabla g(S_jz)\rangle+g^*\big(\nabla g(S_jz)\big)\big] \\
&\quad -\frac{\beta_{n,l}\beta_{n,m}}{(\beta_{n,l}+\beta_{n,m})}\rho_{r_3}^*\big(\|\nabla g(S_lz)-\nabla g(S_mz)\|\big) \\
&= \sum_{j=1,2,\ldots,n}^{n}\beta_{n,j}V\big(w,\nabla g(S_jz)\big) \\
&\quad -\frac{\beta_{n,l}\beta_{n,m}}{(\beta_{n,l}+\beta_{n,m})}\rho_{r_3}^*\big(\|\nabla g(S_lz)-\nabla g(S_mz)\|\big) \\
&= \sum_{j=1,2,\ldots,n}^{n}\beta_{n,j}D_g(w,S_jz) \\
&\quad -\frac{\beta_{n,l}\beta_{n,m}}{(\beta_{n,l}+\beta_{n,m})}\rho_{r_3}^*\big(\|\nabla g(S_lz)-\nabla g(S_mz)\|\big) \\
&= D_g(w,z)-\frac{\beta_{n,l}\beta_{n,m}}{(\beta_{n,l}+\beta_{n,m})}\rho_{r_3}^*\big(\|\nabla g(S_lz)-\nabla g(S_mz)\|\big).
\end{aligned}$$

This implies that for any $l, m \in \mathbb{N}$,

$$\beta_{n,l}\beta_{n,m}\rho_{r_3}^*\big(\big\|\nabla g(S_l z) - \nabla g\big(S_m(z)\big)\big\|\big) = 0$$

for large enough $n \in \mathbb{N}$.

Therefore, from the property of $\rho_{r_3}^*$ we deduce that

$$\big\|\nabla g(S_l z) - \nabla g(S_m z)\big\| = 0, \quad \forall l, m \in \mathbb{N}.$$

Since ∇g^* is uniformly norm-to-norm continuous on bounded subsets of E^*, we arrive at

$$\big\|S_l(z) - S_m(z)\big\| = 0, \quad \forall l, m \in \mathbb{N}.$$

This implies that

$$\lim_{n\to\infty}\big\|S_l(z) - S_m z\big\| = 0, \quad \forall l, m \in \mathbb{N}.$$

Therefore, $S_l z = S_m$ for every $l, m \in \mathbb{N}$, that is, $z \in F$. This completes the proof. □

3 Strong convergence theorems

In this section, we prove strong convergence theorems in a reflexive Banach space. We start with the following simple lemma which was proved in [35].

Lemma 3.1 *Let E be a reflexive Banach space and $g : E \to \mathbb{R}$ be a convex, continuous, strongly coercive and Gâteaux differentiable function which is bounded on bounded subsets and uniformly convex on bounded subsets of E. Let C be a nonempty, closed and convex subset of E. Let $T : C \to C$ be a Bregman quasi-nonexpansive mapping. Then $F(T)$ is closed and convex.*

Theorem 3.1 *Let E be a reflexive Banach space and $g : E \to \mathbb{R}$ be a convex, continuous, strongly coercive and Gâteaux differentiable function which is bounded on bounded subsets and uniformly convex on bounded subsets of E. Let C be a nonempty, closed and convex subset of E. Let $\{S_n\}_{n\in\mathbb{N}}$ be a family of Bregman weak relatively nonexpansive mappings of C into itself such that $F := \bigcap_{n=1}^{\infty} F(S_n) \neq \emptyset$, and let $\{\beta_{n,k} : k, n \in \mathbb{N}, 1 \leq k \leq n\}$ be a sequence of real numbers such that $0 < \beta_{i,j} \leq 1$ and $0 < \beta_{i,j} < 1$ for all $i \in \mathbb{N}$ and every $j = 2, 3, \ldots, n$. Let W_n be the Bregman W-mapping generated by $S_n, S_{n-1}, \ldots, S_1$ and $\beta_{n,n}, \beta_{n,n-1}, \ldots, \beta_{n,1}$. Let $\{\alpha_n\}_{n\in\mathbb{N}\cup\{0\}}$ be a sequence in $[0,1)$ such that $\liminf_{n\to\infty}\alpha_n(1-\alpha_n) > 0$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence generated by*

$$\begin{cases} x_0 = x \in C \quad \text{chosen arbitrarily}, \\ C_0 = C, \\ y_n = \nabla g^*[\alpha_n \nabla g(x_n) + (1-\alpha_n)\nabla g(W_n x_n)], \\ C_{n+1} = \{z \in C_n : D_g(z, y_n) \leq D_g(z, x_n)\}, \\ x_{n+1} = \operatorname{proj}_{C_{n+1}}^g x \quad \text{and} \quad n \in \mathbb{N}\cup\{0\}, \end{cases} \tag{3.1}$$

where ∇g is the gradient of g. Then $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to $\operatorname{proj}_F^g x_0$ as $n \to \infty$.

Proof We divide the proof into several steps.

Step 1. We show that C_n is closed and convex for each $n \in \mathbb{N} \cup \{0\}$.

We proceed by the mathematical induction. It is clear that $C_0 = C$ is closed and convex. Let C_m be closed and convex for some $m \in \mathbb{N}$. For $z \in C_m$, we see that

$$D_g(z, y_m) \le D_g(z, x_m)$$

is equivalent to

$$\langle z, \nabla g(x_m) - \nabla g(y_m)\rangle \le g(y_m) - g(x_m) + \langle x_m, \nabla g(x_m)\rangle - \langle y_m, \nabla g(y_m)\rangle.$$

An easy argument shows that C_{m+1} is closed and convex. Hence C_n is closed and convex for each $n \in \mathbb{N} \cup \{0\}$.

Step 2. We claim that $F \subset C_n$ for all $n \in \mathbb{N} \cup \{0\}$.

It is obvious that $F \subset C_0 = C$. Assume now that $F \subset C_m$ for some $m \in \mathbb{N}$. Take any $w \in F \subset C_m$. Employing Lemma 2.3, we obtain

$$\begin{aligned}
D_g(w, y_m) &= D_g\bigl(w, \nabla g^*\bigl[\alpha_m \nabla g(x_m) + (1-\alpha_m)\nabla g(W_m x_m)\bigr]\bigr) \\
&= V_g\bigl(w, \alpha_m \nabla g(x_m) + (1-\alpha_m)\nabla g(W_m x_m)\bigr) \\
&= g(w) - \bigl\langle w, \alpha_m \nabla g(x_m) + \bigl(1-\alpha_m \nabla g(W_m x_m)\bigr)\bigr\rangle \\
&\quad + g^*\bigl(\alpha_m \nabla g(x_m) + (1-\alpha_m)\nabla g(W_m x_m)\bigr) \\
&\le \alpha_m g(w) + (1-\alpha_m) g(w) \\
&\quad + \alpha_m g^*\bigl(\nabla g(x_m)\bigr) + (1-\alpha_m) g^*\bigl(\nabla g(W_m x_m)\bigr) \\
&= \alpha_m V_g\bigl(w, \nabla g(x_m)\bigr) + (1-\alpha_m) V_g\bigl(w, \nabla g(W_m x_m)\bigr) \\
&= \alpha_m D_g(w, x_m) + (1-\alpha_m) D_g(w, W_m x_m) \\
&\le \alpha_m D_g(w, x_m) + (1-\alpha_m) D_g(w, x_m) \\
&= D_g(w, x_m).
\end{aligned} \tag{3.2}$$

This proves that $w \in C_{m+1}$. Thus, we have $F \subset C_n$ for all $n \in \mathbb{N} \cup \{0\}$.

Step 3. We prove that $\{x_n\}_{n\in\mathbb{N}}$, $\{y_n\}_{n\in\mathbb{N}}$ and $\{W_n x_n\}_{n\in\mathbb{N}}$ are bounded sequences in C.

It is then easily seen from (1.9) that

$$\begin{aligned}
D_g(x_n, x) &= D_g\bigl(\operatorname{proj}^g_{C_n} x, x\bigr) \\
&\le D_g(w, x) - D_g(w, x_n) \le D_g(w, x), \quad \forall w \in F \subset C_n, n \in \mathbb{N} \cup \{0\}.
\end{aligned}$$

This leads immediately to the boundedness of $\{D_g(x_n, x)\}_{n\in\mathbb{N}}$. So, there exists $M_1 > 0$ such that

$$D_g(x_n, x) \le M_1, \quad \forall n \in \mathbb{N}. \tag{3.3}$$

Using Lemma 2.1(3) and (3.3), we have the boundedness of $\{x_n\}_{n\in\mathbb{N}}$. Since $\{W_n\}_{n\in\mathbb{N}}$ is an infinite family of Bregman weak relatively nonexpansive mappings from C into itself, we have for any $q \in F$ that

$$D_g(q, W_n x_n) \leq D_g(q, x_n), \quad \forall n \in \mathbb{N}. \tag{3.4}$$

Then by Definition 2.1, (3.4) and observing that $\{x_n\}_{n\in\mathbb{N}}$ is bounded, we are led to the boundedness of $\{W_n x_n\}_{n\in\mathbb{N}}$.

Step 4. We show that $x_n \to u$ for some $u \in F$, where $u = \mathrm{proj}_F^g x$.

By Step 3, we have that $\{x_n\}_{n\in\mathbb{N}}$ is bounded. By the construction of C_n, we conclude that $C_m \subset C_n$ and $x_m = \mathrm{proj}_{C_m}^g x \in C_m \subset C_n$ for any positive integer $m \geq n$. This, together with (1.9), implies that

$$\begin{aligned} D_g(x_m, x_n) &= D_g\big(x_m, \mathrm{proj}_{C_n}^g x\big) \leq D_g(x_m, x) - D_g\big(\mathrm{proj}_{C_n}^g x, x\big) \\ &= D_g(x_m, x) - D_g(x_n, x). \end{aligned} \tag{3.5}$$

In view of (3.5), we conclude that

$$D_g(x_n, x) \leq D_g(x_n, x) + D_g(x_m, x_n) \leq D_g(x_m, x), \quad \forall m \geq n.$$

This proves that $\{D_g(x_n, x)\}_{n\in\mathbb{N}}$ is an increasing sequence in $\mathbb{R}$ and hence by (3.5) the limit $\lim_{n\to\infty} D_g(x_n, x)$ exists. Letting $m, n \to \infty$ in (3.5), we deduce that $D_g(x_m, x_n) \to 0$. In view of Lemma 2.2, we obtain that $\|x_m - x_n\| \to 0$ as $m, n \to \infty$. This means that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Since E is a Banach space and C is closed and convex, we conclude that there exists $u \in C$ such that

$$\lim_{n\to\infty} \|x_n - u\| = 0. \tag{3.6}$$

Now, we show that $u \in F$. In view of Lemma 2.2, (3.5) and (3.6), we obtain

$$\lim_{n\to\infty} D_g(x_{n+1}, x_n) = 0. \tag{3.7}$$

Since $x_{n+1} \in C_{n+1}$, we conclude that

$$D_g(x_{n+1}, y_n) \leq D_g(x_{n+1}, x_n).$$

This, together with (3.7), implies that

$$\lim_{n\to\infty} D_g(x_{n+1}, y_n) = 0. \tag{3.8}$$

Employing Lemma 2.2 and (3.7)-(3.8), we deduce that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|x_{n+1} - y_n\| = 0.$$

In view of (3.6), we get

$$\lim_{n\to\infty} \|y_n - u\| = 0. \tag{3.9}$$

From (3.6) and (3.9), it follows that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0.$$

Since ∇g is uniformly norm-to-norm continuous on any bounded subset of E, we obtain

$$\lim_{n\to\infty} \|\nabla g(x_n) - \nabla g(y_n)\| = 0. \tag{3.10}$$

Applying Lemma 2.2 we derive that

$$\lim_{n\to\infty} D_g(y_n, x_n) = 0$$

and hence

$$\lim_{n\to\infty} |g(x_n) - g(y_n)| = \lim_{n\to\infty} |D_g(y_n, x_n) - \langle x_n - y_n, \nabla g(x_n)\rangle| = 0.$$

It follows from the definition of Bregman distance that

$$\begin{aligned}
&|D_g(w, x_n) - D_g(w, y_n)| \\
&\quad = |g(w) - g(x_n) + \langle w - x_n, \nabla g(x_n)\rangle - (g(w) - g(y_n) + \langle w - y_n, \nabla g(y_n)\rangle)| \\
&\quad = |g(y_n) - g(x_n) + \langle w - x_n, \nabla g(x_n) - \nabla g(x_n)\rangle + \langle x_n - y_n, \nabla g(x_n)\rangle| \\
&\quad \le |g(y_n) - g(x_n)| + \|w - x_n\| \|\nabla g(y_n) - \nabla g(x_n)\| + \|x_n - y_n\| \|\nabla g(y_n)\| \\
&\quad \to 0
\end{aligned} \tag{3.11}$$

as $n \to \infty$.

The function g is bounded on bounded subsets of E and, thus, ∇g is also bounded on bounded subsets of E^* (see, for example, [13] for more details). This implies that the sequences $\{\nabla g(x_n)\}_{n\in\mathbb{N}}$, $\{\nabla g(y_n)\}_{n\in\mathbb{N}}$ and $\{\nabla g(W_n x_n) : n \in \mathbb{N} \cup \{0\}\}$ are bounded in E^*.

In view of Theorem 2.2(3), we know that $\operatorname{dom} g^* = E^*$ and g^* is strongly coercive and uniformly convex on bounded subsets. Let $r_4 = \sup\{\|\nabla g(x_n)\|, \|\nabla g(W_n x_n)\| : n \in \mathbb{N} \cup \{0\}\}$ and $\rho^*_{r_4} : E^* \to \mathbb{R}$ be the gauge of uniform convexity of the conjugate function g^*. We prove that for any $w \in F$

$$D_g(w, y_n) \le D_g(w, x_n) - \alpha_n(1 - \alpha_n)\rho^*_{r_4}(\|\nabla g(x_n) - \nabla g(W_n x_n)\|). \tag{3.12}$$

Let us show (3.12). For any given $w \in F$, in view of the definition of Bregman distance (see (1.7)), (1.6), Lemma 2.5, we obtain

$$\begin{aligned}
D_g(w, y_n) &= D_g(w, \nabla g^*[\alpha_n \nabla g(x_n) + (1 - \alpha_n)\nabla g(W_n x_n)]) \\
&= V_g(w, \alpha_n \nabla g(x_n) + (1 - \alpha_n)\nabla g(W_n x_n)) \\
&= g(w) - \langle w, \alpha_n \nabla g(x_n) + (1 - \alpha_n)\nabla g(W_n x_n)\rangle \\
&\quad + g^*(\alpha_n \nabla g(x_n) + (1 - \alpha_n)\nabla g(W_n x_n)) \\
&\le \alpha_n g(w) + (1 - \alpha_n)g(w) - \alpha_n\langle w, \nabla g(x_n)\rangle - (1 - \alpha_n)\langle w, \nabla g(W_n x_n)\rangle
\end{aligned}$$

$$
\begin{aligned}
&+ \alpha_n g^*\big(\nabla g(x_n)\big) + (1-\alpha_n) g^*\big(\nabla g(W_n x_n)\big) \\
&- \alpha_n(1-\alpha_n)\rho^*_{r_4}\big(\|\nabla g(x_n) - \nabla g(W_n x_n)\|\big) \\
&= \alpha_n V_g\big(w, \nabla g(x_n)\big) + (1-\alpha_n) V_g\big(w, \nabla g(W_n x_n)\big) \\
&- \alpha_n(1-\alpha_n)\rho^*_{r_4}\big(\|\nabla g(x_n) - \nabla g(W_n x_n)\|\big) \\
&= \alpha_n D_g(w, x_n) + (1-\alpha_n) D_g(w, W_n x_n) - \alpha_n(1-\alpha_n)\rho^*_{r_4}\big(\|\nabla g(x_n) - \nabla g(W_n x_n)\|\big) \\
&\le \alpha_n D_g(w, x_n) + (1-\alpha_n) D_g(w, x_n) - \alpha_n(1-\alpha_n)\rho^*_{r_4}\big(\|\nabla g(x_n) - \nabla g(W_n x_n)\|\big) \\
&= D_g(w, x_n) - \alpha_n(1-\alpha_n)\rho^*_{r_4}\big(\|\nabla g(x_n) - \nabla g(W_n x_n)\|\big).
\end{aligned}
$$

In view of (3.11), we obtain

$$
D_g(w, x_n) - D_g(w, y_n) \to 0 \quad \text{as } n \to \infty. \tag{3.13}
$$

In view of (3.12) and (3.13), we conclude that

$$
\alpha_n(1-\alpha_n)\rho^*_{r_4}\big(\|\nabla g(x_n) - \nabla g(W_n x_n)\|\big) \le D_g(w, x_n) - D_g(w, y_n) \to 0
$$

as $n \to \infty$. From the assumption $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, we have

$$
\lim_{n\to\infty} \rho^*_{r_4} \|\nabla g(x_n) - \nabla g(W_n x_n)\| = 0.
$$

Therefore, from the property of $\rho^*_{r_4}$ we deduce that

$$
\lim_{n\to\infty} \|\nabla g(x_n) - \nabla g(W_n x_n)\| = 0.
$$

Since ∇g^* is uniformly norm-to-norm continuous on bounded subsets of E^*, we arrive at

$$
\lim_{n\to\infty} \|x_n - W_n x_n\| = 0. \tag{3.14}
$$

$$
\begin{aligned}
&D_g(w, U_{n,k} x_n) \\
&= D_g\big(w, \operatorname{proj}^g_C\big(\nabla g^*\big[\beta_{n,k} \nabla g(S_k U_{n,k+1} x_n) + (1-\beta_{n,k}) \nabla g(x_n)\big]\big)\big) \\
&\le D_g\big(w, \nabla g^*\big[\beta_{n,k} \nabla g(S_k U_{n,k+1} x_n) + (1-\beta_{n,k}) \nabla g(x_n)\big]\big) \\
&- D_g\big(U_{n,k} x_n, \nabla g^*\big[\beta_{n,k} \nabla g(S_k U_{n,k+1} x_n) + (1-\beta_{n,k}) \nabla g(x_n)\big]\big) \\
&= g(w) - \big\langle u, \nabla g(S_k U_{n,k+1} x_n) + (1-\beta_{n,k}) \nabla g(x_n)\big\rangle \\
&+ g^*\big(\beta_{n,k} \nabla g(S_k U_{n,k+1} x_n) + (1-\beta_{n,k}) \nabla g(x_n)\big) \\
&- D_g\big(U_{n,k} x_n, \nabla g^*\big[\beta_{n,k} \nabla g(S_k U_{n,k+1} x_n) + (1-\beta_{n,k}) \nabla g(x_n)\big]\big) \\
&\le \beta_{n,k} g(w) + (1-\beta_{n,1}) g(w) + \beta_{n,k} g^*\big(\nabla g(S_k U_{n,3} x_n)\big) + (1-\beta_{n,1}) g^*\big(\nabla g(x_n)\big) \\
&- \beta_{n,k}(1-\beta_{n,k})\rho^*_{r_3}\big(\|\nabla g(S_k U_{n,2} x_n) - \nabla g(x_n)\|\big) \\
&- D_g\big(U_{n,k} x_n, \nabla g^*\big[\beta_{n,k} \nabla g(S_k U_{n,k+1} x_n) + (1-\beta_{n,k}) \nabla g(x_n)\big]\big) \\
&= \beta_{n,k} V_g\big(v, \nabla g(S_k U_{n,2} x_n)\big) + (1-\beta_{n,k}) V_g\big(u, \nabla g(x_n)\big)
\end{aligned}
$$

$$
\begin{aligned}
&\quad - D_g\big(U_{n,k}x_n, \nabla g^*\big[\beta_{n,k}\nabla g(S_kU_{n,k+1}x_n) + (1-\beta_{n,k})\nabla g(x_n)\big]\big) \\
&\quad - \beta_{n,k}(1-\beta_{n,k})\rho^*_{r_3}\big(\big\|\nabla g(S_kU_{n,2}x_n) - \nabla g(x_n)\big\|\big) \\
&\quad - D_g\big(U_{n,k}x_n, \nabla g^*\big[\beta_{n,k}\nabla g(S_kU_{n,k+1}x_n) + (1-\beta_{n,k})\nabla g(x_n)\big]\big) \\
&= \beta_{n,k}D_g(w, S_kU_{n,k+1}x_n) + (1-\beta_{n,1})D_g(w,x_n) \\
&\quad - \beta_{n,k}(1-\beta_{n,k})\rho^*_{r_3}\big(\big\|\nabla g(S_1U_{n,k+1}x_n) - \nabla g(x_n)\big\|\big) \\
&\quad - D_g\big(U_{n,k}x_n, \nabla g^*\big[\beta_{n,k}\nabla g(S_kU_{n,k+1}x_n) + (1-\beta_{n,k})\nabla g(x_n)\big]\big) \\
&\leq \beta_{n,k}D_g(w, U_{n,k+1}x_n) + (1-\beta_{n,1})D_g(w,x_n) \\
&\quad - \beta_{n,k}(1-\beta_{n,k})\rho^*_{r_3}\big(\big\|\nabla g(S_kU_{n,k+1}x_n) - \nabla g(x_n)\big\|\big) \\
&\quad - D_g\big(U_{n,k}z_n, \nabla g^*\big[\beta_{n,k}\nabla g(S_kU_{n,k+1}x_n) + (1-\beta_{n,k})\nabla g(x_n)\big]\big) \\
&\leq \beta_{n,k}D_g(w, U_{n,k+1}x_n) + (1-\beta_{n,k})D_g(w,x_n) \\
&\quad - \beta_{n,k}(1-\beta_{n,k})\rho_{s_3}\big(\big\|\nabla g(S_kU_{n,k+1}x_n) - \nabla g(x_n)\big\|\big) \\
&\quad - D_g\big(U_{n,k}x_n, \nabla g^*\big[\beta_{n,k}\nabla g(S_kU_{n,k+1}x_n) + (1-\beta_{n,k})\nabla g(x_n)\big]\big).
\end{aligned}
$$

Let $r_5 = \sup\{\|\nabla g(x_n)\|, \|\nabla g(W_nx_n)\| : n \in \mathbb{N}\cup\{0\}\}$ and $\rho^*_{r_5} : E^* \to \mathbb{R}$ be the gauge of uniform convexity of the conjugate function g^*. Then we have

$$
\begin{aligned}
&D_g(w, W_nx_n) \\
&\quad = D_g(w, U_{n,1}x_n) \\
&\quad \leq \beta_{n,1}D_g(w, U_{n,2}x_n) + (1-\beta_{n,1})D_g(w,x_n) \\
&\qquad - \beta_{n,1}(1-\beta_{n,1})\rho^*_{r_5}\big(\big\|\nabla g(S_1U_{n,2}x_n) - \nabla g(x_n)\big\|\big) \\
&\quad \leq \beta_{n,1}\big[\beta_{n,2}D_g(w, U_{n,3}x_n) + (1-\beta_{n,2})D_g(w,x_n) \\
&\qquad - \beta_{n,2}(1-\beta_{n,2})\rho^*_{r_5}\big(\big\|\nabla g(S_2U_{n,3}x_n) - \nabla g(x_n)\big\|\big) \\
&\qquad - D_g\big(U_{n,2}x_n, \nabla g^*\big[\beta_{n,2}\nabla g(S_2U_{n,3}x_n) + (1-\beta_{n,2})\nabla g(x_n)\big]\big)\big] \\
&\qquad + (1-\beta_{n,1})D_g(w,x_n) - \beta_{n,1}(1-\beta_{n,1})\rho^*_{r_5}\big(\big\|\nabla g(S_1U_{n,2}x_n) - \nabla g(x_n)\big\|\big) \\
&\qquad - \beta_{n,1}(1-\beta_{n,1})\rho^*_{r_5}\big(\big\|\nabla g(S_1U_{n,2}x_n) - \nabla g(x_n)\big\|\big) \\
&\quad \leq \cdots \\
&\quad \leq D_g(w,x_n) - \beta_{n,1}(1-\beta_{n,1})\rho^*_{r_5}\big(\big\|\nabla g(S_1U_{n,2}x_n) - \nabla g(x_n)\big\|\big) \\
&\qquad - \beta_{n,1}\beta_{n,2}(1-\beta_{n,2})\rho^*_{r_5}\big(\big\|\nabla g(S_2U_{n,3}x_n) - \nabla g(x_n)\big\|\big) \\
&\qquad - \beta_{n,1}D_g\big(U_{n,2}x_n, \nabla g^*\big[\beta_{n,2}\nabla g(S_2U_{n,3}x_n) + (1-\beta_{n,2})\nabla g(x_n)\big]\big) - \cdots \\
&\qquad - \beta_{n,1}\beta_{n,2}\cdots\beta_{n,n-1}D_g\big(U_{n,n}x_n, \nabla g^*\big[\beta_{n,n}\nabla g(S_nU_{n,n+1}x_n) + (1-\beta_{n,n})\nabla g(x_n)\big]\big)
\end{aligned} \tag{3.15}
$$

for all $n \in \mathbb{N}$. Since ∇g is uniformly norm-to-norm continuous on bounded subsets of E, we obtain

$$
\lim_{n\to\infty} \beta_{n,1}\big\|\nabla g(S_1U_{n,2}x_n) - \nabla g(x_n)\big\| = \lim_{n\to\infty}\big\|\nabla g(W_nx_n) - \nabla g(x_n)\big\| = 0.
$$

This implies that

$$\lim_{n\to\infty}\left\|\nabla g(S_1U_{n,2}x_n)-\nabla g(x_n)\right\|=0.$$

Now, in view of (3.14) and (3.15), we conclude that

$$\lim_{n\to\infty}\left\|\nabla g(S_kU_{n,k+1}x_n)-\nabla g(x_n)\right\|=0,\quad \forall k\in\mathbb{N}. \tag{3.16}$$

Since ∇g^* is uniformly norm-to-norm continuous on bounded subsets of E^*, we deduce that

$$\lim_{n\to\infty}\|S_kU_{n,k+1}x_n-x_n\|=0,\quad \forall k\in\mathbb{N}. \tag{3.17}$$

On the other hand, we have

$$\lim_{n\to\infty}D_g\big(U_{n,k}x_n,\nabla g^*\big[\beta_{n,k}\nabla g(S_kU_{n,k+1}x_n)+(1-\beta_{n,k})\nabla g(x_n)\big]\big)=0,\quad \forall k\in\mathbb{N}\text{ with }k\geq 2.$$

This, together with Lemma 2.2, implies that

$$\begin{aligned}&\lim_{n\to\infty}\left\|U_{n,k}x_n-\nabla g^*\big[\beta_{n,k}\nabla g(S_kU_{n,k+1}x_n)+(1-\beta_{n,k})\nabla g(x_n)\big]\right\|=0,\\&\quad\forall k\in\mathbb{N}\text{ with }k\geq 2.\end{aligned} \tag{3.18}$$

In view of (3.17), we obtain

$$\lim_{n\to\infty}\left\|\big[\beta_{n,k}\nabla g(S_kU_{n,k+1}x_n)+(1-\beta_{n,k})\nabla g(x_n)\big]-\nabla g(x_n)\right\|=0,\quad \forall k\in\mathbb{N}.$$

Therefore,

$$\lim_{n\to\infty}\left\|\nabla g^*\big[\beta_{n,k}\nabla g(S_kU_{n,k+1}x_n)+(1-\beta_{n,k})\nabla g(x_n)\big]-x_n\right\|=0,\quad \forall k\in\mathbb{N}.$$

From (3.14) and (3.17), we get

$$\lim_{n\to\infty}\|U_{n,k}x_n-x_n\|=0,\quad \forall k\in\mathbb{N}.$$

This, together with (3.18), implies that

$$\lim_{n\to\infty}\|S_kU_{n,k+1}x_n-U_{n,k+1}x_n\|=0,\quad \forall k\in\mathbb{N}.$$

Since $U_{n,k+1}x_n\to u$ and S_k is Bregman weak relatively nonexpansive, we obtain $u\in F(S_k)$ for every $k\in\mathbb{N}$. Thus, $x_n\to \operatorname{proj}_F^g x_0$ as $n\to\infty$.

Finally, we show that $u=\operatorname{proj}_F^g x$. From $x_n=\operatorname{proj}_{C_n}^g x$, we conclude that

$$\big\langle z-x_n,\nabla g(x_n)-\nabla g(x)\big\rangle\geq 0,\quad \forall z\in C_n.$$

Since $F\subset C_n$ for each $n\in\mathbb{N}$, we obtain

$$\big\langle z-x_n,\nabla g(x_n)-\nabla g(x)\big\rangle\geq 0,\quad \forall z\in F. \tag{3.19}$$

Letting $n \to \infty$ in (3.19), we deduce that

$$\langle z - u, \nabla g(u) - \nabla g(x) \rangle \geq 0, \quad \forall z \in F.$$

In view of (1.8), we have $u = \mathrm{proj}_F^g x$, which completes the proof. □

Theorem 3.2 *Let E be a reflexive Banach space and $g : E \to \mathbb{R}$ be a convex, continuous, strongly coercive and Gâteaux differentiable function which is bounded on bounded subsets and uniformly convex on bounded subsets of E. Let C be a nonempty, closed and convex subset of E. Let $\{S_n\}_{n\in\mathbb{N}}$ be a family of Bregman weak relatively nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(S_n) \neq \emptyset$, and let $T_n x = \nabla g^*(\sum_{j=1}^{n} \beta_{n,j} \nabla g(S_j x))$ for every $n \in \mathbb{N}$ and $x \in C$, where $0 \leq \beta_{n,j} \leq 1$ ($n \in \mathbb{N}$, $j = 1, 2, \dots, n$) with $\sum_{j=1}^{n} \beta_{n,j} = 1$ for all $n \in \mathbb{N}$ and $\liminf_{n\to\infty} \beta_{n,j} > 0$ for each $j \in \{1, 2, \dots, n\}$. Let $\{\alpha_n\}_{n\in\mathbb{N}\cup\{0\}}$ and $\{\beta_n\}_{n\in\mathbb{N}\cup\{0\}}$ be sequences in $[0, 1)$ such that $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence generated by*

$$\begin{cases} x_0 = x \in C \quad \text{chosen arbitrarily}, \\ C_0 = C, \\ y_n = \nabla g^*[\alpha_n \nabla g(x_n) + (1 - \alpha_n)\nabla g(T_n x_n)], \\ C_{n+1} = \{z \in C_n : D_g(z, y_n) \leq D_g(z, x_n)\}, \\ x_{n+1} = \mathrm{proj}_{C_{n+1}}^g x \quad \text{and} \quad n \in \mathbb{N} \cup \{0\}, \end{cases} \tag{3.20}$$

where ∇g is the gradient of g. Then $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to $\mathrm{proj}_F^g x_0$ as $n \to \infty$.

Remark 3.1 Theorem 3.1 improves Theorem 1.1 in the following aspects.

(1) For the structure of Banach spaces, we extend the duality mapping to a more general case, that is, a convex, continuous and strongly coercive Bregman function which is bounded on bounded subsets, and uniformly convex and uniformly smooth on bounded subsets.
(2) For the mappings, we extend the mapping from a relatively nonexpansive mapping to a countable family of Bregman W-mappings. We remove the assumption $\hat{F}(T) = F(T)$ on the mapping T and extend the result to a countable family of Bregman weak relatively nonexpansive mappings, where $\hat{F}(T)$ is the set of asymptotic fixed points of the mapping T.
(3) For the algorithm, we remove the set W_n in Theorem 1.1.

The following result was proved in [29].

Lemma 3.2 *Let E be a reflexive Banach space and $g : E \to \mathbb{R}$ be a strongly coercive Bregman function which is bounded on bounded subsets, and uniformly convex and uniformly smooth on bounded subsets of E. Let A be a maximal monotone operator from E to E^* such that $A^{-1}(0) \neq \emptyset$. Let $r > 0$ and $\mathrm{Res}_{rA}^g = (\nabla g + rA)^{-1}\nabla g$ be the g-resolvent of A. Then Res_{rA}^g is a Bregman weak relatively nonexpansive mapping.*

As an application of our main result, we include a concrete example in support of Theorem 3.1. Using Theorem 3.1, we obtain the following strong convergence theorem for maximal monotone operators.

Theorem 3.3 *Let E be a reflexive Banach space and $g : E \to \mathbb{R}$ be a strongly coercive Bregman function which is bounded on bounded subsets, and uniformly convex and uniformly smooth on bounded subsets of E. Let $\{A_n\}_{n\in\mathbb{N}}$ be an infinite family of maximal monotone operators from E to E^* such that $Z = \bigcap_{n=1}^{\infty} A_n^{-1}(0) \neq \emptyset$. Let $r > 0$ and $\mathrm{Res}^g_{rA_n} = (\nabla g + rA_n)^{-1}\nabla g$ be the g-resolvent of A_n. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence generated by*

$$\begin{cases} x_0 = x \in E \quad \textit{chosen arbitrarily}, \\ C_0 = E, \\ y_n = \nabla g^*[\alpha_n \nabla g(x_n) + (1-\alpha_n)\nabla g(W_n x_n)], \\ C_{n+1} = \{z \in C_n : D_g(z, y_n) \le D_g(z, x_n)\}, \\ x_{n+1} = \mathrm{proj}^g_{C_{n+1}} x \quad \textit{and} \quad n \in \mathbb{N} \cup \{0\}, \end{cases} \tag{3.21}$$

where ∇g is the right-hand derivative of g and W_n is the W-mapping generated by $\mathrm{Res}^g_{rA_n}$, $\mathrm{Res}^g_{rA_{n-1}}, \ldots, \mathrm{Res}^g_{rA_1}$ and $\beta_{n,n}, \beta_{n,n-1}, \ldots, \beta_{n,1}$. Let $\{\alpha_n\}_{n\in\mathbb{N}\cup\{0\}}$ and $\{\beta_n\}_{n\in\mathbb{N}\cup\{0\}}$ be sequences in $[0,1)$ *satisfying the following control conditions*:

(1) $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$;

(2) $0 \le \beta_n < 1$ *for all* $n \in \mathbb{N} \cup \{0\}$ *and* $\liminf_{n\to\infty} \beta_n < 1$.

Then the sequence $\{x_n\}_{n\in\mathbb{N}}$ defined in (3.21) *converges strongly to* $\mathrm{proj}^g_Z x$ *as* $n \to \infty$.

Proof Letting $S_n = \mathrm{Res}^g_{rA_n}$, $\forall n \in \mathbb{N}$, in Theorem 3.1, from (3.1) we obtain (3.19). We need only to show that S_n satisfies all the conditions in Theorem 3.1 for all $n \in \mathbb{N}$. In view of Lemma 3.2, we conclude that S_n is a Bregman relatively nonexpansive mapping for each $n \in \mathbb{N}$. Thus, we obtain

$$D_g\big(p, \mathrm{Res}^g_{rA_n} v\big) \le D_g(p, v), \quad \forall v \in E, p \in F\big(\mathrm{Res}^g_{rA_n}\big)$$

and

$$\tilde{F}\big(\mathrm{Res}^g_{rA}\big) = F\big(\mathrm{Res}^g_{rA_n}\big) = A_n^{-1}(0),$$

where $\tilde{F}(\mathrm{Res}^g_{rA_n})$ is the set of all strong asymptotic fixed points of $\mathrm{Res}^g_{rA_n}$. Therefore, in view of Theorem 3.1, we have the conclusions of Theorem 3.2. This completes the proof. □

Below we include a nontrivial example of an infinite family of Bregman weak relatively nonexpansive mappings in order to reconstruct a Bregman W-mapping in the setting of Hilbert spaces.

Example 3.1 Let $E = l^2$, where

$$l^2 = \left\{\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n, \ldots) : \sum_{n=1}^{\infty} \|\sigma_n\|^2 < \infty\right\}, \qquad \|\sigma\| = \left(\sum_{n=1}^{\infty} \|\sigma_n\|^2\right)^{\frac{1}{2}}, \forall \sigma \in l^2,$$

$$\langle \sigma, \eta \rangle = \sum_{n=1}^{\infty} \sigma_n \eta_n, \quad \forall \delta = (\sigma_1, \sigma_2, \ldots, \sigma_n, \ldots), \eta = (\eta_1, \eta_2, \ldots, \eta_n, \ldots) \in l^2.$$

Let $\{x_n\}_{n\in\mathbb{N}\cup\{0\}} \subset E$ be a sequence defined by

$$\begin{aligned}
&x_0 = (1,0,0,0,\ldots),\\
&x_1 = (1,1,0,0,0,\ldots),\\
&x_2 = (1,0,1,0,0,0,\ldots),\\
&x_3 = (1,0,0,1,0,0,0,\ldots),\\
&\ldots\\
&x_n = (\sigma_{n,1},\sigma_{n,2},\ldots,\sigma_{n,k},\ldots),\\
&\ldots,
\end{aligned}$$

where

$$\sigma_{n,k} = \begin{cases} 1 & \text{if } k = 1, n+1,\\ 0 & \text{if } k \neq 1, k \neq n+1,\end{cases}$$

for all $n \in \mathbb{N}$. It is clear that the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges weakly to x_0. Indeed, for any $\Lambda = (\lambda_1,\lambda_2,\ldots,\lambda_n,\ldots) \in l^2 = (l^2)^*$, we have

$$\begin{aligned}
\Lambda(x_n - x_0) &= \langle x_n - x_0, \Lambda\rangle\\
&= \sum_{k=2}^{\infty} \lambda_k \sigma_{n,k} \to 0
\end{aligned}$$

as $n \to \infty$. It is also obvious that $\|x_n - x_m\| = \sqrt{2}$ for any $n \neq m$ with n, m sufficiently large. Thus, $\{x_n\}_{n\in\mathbb{N}}$ is not a Cauchy sequence. Let k be an even number in $\mathbb{N}$ and let $g : E \to \mathbb{R}$ be defined by

$$g(x) = \frac{1}{k}\|x\|^k, \quad x \in E.$$

It is easy to show that $\nabla g(x) = J_k(x)$ for all $x \in E$, where

$$J_k(x) = \bigl\{x^* \in E^* : \bigl\langle x, x^*\bigr\rangle = \|x\|\bigl\|x^*\bigr\|, \bigl\|x^*\bigr\| = \|x\|^{k-1}\bigr\}.$$

It is also obvious that

$$J_k(\lambda x) = \lambda^{k-1} J_k(x), \quad \forall x \in E, \lambda \in \mathbb{R}.$$

We define a countable family of mappings $S_j : E \to E$ by

$$S_j(x) = \begin{cases} \frac{n}{n+1}x & \text{if } x = x_n;\\ \frac{-j}{j+1}x & \text{if } x \neq x_n,\end{cases}$$

for all $j \geq 1$ and $n \geq 0$. It is clear that $F(S_j) = \{0\}$ for all $j \geq 1$. Choose $j \in \mathbb{N}$, then for any $n \in \mathbb{N}$

$$\begin{aligned} D_g(0, S_j x_n) &= g(0) - g(S_j x_n) - \langle 0 - S_j x_n, \nabla g(S_j x_n)\rangle \\ &= -\frac{n^k}{(n+1)^k} g(x_n) + \frac{n^k}{(n+1)^k}\langle x_n, \nabla g(x_n)\rangle \\ &= \frac{n^k}{(n+1)^k}\bigl[-g(x_n) + \langle x_n, \nabla g(x_n)\rangle\bigr] \\ &= \frac{n^k}{(n+1)^k}\bigl[D_g(0, x_n)\bigr] \\ &\leq D_g(0, x_n). \end{aligned}$$

If $x \neq x_n$, then we have

$$\begin{aligned} D_g(0, S_j x) &= g(0) - g(S_j x) - \langle 0 - S_j x, \nabla g(S_j x)\rangle \\ &= -\frac{j^k}{(j+1)^k} g(x) - \frac{j^k}{(j+1)^k}\langle x, -\nabla g(x)\rangle \\ &= \frac{j^k}{(j+1)^k}\bigl[-g(x) - \langle -x, \nabla g(x)\rangle\bigr] \\ &\leq D_g(0, x). \end{aligned}$$

Therefore, S_j is a Bregman quasi-nonexpansive mapping. Next, we claim that S_j is a Bregman weak relatively nonexpansive mapping. Indeed, for any sequence $\{z_n\}_{n\in\mathbb{N}} \subset E$ such that $z_n \to z_0$ and $\|z_n - S_j z_n\| \to 0$ as $n \to \infty$, there exists a sufficiently large number $N_0 \in \mathbb{N}$ such that $z_n \neq x_m$ for any $n, m > N_0$. This implies that $S_j z_n = -\frac{j}{j+1} z_n$ for all $n > N_0$. It follows from $\|z_n - S_j z_n\| \to 0$ that $\frac{2j+1}{j+1} z_n \to 0$ and hence $z_n \to z_0 = 0$. Since $z_0 \in F(S_j)$, we conclude that S_j is a Bregman weak relatively nonexpansive mapping. It is clear that $\bigcap_{j=1}^{\infty} \tilde{F}(S_j) = \bigcap_{j=1}^{\infty} F(S_j) = \{0\}$. Thus $\{S_j\}_{j\in\mathbb{N}}$ is a countable family of Bregman weak relatively nonexpansive mappings. Next, we show that $\{S_j\}_{j\in\mathbb{N}}$ is not a countable family of Bregman relatively nonexpansive mappings. In fact, though $x_n \rightharpoonup x_0$ and

$$\|x_n - S_j x_n\| = \left\| x_n - \frac{n}{n+1} x_n \right\| = \frac{1}{n+1}\|x_n\| \to 0$$

as $n \to \infty$, but $x_0 \notin F(S_j)$ for all $j \in \mathbb{N}$. Therefore, $\hat{F}(S_j) \neq F(S_j)$ for all $j \in \mathbb{N}$. This implies that $\bigcap_{j=1}^{\infty} \hat{F}(S_j) \neq \bigcap_{j=1}^{\infty} F(S_j)$. Let $\{\beta_{n,k} : k, n \in \mathbb{N}, 1 \leq k \leq n\}$ be a sequence of real numbers such that $0 < \beta_{n,1} \leq 1$ and $0 < \beta_{n,i} < 1$ for every $i = 2, 3, \ldots, n$. Let W_n be the Bregman W-mapping generated by $S_n, S_{n-1}, \ldots, S_1$ and $\beta_{n,n}, \beta_{n,n-1}, \ldots, \beta_{n,1}$. Finally, it is obvious that the family $\{S_j\}_{j\in\mathbb{N}}$ satisfies all the aspects of the hypothesis of Theorem 3.1.

4 Applications to convex feasibility problems

Let $\{D_n\}_{n\in\mathbb{N}}$ be a family of nonempty, closed and convex subsets of a Banach space E. The convex feasibility problem is to find an element in the assumed nonempty intersection

$\bigcap_{n=1}^{\infty} D_n$ (see [36]). In the following, we prove a strong convergence theorem concerning convex feasibility problems in a reflexive Banach space.

Theorem 4.1 *Let E be a reflexive Banach space and $g : E \to \mathbb{R}$ be a convex, continuous, strongly coercive and Gâteaux differentiable function which is bounded on bounded subsets and uniformly convex on bounded subsets of E. Let C be a nonempty, closed and convex subset of E. Let $F := \{D_n\}_{n\in\mathbb{N}}$ be an infinite family of nonempty, closed and convex subsets of E such that $\bigcap_{n=1}^{\infty} D_n \neq \emptyset$, and let $\{\beta_{n,k} : k, n \in \mathbb{N}, 1 \leq k \leq n\}$ be a sequence of real numbers such that $0 < \beta_{i,j} \leq 1$ and $0 < \beta_{i,j} < 1$ for every $i = 2, 3, \ldots, n$. Let W_n be the Bregman W-mapping generated by $\operatorname{proj}^g_{D_n}, \operatorname{proj}^g_{D_{n-1}}, \ldots, \operatorname{proj}^g_{D_1}$ and $\beta_{n,n}, \beta_{n,n-1}, \ldots, \beta_{n,1}$. Let $\{\alpha_n\}_{n\in\mathbb{N}\cup\{0\}}$ and $\{\beta_n\}_{n\in\mathbb{N}\cup\{0\}}$ be sequences in $[0, 1)$ such that $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence generated by*

$$\begin{cases} x_0 = x \in C \quad \text{chosen arbitrarily}, \\ C_0 = C, \\ y_n = \nabla g^*[\alpha_n \nabla g(x_n) + (1 - \alpha_n)\nabla g(W_n x_n)], \\ C_{n+1} = \{z \in C_n : D_g(z, y_n) \leq D_g(z, x_n)\}, \\ x_{n+1} = \operatorname{proj}^g_{C_{n+1}} x \quad \text{and} \quad n \in \mathbb{N} \cup \{0\}, \end{cases} \tag{4.1}$$

where ∇g is the gradient of g. Then $\{x_n\}_{n\in\mathbb{N}}$ defined in (4.1) converges strongly to $\operatorname{proj}^g_F x_0$ as $n \to \infty$.

Proof For each $j \in \mathbb{N}$, let $S_j = \operatorname{proj}^g_{D_j}$. We will prove that S_j is a Bregman weak relatively nonexpansive mapping. Indeed, for any sequence $\{z_n\}_{n\in\mathbb{N}} \subset E$ such that $z_n \to z_0$ and $\|z_n - S_j z_n\| \to 0$ as $n \to \infty$, in view of Lemma 2.2, we conclude that

$$\begin{aligned} &\lim_{n\to\infty} D_g(z_n, S_j z_n) = 0, \\ &\lim_{n\to\infty} D_g(z_n, z_0) = 0. \end{aligned} \tag{4.2}$$

It follows from (1.9) that

$$D_g\bigl(z_n, \operatorname{proj}^g_{D_j} z_n\bigr) + D_g\bigl(\operatorname{proj}^g_{D_j} z_n, z_0\bigr) \leq D_g(z_n, z_0).$$

This, together with (4.2), amounts to

$$\lim_{n\to\infty} D_g\bigl(\operatorname{proj}^g_{D_j} z_n, z_0\bigr) = 0$$

and hence by Lemma 2.2

$$\lim_{n\to\infty} \bigl\|\operatorname{proj}^g_{D_j} z_n - z_0\bigr\| = 0.$$

Thus we obtain $z_0 \in F(S_j) = D_j$ and hence S_j is a Bregman weak relatively nonexpansive mapping. By a similar argument as in the proof of Theorem 3.1, we get the desired conclusion, which completes the proof. □

5 Numerical example

In this section, in order to demonstrate the effectiveness, realization and convergence of algorithm of Theorem 3.1, we consider the following simple example.

Example 5.1 Let $S : [0,2] \to [0,2]$ be defined by

$$Sx = \begin{cases} 0 & \text{if } x \neq 2, \\ 1 & \text{if } x = 2. \end{cases}$$

Then T is a quasi-nonexpansive mapping. Indeed, for any $x \in [0,2)$, we have that $Sx = 0$. Thus,

$$|Sx - 0|^2 = 0 \leq |x - 0|^2.$$

The other cases can be verified similarly. It is worth mentioning that S is neither nonexpansive nor continuous. Let $\beta_{n,k} = 0$ and $\alpha_n = \frac{1}{4}$ for all $n, k \geq 1$. Under the above assumptions, the given algorithm (3.1) in Theorem 3.1 is simplified as follows:

$$\begin{cases} x_0 = x \in [0,2] \quad \text{chosen arbitrarily,} \\ C_0 = [0,2], \\ y_n = \frac{1}{4}x_n + \frac{3}{4}Sx_n, \\ C_{n+1} = \{z \in C_n : |z - y_n| \leq |z - x_n|\}, \\ x_{n+1} = P_{C_{n+1}}x \quad \text{and} \quad n \in \mathbb{N} \cup \{0\}. \end{cases} \tag{5.1}$$

We know that, in a one-dimensional case, the set C_{n+1} is a closed interval. If we set $[a_{n+1}, b_{n+1}] := C_{n+1}$, then the projection point x_{n+1} of $x \in C$ onto C_{n+1} can be expressed as

$$x_{n+1} := P_{C_{n+1}}x = \begin{cases} x & \text{if } x \in [a_{n+1}, b_{n+1}]; \\ b_{n+1} & \text{if } x > b_{n+1}; \\ a_{n+1} & \text{if } x < a_{n+1}. \end{cases}$$

Choose $x_0 = x = 1$. Then the iteration process (5.1) becomes

$$\begin{gathered} C_0 = [0,2], \qquad u_n = \frac{1}{2}x_n, \qquad y_n = \frac{1}{4}x_n, \\ C_{n+1} = \left[0, \frac{5}{8}x_n\right], \qquad x_{n+1} = \left(\frac{5}{8}\right)^n. \end{gathered} \tag{5.2}$$

In this section, we give some numerical experiment results (based on Matlab) as follows.

6 Conclusion

Table 1 and Figure 1 show that the sequence $\{x_n\}_{n\in\mathbb{N}}$ generated by (5.2) converges to 0, which solves the fixed point problem.

Table 1 This table shows the values of the sequence $\{x_n\}_{n\in\mathbb{N}}$ on 30th iteration steps (initial value $x_0 = 1$)

n	x_n	u_n	y_n
1	1.000000000000000e+000	5.000000000000000e–001	2.500000000000000e–001
2	6.250000000000000e–001	3.125000000000000e–001	1.562500000000000e–001
3	3.906250000000000e–001	1.953125000000000e–001	9.765625000000000e–002
4	2.441406250000000e–001	1.220703125000000e–001	6.103515625000000e–002
5	1.525878906250000e–001	7.629394531250000e–002	3.814697265625000e–002
6	9.536743164062500e–002	4.768371582031250e–002	2.384185791015625e–002
7	5.960464477539063e–002	2.980232238769531e–002	1.490116119384766e–002
8	3.725290298461914e–002	1.862645149230957e–002	9.313225746154785e–003
9	2.328306436538696e–002	1.164153218269348e–002	5.820766091346741e–003
10	1.455191522836685e–002	7.275957614183426e–003	3.637978807091713e–003
11	9.094947017729282e–003	4.547473508864641e–003	2.273736754432321e–003
12	5.684341886080802e–003	2.842170943040401e–003	1.421085471520200e–003
13	3.552713678800501e–003	1.776356839400251e–003	8.881784197001252e–004
14	2.220446049250313e–003	1.110223024625157e–003	5.551115123125783e–004
15	1.387778780781446e–003	6.938893903907228e–004	3.469446951953614e–004
16	8.673617379884036e–004	4.336808689942018e–004	2.168404344971009e–004
17	5.421010862427522e–004	2.710505431213761e–004	1.355252715606881e–004
18	3.388131789017201e–004	1.694065894508601e–004	8.470329472543003e–005
19	2.117582368135751e–004	1.058791184067875e–004	5.293955920339377e–005
20	1.323488980084844e–004	6.617444900424221e–005	3.308722450212111e–005
21	8.271806125530277e–005	4.135903062765138e–005	2.067951531382569e–005
22	5.169878828456423e–005	2.584939414228212e–005	1.292469707114106e–005
23	3.231174267785264e–005	1.615587133892632e–005	8.077935669463161e–006
24	2.019483917365790e–005	1.009741958682895e–005	5.048709793414475e–006
25	1.262177448353619e–005	6.310887241768094e–006	3.155443620884047e–006
26	7.888609052210119e–006	3.944304526105059e–006	1.972152263052530e–006
27	4.930380657631324e–006	2.465190328815662e–006	1.232595164407831e–006
28	3.081487911019577e–006	1.540743955509789e–006	7.703719777548944e–007
29	1.925929944387236e–006	9.629649721936179e–007	4.814824860968089e–007
30	1.203706215242023e–006	6.018531076210113e–007	3.009265538105056e–007

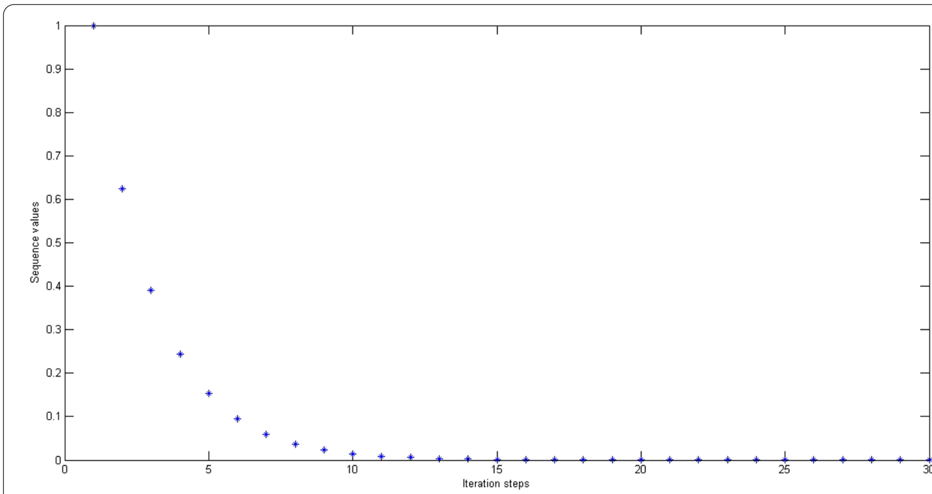

Figure 1 Iteration chart of the sequence $\{x_n\}_{n\in\mathbb{N}}$ in Example 5.1 with initial value $x_0 = 1$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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