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The method of finding solutions of partial dynamic equations on time scales

Hsuan-Ku Liu*

*Correspondence:
HKLiu.nccu@gmail.com
Department of Mathematics and
Information Education, National
Taipei University of Education,
Taipei, Taiwan

Abstract

On time scales, one area lacking of development is the method of finding solutions on partial dynamic equations. This paper proposes a method for finding the exact solution of linear partial dynamic equations on arbitrage time scales. We modify the variational iteration method on \mathbb{R} to find an approximation of the nonlinear partial dynamic equation on $\overline{q^{\mathbb{N}}}$. As an example, the modified variational iteration method is applied to q -Berger equations and to q -Fisher equations. Their numerical results reveal that the proposed method is very effective.

Keywords: partial dynamic equations on time scales; nonlinear q -difference equation; variational iterative method; approximate solutions

1 Introduction

A time scale is a nonempty closed subset of real numbers. On time scale calculus, notations and theorems have been well established for the univariate case [1]. Solutions of ordinary differential equations, such as initial value problems and boundary value problems, have been studied and published during the past two decades on time scales. In recent years, Hoffacker [2] and Ahlbrandt and Morian [3] demonstrated the related ideas to the multivariate case and studied partial dynamic equations on time scales. Notations and definitions on multivariate time scales calculus can be found in Bohner and Guseinov [4, 5]. Jackson [6] extended the existing ideas of the time scales calculus [3] to the multivariate case. The method of generalized Laplace transform on time scales is applied to find solutions of the homogeneous and nonhomogeneous heat and wave equations. Recent developments in the method of finding solutions have aroused further interest in the discussion of partial dynamic equations on time scales.

For the nonlinear cases, methods of finding solutions are not mentioned for partial dynamic equations on time scales. One of the difficulties for developing a theory of series solutions to nonlinear equations on time scales is that the formula for multiplications of two generalized polynomials is not easily found. If a time scale has constant graininess, Haile and Hall [7] provided an exact formula for the multiplication of two generalized polynomials. Using the obtained results, the series solutions for linear dynamic equations are proposed on the time scales \mathbb{R} and $\mathbb{T} = h\mathbb{Z}$ (difference equations with step size h). On generalized time scales, Mozyrska and Pawtuszewicz [8] presented a formula for the multiplication of generalized polynomials of degree one and degree $n \in \mathbb{N}$. Liu [9] provided a product rule of two generalized polynomials on the time scale $\overline{q^{\mathbb{Z}}} = \{q^n \mid n \in \mathbb{N}\} \cup \{0\}$.

The variational iteration method proposed by He [10] is a powerful mathematical tool in analyzing the nonlinear problems on \mathbb{R} (the set of real numbers). Over the last few years, the variational iteration method (VIM) has been widely applied to analyze the nonlinear boundary value problems [11], the nonlinear heat diffusion equations [12] and the nonlinear reaction-diffusion equations [13]. An advantage of the VIM is that there is no need to make the assumption of the small parameters. On nonlinear partial dynamic equations, approximate solutions obtained by the variational iteration method are not found yet.

In this paper, we first explore a simple method to find the exact solution of linear partial dynamic equations on time scales. For the nonlinear cases, we derive a product rule of two generalized polynomials on $\overline{q^{\mathbb{Z}}}$, which provides an idea for developing a series solutions on q -calculus. Applying the product rules, we extend the variational iteration method from the set of real numbers \mathbb{R} to the time scales $\overline{q^{\mathbb{Z}}}$. The extension provides a method to find an approximate solution on the nonlinear partial dynamic equation on $\overline{q^{\mathbb{Z}}}$. Moreover, the VIM is applied to find an approximation of the q -Berger equation and the q -Fisher equation. By the numerical results, we found that the modified VIM is very effective. The VIM can be applied to other time scales when the multiplication rule of two generalized polynomials on these time scales is obtained.

This paper is organized as follows. In Section 2, the basic ideas of partial dynamic equations on time scales are introduced. In Section 3, a method is explored to find an exact solution of linear initial value problems on time scales. In Section 4, a product rule of two generalized polynomials at 0 is derived on $\overline{q^{\mathbb{Z}}}$ and the variational iteration method is applied to find an approximate solution of the Burger equation and the Fisher equation. In Section 5, numerical examples reveal that the proposed method is very effective. Finally, a concise conclusion and future directions are provided in Section 6.

2 Basic concepts on time scales

A time scale is an arbitrary nonempty closed subset of the real numbers. The calculus of time scales was introduced by Hilger [14] in order to create a theory that can unify discrete and continuous analysis.

2.1 An introduction to time scales

In this subsection, we first define the forward and backward jump operators on time scales and then introduce the delta derivative and the integration.

Definition 1 Let \mathbb{T} be a time scale. For $t \in \mathbb{T}$ the forward jump operator $\sigma : \mathbb{T} \rightarrow \mathbb{T}$ is defined by

$$\sigma(t) := \inf\{s > t \mid s \in \mathbb{T}\}$$

and the backward jump operator $\rho : \mathbb{T} \rightarrow \mathbb{T}$ is defined by

$$\rho(t) := \sup\{s < t \mid s \in \mathbb{T}\}.$$

The grain function $\mu : \mathbb{T} \rightarrow [0, \infty)$ is defined by

$$\mu(t) := \sigma(t) - t.$$

According to the forward jump operator and the gain function, the delta derivative on the time scale \mathbb{T} is given as follows.

Definition 2 Assume that $f : \mathbb{T} \rightarrow \mathbb{R}$ is a function and let $t \in \mathbb{T}$. If $\sigma(t) > t$, the delta derivative of $f(t)$ at t on the time scale \mathbb{T} is defined as

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

A function $f(t)$ on \mathbb{T} is said to be differentiable at t if its derivative exists at t , $\forall t \in \mathbb{T}$.

Integration on a time scale can be viewed as an anti-derivative.

Definition 3 If we have delta derivative $g(t) = f^\Delta(t)$ on the time scale \mathbb{T} , then the anti-derivative is

$$f(t) = \int_a^t g(s) \Delta s + \text{constant}, \quad a, t \in \mathbb{T}$$

and the definite integral on the time scale \mathbb{T} follows as

$$\int_a^b g(s) \Delta s = f(b) - f(a), \quad a, b \in \mathbb{T}.$$

Following the delta derivative and integration, we define the generalized polynomials as follows.

Definition 4 On the time scale \mathbb{T} , the generalized polynomials $h_k(\cdot, t_0) : \mathbb{T} \rightarrow \mathbb{R}$ are defined recursively as follows:

$$h_0(t, s) = 1, \quad h_{k+1}(t, s) = \int_s^t h_k(\tau, s) \Delta \tau.$$

Hence, for each fixed s , the delta derivative of h_k with respect to t satisfies

$$h_k^{\Delta^j}(t, 0) = \begin{cases} h_{k-j}(t, 0) & \text{if } k \geq j, \\ 0 & \text{if } k < j. \end{cases}$$

2.2 An introduction to q -calculus

Let

$$q^{\mathbb{N}} = \{q^n \mid n \in \mathbb{N}\} \quad \text{and} \quad \overline{q^{\mathbb{N}}} = q^{\mathbb{N}} \cup \{0\},$$

where \mathbb{N} denotes the set of positive integers.

If a and q are real numbers such that $0 < q < 1$, then the q -shift factorial [15] is defined by

$$(a; q)_0 = 1 \quad \text{and} \quad (a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k), \quad n \in \mathbb{N}.$$

Following Definition 2, the q -derivative is given as follows.

Definition 5 Assume that $f: \overline{q^{\mathbb{N}}} \rightarrow \mathbb{R}$ is a function and $t \in \overline{q^{\mathbb{N}}}$. The q -derivative [1] at t is defined by

$$f^{\Delta}(t) = \frac{f(qt) - f(t)}{(q-1)t}, \quad t \neq 0$$

and

$$f^{\Delta}(0) = \lim_{n \rightarrow \infty} \frac{f(q^n) - f(0)}{q^n}.$$

By computing the recurrence relation, the q -polynomials are represented as

$$h_k(t, s) = \prod_{v=0}^{k-1} \frac{t - sq^v}{\sum_{j=0}^v q^j}$$

on $\overline{q^{\mathbb{N}}}$ [1].

A q -difference equation is an equation that contains q -derivatives of a function defined on $\overline{q^{\mathbb{N}}}$.

2.3 Multivariable calculus on time scales

The differentiation and integrations are introduced for functions of two variables on time scales [3]. Definitions on multivariate calculus on time scales can be found in Bohner and Guseinov [4, 5]. Following the line of ideas, the dynamic equations on time scales are extended to partial dynamic equations on time scales [2, 3, 14].

Let \mathbb{T}_1 and \mathbb{T}_2 be any two time scales. Consider the 'rectangle' $\mathbb{T} = \mathbb{T}_1 \times \mathbb{T}_2$. For any $\mathbf{t} \in \mathbb{T}$, the jump operator of $\mathbf{t} = (t_1, t_2)$ for $t_1 \in \mathbb{T}_1$ and $t_2 \in \mathbb{T}_2$ is given as follows:

1. The forward jump operators $\sigma: \mathbb{T} \rightarrow \mathbb{T}$ by $\sigma(\mathbf{t}) = (\sigma(t_1), \sigma(t_2))$ are defined as $\sigma(t_1) = \inf\{s \in \mathbb{T}_1 \mid s > t_1\}$ and $\sigma(t_2) = \inf\{s \in \mathbb{T}_2 \mid s > t_2\}$.
2. The backward jump operators $\tau: \mathbb{T} \rightarrow \mathbb{T}$ by $\tau(\mathbf{t}) = (\tau(t_1), \tau(t_2))$ are defined as $\tau(t_1) = \sup\{s \in \mathbb{T}_1 \mid s < t_1\}$ and $\tau(t_2) = \sup\{s \in \mathbb{T}_2 \mid s < t_2\}$.

To use the notations for partial derivatives with respect to time scale variables t_1 and t_2 , respectively, we employ lexicographic ordering for consistency. Let f^{Δ_1} denote the time scale partial derivative with respect to t_1 and let f^{Δ_2} denote the time scale partial derivative with respect to t_2 . Definitions of these partial derivatives are given as below [2, 3].

Definition 6 Let f be a real-valued function on \mathbb{T} . At $(t_1, t_2) \in \mathbb{T} = \mathbb{T}_1 \times \mathbb{T}_2$ we say f has a Δ_1 -partial derivative $f^{\Delta_1}(t_1, t_2)$ if for each $\varepsilon > 0$, there exists a neighborhood U of t_1 , with $U = (t_1 - \delta, t_1 + \delta) \cap \mathbb{T}_1$ for $\delta > 0$, such that

$$|f(\sigma(t_1), t_2) - f(s, t_2) - f^{\Delta_1}(t_1, t_2)(\sigma(t_1) - s)| \leq |\sigma(t_1) - s|$$

for all $s \in U$. On the other hand, we say f has a Δ_2 -partial derivative $f^{\Delta_2}(t_1, t_2)$ if for each $\varepsilon > 0$, there exists a neighborhood V of t_2 , with $V = (t_2 - \delta, t_2 + \delta) \cap \mathbb{T}_2$ for $\delta > 0$, such that

$$|f(t_1, \sigma(t_2)) - f(t_1, s) - f^{\Delta_2}(t_1, t_2)(\sigma(t_2) - s)| \leq |\sigma(t_2) - s|$$

for all $s \in V$.

Using the ideas of time scale partial derivatives, notations of mixed partial and high order partial derivatives are given as follows:

1. $f^{\Delta_{ij}}(\mathbf{t})$ (if this value exists) denotes first taking the partial derivative with respect to t_i and then taking the partial derivative with respect to t_j , so that $f^{\Delta_{ij}} = (f^{\Delta_i})^{\Delta_j}$, $i, j = 1, 2$.
2. $f^{\Delta_i^n}(\mathbf{t})$ (if this value exists) denotes taking the partial derivative of $f(\mathbf{t})$ with respect to t_i n times.

The details and examples can be found in [3] and [6].

3 The exact solution of linear initial value problems on time scales

Let $h_k(t_1, 0)$ and $g_k(t_2, 0)$ be the generalized polynomials on \mathbb{T}_1 and \mathbb{T}_2 , respectively. In this section, the variational iteration method on \mathbb{R} is extended to provide a method of finding the exact solution of linear partial dynamic equations on time scales. The introduction and the details of the variational iteration method can be found in the Appendix and in [10].

3.1 The exact solution of the first-order linear partial dynamic equations

We first consider the first-order linear partial dynamic equations as the form

$$\begin{cases} u^{\Delta_1} = c^2 u^{\Delta_2} & \text{on } \mathbb{T}_1 \times \mathbb{T}_2, \\ u(0, t_2) = f(t_2) & \text{on } \mathbb{T}_2, \end{cases} \quad (1)$$

where $f(t_2) = \sum_{i=0}^K a_i g_i(t_2, 0)$ on \mathbb{T}_2 and $a_i, i = 1, \dots, K$ are real numbers.

The basic character of the variational iteration method is to construct a correction functional for the system, which reads

$$u_{n+1}(t_1, t_2) = u_n(t_1, t_2) + \int_{t_0}^{t_1} \lambda \{Lu_n(s, t_2) + N\tilde{u}_n(s, t_2)\} \Delta s,$$

where L is a linear operator on \mathbb{T}_1 , N is a linear (or nonlinear) operator on \mathbb{T}_2 (or $\mathbb{T}_1 \times \mathbb{T}_2$), λ is a Lagrange multiplier which can be identified optimally by variational theory, u_n is the n th approximation, and \tilde{u}_n denotes a restricted variation, that is, $\delta \tilde{u}_n = 0$.

The linear operator L is selected as

$$Lu = u^{\Delta_1}$$

and the other operator N is selected as

$$Nu = -c^2 u^{\Delta_2}.$$

Make the above correction functional stationary with respect to u_n

$$\begin{aligned} \delta u_{n+1}(t_1, t_2) &= \delta u_n(t_1, t_2) + \delta \int_0^{t_1} \lambda \{u^{\Delta_1}(s, t_2) + N\tilde{u}_n(s, t_2)\} \Delta s \\ &= (1 + \lambda(t_1)) \delta u_n(t_1, t_2) + \int_0^{t_1} \lambda^{\Delta}(s) \delta u_n(\sigma(s), t_2) \Delta s. \end{aligned}$$

We, therefore, have the following stationary conditions:

$$\begin{aligned}1 + \lambda(t_1) &= 0, \\ \lambda^{\Delta}(s) &= 0.\end{aligned}$$

The Lagrange multiplier can be readily identified

$$\lambda(s) = -1.$$

As a result, the variational iteration formula is obtained

$$u_{n+1}(t_1, t_2) = u_n(t_1, t_2) - \int_0^t \{u_n^{\Delta_1}(s, t_2) + Nu_n(s, t_2)\} \Delta s. \quad (2)$$

Using the initial condition $u_0 = f(t_2)$ and the iteration formula (2), we have the following equations:

$$\begin{aligned}u_1(t_1, t_2) &= f(t_2) - \int_0^{t_1} f^{\Delta_2}(t_2) \Delta s \\ &= f(t_2) + c^2 f^{\Delta_2}(t_2) h_1(t_1, 0), \\ u_2(t_1, t_2) &= u_1(t_1, t_2) - \int_0^{t_1} c^2 f^{\Delta_2^2}(t_2) h_1(s, 0) \Delta s \\ &= u_1(t_1, t_2) + c^4 f^{\Delta_2^2}(t_2) h_2(t_1, 0) \\ &= f(t_2) + c^2 f^{\Delta_2}(t_2) h_1(t_1, 0) + c^4 f^{\Delta_2^2}(t_2) h_2(t_1, 0)\end{aligned}$$

and

$$u_k(t_1, t_2) = \sum_{j=0}^k c^{2j} f^{\Delta_2^j}(t_2) h_j(t_1, 0).$$

As k is large enough such that $f^{\Delta_2^k}$ equals to zero, the series solution u_k is the exact solution of (1).

Example 1 Consider the initial value problem

$$\begin{cases} u^{\Delta_1} = c^2 u^{\Delta_2} & \text{on } \mathbb{T}_1 \times \mathbb{T}_2, \\ u(0, t_2) = g_k(t_2, 0) & \text{on } \mathbb{T}_2, \end{cases} \quad (3)$$

where $g_k(t_2, 0)$ is a generalized polynomial on \mathbb{T}_2 . The function

$$u_k(t_1, t_2) = \sum_{j=0}^k c^{2j} g_{k-j}(t_2, 0) h_j(t_1, 0)$$

is the exact solution of (3).

Proof We now verify that the obtained function u_k actually solves the initial value problem (3). First, we show that the obtained function satisfies the initial condition. Since $h_0(t, s) \equiv 1$ for all t, s and $h_j(0, 0) \equiv 1$ for $j > 0$, we have

$$u_k(0, t_2) = g_k(t_2, 0).$$

Second, we display the obtained function u_k satisfying the equation by

$$\begin{aligned} u_k^{\Delta_1}(t_1, t_2) &= \sum_{j=1}^k c^{2j} g_{k-j}(t_2, 0) h_{j-1}(t_1, 0), \\ c^2 u_k^{\Delta_2}(t_1, t_2) &= \sum_{j=0}^{k-1} c^{2(j+1)} g_{k-j-1}(t_2, 0) h_j(t_1, 0) = \sum_{j=1}^k c^{2j} g_{k-j}(t_2, 0) h_{j-1}(t_1, 0). \end{aligned}$$

This implies that $u_k^{\Delta_1}(t_1, t_2) - c^2 u_k^{\Delta_2}(t_1, t_2) = 0$ on $\mathbb{T}_1 \times \mathbb{T}_2$. \square

3.2 The exact solution of the second-order linear partial dynamic equations

Consider the second-order partial dynamic equation as the form

$$\begin{cases} u^{\Delta_1} = c^2 u^{\Delta_2^2} & \text{on } \mathbb{T}_1 \times \mathbb{T}_2, \\ u(0, t_2) = f(t_2) & \text{on } \mathbb{T}_2, \end{cases} \quad (4)$$

where $f(t_2) = \sum_{i=0}^K a_i g_i(t_2, 0)$ on \mathbb{T}_2 and $a_i, i = 1, \dots, K$ are real numbers.

In this work, the linear operator L is selected as

$$Lu = u^{\Delta_1}$$

and the other operator N is selected as

$$Nu = -c^2 u^{\Delta_2^2}.$$

Using the initial condition $u_0(t_1, t_2) = f(t_2)$ and the iteration formula (2), we have the following equations:

$$\begin{aligned} u_1(t_1, t_2) &= f(t_2) - \int_0^{t_1} c^2 f^{\Delta_2^2}(t_2) \Delta s \\ &= f(t_2) + c^2 f^{\Delta_2^2}(t_2) h_1(t_1, 0), \\ u_2(t_1, t_2) &= u_1(t_1, t_2) - \int_0^{t_1} c^2 f^{\Delta_2^4}(t_2) h_1(s, 0) \Delta s \\ &= u_1(t_1, t_2) + f^{\Delta_2^4}(t_2) h_2(t_1, 0) \\ &= f(t_2) + c^2 f^{\Delta_2^2}(t_2) h_1(t_1, 0) + c^4 f^{\Delta_2^4}(t_2) h_2(t_1, 0) \end{aligned}$$

and

$$u_k(t_1, t_2) = \sum_{j=0}^k c^{2j} f^{\Delta_2^{(2j)}}(t_2) h_j(t_1, 0).$$

As k is large enough such that $f^{\Delta_2^{(2k)}}$ equals to zero, the series solution u_k is the exact solution of (4).

Example 2 Consider the IVP

$$\begin{cases} u^{\Delta_1} = c^2 u^{\Delta_2^2} & \text{on } \mathbb{T}_1 \times \mathbb{T}_2, \\ u(0, t_2) = g_k(t_2, 0) & \text{on } \mathbb{T}_2, \end{cases} \quad (5)$$

where $g_k(t_2, 0)$ is a generalized polynomial of \mathbb{T}_2 . The function

$$u_{\lfloor k/2 \rfloor}(t_1, t_2) = \sum_{j=0}^{\lfloor k/2 \rfloor} c^{2j} g_{k-2j}(t_2, 0) h_j(t_1, 0)$$

is the exact solution of (5).

The exact solution of Example 2 is also obtained by Jackson [6]. He transformed the IVP into an ODE and obtained the exact solution as

$$u(t_1, t_2) = \sum_{j=0}^{\lfloor k/2 \rfloor} c^{2j} g_{k-2j}(t_2, 0) h_j(t_1, 0),$$

where $\lfloor k/2 \rfloor$ denotes the floor of $k/2$.

When the initial condition can be represented as a finite series of generalized polynomials, we have proposed a useful method of finding the exact solution of partial dynamic equations on time scales. When the initial condition is represented as an infinite series of generalized polynomials, the approximate solution can be obtained by the same manner. In the following section, we consider the nonlinear partial dynamic equation on the specific time scales.

4 Approximation solutions of nonlinear q -partial dynamic equations

In this section, we extend the variational iteration method to find an approximate solution of nonlinear initial value problems on the time scale $\overline{q^{\mathbb{N}}}$.

To extend the variational iteration method, we first display a production rule [9] of two q -polynomials at 0 which will be used to derive an approximate solution in the following discussion.

Theorem 1 Let $h_i(t, 0)$ and $h_j(t, 0)$ be two q -polynomials at zero. We have

$$h_i(t, 0) h_j(t, 0) = \frac{(q^{i+1}; q)_j}{(q; q)_j} h_{i+j}(t, 0).$$

Proof Since

$$h_{i+j}(t, 0) = \prod_{v=0}^{i+j-1} \frac{t}{\sum_{\mu=0}^v q^{\mu}},$$

we have

$$\begin{aligned} h_{i+j}(t, 0) &= \left(\prod_{v=0}^{i-1} \frac{t}{\sum_{\mu=0}^v q^\mu} \right) \left(\prod_{v=i}^{i+j-1} \frac{t}{\sum_{\mu=0}^v q^\mu} \right) \\ &= h_i(t, 0) \left(\frac{\prod_{v=0}^{j-1} \sum_{\mu=0}^v q^\mu}{\prod_{v=0}^{i-1} \sum_{\mu=0}^v q^\mu} \right) t^j \left(\prod_{v=i}^{i+j-1} \frac{1}{\sum_{\mu=0}^v q^\mu} \right) \\ &= h_i(t, 0) \left(\prod_{v=0}^{j-1} \frac{t}{\sum_{\mu=0}^v q^\mu} \right) \left(\prod_{v=0}^{j-1} \sum_{\mu=0}^v q^\mu \right) \left(\prod_{v=i}^{i+j-1} \frac{1}{\sum_{\mu=0}^v q^\mu} \right) \\ &= h_i(t, 0) h_j(t, 0) \left(\prod_{v=0}^{j-1} \frac{\sum_{\mu=0}^v q^\mu}{\sum_{\mu=0}^{v+i} q^\mu} \right). \end{aligned}$$

This implies that

$$\begin{aligned} h_i(t, 0) h_j(t, 0) &= \left(\prod_{v=0}^{j-1} \frac{\sum_{\mu=0}^{v+i} q^\mu}{\sum_{\mu=0}^v q^\mu} \right) h_{i+j}(t, 0) = \prod_{v=0}^{j-1} \frac{(1 - q^{v+i+1})}{(1 - q^{v+1})} h_{i+j}(t) \\ &= \frac{(q^{i+1}; q)_j}{(q; q)_j} h_{i+j}(t, 0). \end{aligned}$$

□

Proposition 1 Let $h_i(t, 0)$ and $h_j(t, 0)$ be any two q -polynomials. We have

$$h_i(t, 0) h_j(t, 0) = h_j(t, 0) h_i(t, 0).$$

Proof It suffices to show that

$$\frac{(q^{i+1}; q)_j}{(q, q)_j} = \frac{(q^{j+1}; q)_i}{(q, q)_i}.$$

Suppose $i > j$, we have

$$\begin{aligned} &\frac{(q^{i+1}; q)_j}{(q, q)_j} - \frac{(q^{j+1}; q)_i}{(q, q)_i} \\ &= \frac{(1 - q^{i+1}) \cdots (1 - q^{i+j})}{(1 - q) \cdots (1 - q^i)} - \frac{(1 - q^{i+1}) \cdots (1 - q^{i+j})}{(1 - q) \cdots (1 - q^j)} \\ &= \frac{(1 - q^{i+1}) \cdots (1 - q^{i+j})}{(1 - q) \cdots (1 - q^i)} - \frac{(1 - q^{i+1}) \cdots (1 - q^{i+j})(1 - q^{j+1}) \cdots (1 - q^i)}{(1 - q) \cdots (1 - q^j)(1 - q^{j+1}) \cdots (1 - q^i)} = 0. \end{aligned}$$

□

Let h_k and g_k be generalized polynomials of $\overline{q_1^{\mathbb{N}}}$ and $\overline{q_2^{\mathbb{N}}}$. The variational iteration method is now applied to find an approximate solution of the nonlinear partial dynamic equations as the form

$$\begin{cases} u^{\Delta_1} = Nu & \text{on } \overline{q_1^{\mathbb{N}}} \times \overline{q_2^{\mathbb{N}}}, \\ u(0, t_2) = g_k(t_2, 0) & \text{on } \overline{q_2^{\mathbb{N}}}. \end{cases}$$

When the linear operator L is selected as

$$Lu = u^{\Delta_1}$$

and the other operator N is selected as $-Nu$, the variational iteration formula is obtained as

$$u_{n+1}(t_1, t_2) = u_n(t_1, t_2) - \int_0^{t_1} \{u_n^{\Delta_1}(s, t_2) - Nu_n(s, t_2)\} \Delta s$$

with the initial approximation

$$u_0 = g_k(t_2, 0).$$

Example 3 Consider the partial dynamic equations as the form

$$\begin{cases} u^{\Delta_1} = uu^{\Delta_2} & \text{on } \mathbb{T}_1 \times \mathbb{T}_2, \\ u(0, t_2) = g_k(t_2, 0) & \text{on } \mathbb{T}_2. \end{cases}$$

With the variational iteration formula, we obtain the first few components of $u_n(t_1, t_2)$:

$$\begin{aligned} u_1(t_1, t_2) &= u_0(t_1, t_2) - \int_0^{t_1} \{u_0^{\Delta_1}(s, t_2) - u_0(s, t_2)u_0^{\Delta_2}(s, t_2)\} \Delta s \\ &= g_k(t_2, 0) + \int_0^{t_1} [g_k(t_2, 0)g_{k-1}(t_2, 0)] \Delta s \\ &= g_k(t_2, 0) + H_2(k, k-1)g_{2k-1}(t_2, 0)h_1(t_1, 0), \\ u_2(t_1, t_2) &= u_1(t_1, t_2) - \int_0^{t_1} \{u_1^{\Delta_1}(s, t_2) - u_1(s, t_2)u_1^{\Delta_2}(s, t_2)\} \Delta s \\ &= g_k(t_2, 0) + H_2(k, k-1)g_{2k-1}(t_2, 0)h_1(t_1, 0) \\ &\quad - \int_0^{t_1} [H_2(k, k-1)g_{2k-1}(t_2, 0) - (g_k(t_2, 0) + H_2(k, k-1)g_{2k-1}(t_2, 0)h_1(s, 0)) \\ &\quad \times (g_{k-1}(t_2, 0) + H_2(k, k-1)g_{2k-2}(t_2, 0)h_1(s, 0))] \Delta s \\ &= g_k(t_2, 0) + H_2(k, k-1)g_{2k-1}(t_2, 0)h_1(t_1, 0) \\ &\quad + H_2(k, k-1)H_2(k, 2k-2)g_{3k-2}(t_2, 0)h_2(t_1, 0) \\ &\quad + H_2(k, k-1)H_2(k-1, 2k-1)g_{3k-2}(t_2, 0)h_2(t_1, 0) \\ &\quad + H_2(k, k-1)H_2(k, k-1)H_2(2k-1, 2k-2)g_{4k-3}(t_2, 0)H_1(1, 1)h_3(t_1, 0), \end{aligned}$$

where $H_1(k, l) = \frac{(q_1^{k+1}; q_1)_l}{(q_1; q_1)_l}$ and $H_2(k, l) = \frac{(q_2^{k+1}; q_2)_l}{(q_2; q_2)_l}$.

In the same manner, the rest of components of the iteration formula are obtained iteratively.

4.1 Applications to the q -Burger equation and the Fisher equation

q -Burger equation

First of all, we consider the q -Burger equation as the form

$$\begin{cases} u^{\Delta_1} - uu^{\Delta_2} - \alpha u^{\Delta_2} = 0 & \text{on } \overline{q_1^{\mathbb{N}}} \times \overline{q_2^{\mathbb{N}}}, \\ u(0, t_2) = g_k(t_2, 0) & \text{on } \overline{q_2^{\mathbb{N}}}. \end{cases}$$

When the linear operator and the nonlinear operator are selected as $Lu = u^{\Delta_1}$ and $-Nu = -uu^{\Delta_2} - \alpha u^{\Delta_2^2}$, respectively, the variational iteration formula is obtained as

$$u_{n+1}(t_1, t_2) = u_n(t_1, t_2) - \int_0^{t_1} \{u_n^{\Delta_1}(s, t_2) - u_n(s, t_2)u_n^{\Delta_2}(s, t_2) - \alpha u_n^{\Delta_2^2}(s, t_2)\} \Delta s. \quad (6)$$

Let $G(t_2) = H_2(k, k-1)g_{2k-1}(t_2, 0) + \alpha g_{k-2}(t_2, 0)$ and $H_2(k, l) = \frac{(q_2^{k+1}; q_2)_l}{(q_2; q_2)_l}$. With the initial condition $u_0(t_1, t_2) \equiv u(0, t_2) = g_k(t_2, 0)$, we have

$$\begin{aligned} u_1(t_1, t_2) &= u_0(t_1, t_2) - \int_0^{t_1} \{u_0^{\Delta_1}(s, t_2) - u_0(s, t_2)u_0^{\Delta_2}(s, t_2) - \alpha u_0^{\Delta_2^2}(s, t_2)\} \Delta s \\ &= g_k(t_2, 0) - \int_0^{t_1} \{-g_k(t_2, 0)g_{k-1}(t_2, 0) - \alpha g_{k-2}(t_2, 0)\} \Delta s \\ &= g_k(t_2, 0) + (H_2(k, k-1)g_{2k-1}(t_2, 0) + \alpha g_{k-2}(t_2, 0))h_1(t_1, 0) \\ &= g_k(t_2, 0) + G(t_2)h_1(t_1, 0), \\ u_2(t_1, t_2) &= u_1(t_1, t_2) - \int_0^{t_1} \{u_1^{\Delta_1}(s, t_2) - u_1(s, t_2)u_1^{\Delta_2}(s, t_2) - \alpha u_1^{\Delta_2^2}(s, t_2)\} \Delta s \\ &= g_k(t_2, 0) + G(t_2)h_1(t_1, 0) \\ &\quad - \int_0^{t_1} \{G(t_2) - (g_k(t_2, 0) + G(t_2)h_1(s, 0)) \\ &\quad \times (g_{k-1}(t_2, 0) + g^{\Delta_2}(t_2)h_1(s, 0)) \\ &\quad - \alpha (g_{k-2}(t_2, 0) + g^{\Delta_2^2}(t_2)h_1(s, 0))\} \Delta s \\ &= g_k(t_2, 0) + H_2(k, k-1)g_{2k-1}(t_2, 0)h_1(t_1, 0) \\ &\quad + G(t_2)g_{k-1}(t_2, 0)h_2(t_1, 0) + g^{\Delta_2}(t_2)g_k(t_2, 0)h_2(t_1, 0) \\ &\quad + G(t_2)g^{\Delta_2}(t_2)H_1(1, 1)h_3(t_1, 0) \\ &\quad + \alpha g_{k-2}(t_2, 0)h_1(t_1, 0) + \alpha g^{\Delta_2^2}(t_2)h_2(t_1, 0) \\ &= g_k(t_2, 0) + [H_2(k, k-1)g_{2k-1}(t_2, 0) + \alpha g_{k-2}(t_2, 0)]h_1(t_1, 0) \\ &\quad + [G(t_2)g_{k-1}(t_2, 0) + g^{\Delta_2}(t_2)g_k(t_2, 0) + \alpha g^{\Delta_2^2}(t_2)]h_2(t_1, 0) \\ &\quad + G(t_2)g^{\Delta_2}(t_2)H_1(1, 1)h_3(t_1, 0). \end{aligned}$$

In the same manner, the rest of components of the iteration formula are obtained iteratively.

q-Fisher equation

Secondly, we consider the q -Fisher equation, which is a nonlinear reaction diffusion equation, as the form

$$\begin{cases} u^{\Delta_1} - \alpha u^{\Delta_2^2} - \beta u(1-u) = 0 & \text{on } \overline{q_1^{\mathbb{N}}} \times \overline{q_2^{\mathbb{N}}}, \\ u(0, t_2) = g_k(t_2, 0) & \text{on } \overline{q_2^{\mathbb{N}}}. \end{cases}$$

The variational iteration formula is obtained as

$$u_{n+1}(t_1, t_2) = u_n(t_1, t_2) - \int_0^{t_1} \left\{ u_n^{\Delta_1}(s, t_2) - \alpha u_n^{\Delta_2^2}(s, t_2) - \beta u_n(s, t_2) + \beta u_n(s, t_2) u_n(s, t_2) \right\} \Delta s. \quad (7)$$

Let $G(t_2) = \alpha h_{k-2}(t_2) + \beta g_k(t_2, 0) - \beta H_2(k, k) g_{2k}(t_2, 0)$ and $H_2(k, l) = \frac{(q_2^{k+1}, q_2)_l}{(q_2, q_2)_l}$. With the initial condition $u_0(t_1, t_2) \equiv u(0, t_2) = g_k(t_2, 0)$, we have

$$\begin{aligned} u_1(t_1, t_2) &= u_0(t_1, t_2) - \int_0^{t_1} \left\{ u_0^{\Delta_1}(s, t_2) - \alpha u_0^{\Delta_2^2}(s, t_2) - \beta u_0(s, t_2) + \beta u_0(s, t_2) u_0(s, t_2) \right\} \Delta s \\ &= g_k(t_2, 0) + \int_0^{t_1} \left\{ \alpha g_{k-2}(t_2, 0) + \beta g_k(t_2, 0) - \beta H_2(k, k) g_{2k}(t_2, 0) \right\} \Delta s \\ &= g_k(t_2, 0) + (\alpha h_{k-2}(t_2) + \beta g_k(t_2, 0) - \beta H_2(k, k) g_{2k}(t_2, 0)) h_1(t_1, 0) \\ &= g_k(t_2, 0) + G(t_2) h_1(t_1, 0), \\ u_2(t_1, t_2) &= u_1(t_1, t_2) - \int_0^{t_1} \left\{ u_1^{\Delta_1}(s, t_2) - \alpha u_1^{\Delta_2^2}(s, t_2) - \beta u_1(s, t_2) + \beta u_1(s, t_2) u_1(s, t_2) \right\} \Delta s \\ &= g_k(t_2, 0) + G(t_2) h_1(t_1, 0) \\ &\quad - \int_0^{t_1} \left\{ G(t_2) - \alpha (g_{k-2}(t_2, 0) + g^{\Delta_2^2}(t_2) h_1(s, 0)) - \beta (g_k(t_2, 0) + G(t_2) h_1(s, 0)) \right. \\ &\quad \left. + \beta (g_k(t_2, 0) + G(t_2) h_1(s, 0)) (g_k(t_2, 0) + G(t_2) h_1(s, 0)) \right\} \Delta s \\ &= g_k(t_2, 0) + \alpha g_{k-2}(t_2, 0) h_1(t_1, 0) + \alpha g^{\Delta_2^2}(t_2) h_2(t_1, 0) \\ &\quad + \beta g_k(t_2, 0) h_1(t_1, 0) + \beta G(t_2) h_2(t_1, 0) \\ &\quad - \int_0^{t_1} \left\{ \beta (H_2(k, k) g_{2k}(t_2, 0) + 2G(t_2) g_k(t_2, 0) h_1(s, 0) \right. \\ &\quad \left. + g^2(t_2) H_1(1, 1) h_2(s, 0)) \right\} \Delta s \\ &= g_k(t_2, 0) + [\alpha g_{k-2}(t_2, 0) + \beta g_k(t_2, 0) - \beta H_2(k, k) g_{2k}(t_2, 0)] h_1(t_1, 0) \\ &\quad + (\alpha g^{\Delta_2^2}(t_2) + \beta G(t_2) - 2\beta G(t_2) g_k(t_2, 0)) h_2(t_1, 0) \\ &\quad - \beta g^2(t_2) H_1(1, 1) h_3(t_1, 0). \end{aligned}$$

In the same manner, the rest of components of the iteration formula are obtained iteratively.

5 Numerical results

The approximate solutions introduced in the previous sections will be illustrated with some examples.

Let $\mathbb{T}_1 = \mathbb{T}_2 = \overline{0.9^{\mathbb{N}}} = \{0.9, 0.81, 0.729, \dots, 0\}$, where 0 is the cluster point of $\overline{q^{\mathbb{N}}}$. The q -shift factorial with $q = 0.9$ is given as

$$(a; 0.9)_0 = 1 \quad \text{and} \quad (a; 0.9)_n = \prod_{k=0}^{n-1} (1 - 0.9^k a), \quad n \in \mathbb{N}$$

and the q -polynomials are represented as

$$h_k(t, 0) = \prod_{v=0}^{k-1} \frac{t}{\sum_{j=0}^v 0.9^j} = \frac{t^k}{\prod_{v=0}^{k-1} \sum_{j=0}^v 0.9^j} = \frac{t^k}{\prod_{v=1}^{k-1} 9(1 - 0.9^v)},$$

$t \in \{0.9^n, n \in \mathbb{N}\} \cup \{0\}$ and $h_k(0, 0) = 1$. The multiplication of two generalized polynomials $h_k(t, 0)$ and $h_l(t, 0)$ is obtained as

$$h_k(t, 0)h_l(t, 0) = H(k, l)h_{k+l}(t, 0),$$

where $H(k, l) = \frac{(0.9^{k+1}; 0.9)_l}{(0.9; 0.9)_l}$. For example, $H(1, 1) = \frac{(0.9^2; 0.9)_1}{(0.9; 0.9)_1} = \frac{(1-0.9^2)}{(1-0.9)} = \frac{1.9}{0.1} = 1.9$.

Example 4 Consider the nonlinear partial dynamic equation as the form

$$\begin{cases} u^{\Delta_1} - uu^{\Delta_2} - u^{\Delta_2^2} = 0 & \text{on } \overline{0.9^{\mathbb{N}}} \times \overline{0.9^{\mathbb{N}}}, \\ u(0, t_2) = g_1(t_2, 0) & \text{on } \overline{0.9^{\mathbb{N}}}. \end{cases} \quad (8)$$

The initial approximation can be given as

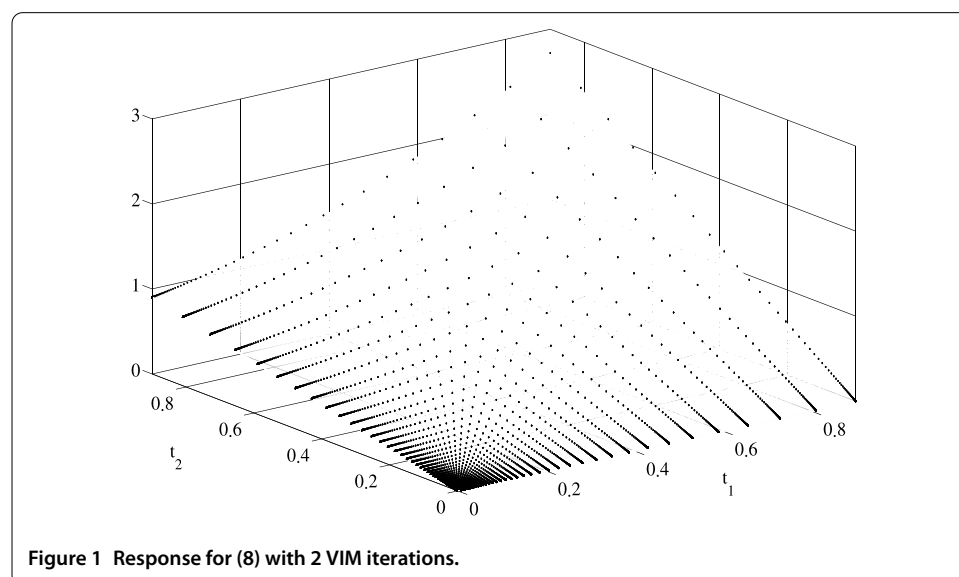
$$u_0(t_1, t_2) = g_1(t_2, 0)$$

according to the initial condition. By the variational iteration formula (2), the first two components of $u_n(t_1, t_2)$ are obtained:

$$u_1(t_1, t_2) = (1 + h_1(t_1, 0))g_1(t_2, 0),$$

$$u_2(t_1, t_2) = (1 + h_1(t_1, 0) + 2h_2(t_1, 0) + 1.9h_3(t_1, 0))g_1(t_2, 0).$$

The rest of components of the iteration formula are obtained in the same manner. The responses of $u(t_1, t_2)$ are shown in Figure 1.



Example 5 Consider the Fisher equation as the form

$$\begin{cases} u^{\Delta_1} - u^{\Delta_2} - u(1-u) = 0 & \text{on } \overline{0.9^{\mathbb{N}}} \times \overline{0.9^{\mathbb{N}}}, \\ u(0, t_2) = g_1(t_2, 0) & \text{on } \overline{0.9^{\mathbb{N}}}. \end{cases} \quad (9)$$

With the initial condition $u_0(t_1, t_2) = g_1(t_2, 0)$, the first two components are obtained

$$\begin{aligned} u_1(t_1, t_2) &= g_1(t_2, 0) + (g_1(t_2, 0) - 1.9g_2(t_2, 0))_1(t_2, 0) = g_1(t_2, 0) + G(t_2)g_1(t_2, 0), \\ u_2(t_1, t_2) &= g_1(t_2, 0) + G(t_2)h_1(t_1, 0) + [g^{\Delta_2}(t_2) + G(t_2) - 2G(t_2)g_1(t_2, 0)]h_2(t_1, 0) \\ &\quad - g^2(t_2)H_1(1, 1)h_3(t_1, 0) \\ &= g_1(t_2, 0) + [g_1(t_2, 0) - 1.9g_2(t_2, 0)]h_1(t_1, 0) \\ &\quad + [-1.9 + g_1(t_2, 0) - 1.9g_2(t_2, 0) - 2(g_1(t_2, 0) - 1.9g_2(t_2, 0))g_1(t_2, 0)]h_2(t_1, 0) \\ &\quad - 1.9(g_1(t_2, 0) - 1.9g_2(t_2, 0))^2h_3(t_1, 0) \\ &= g_1(t_2, 0) + [g_1(t_2, 0) - 1.9g_2(t_2, 0)]h_1(t_1, 0) \\ &\quad + [-1.9 + g_1(t_2, 0) - 5.7g_2(t_2, 0) + 10.298g_3(t_2, 0)]h_2(t_1, 0) \\ &\quad - 1.9[1.9g_2(t_2, 0) - 10.298g_3(t_2, 0) + 4.9051g_4(t_2, 0)]h_3(t_1, 0), \end{aligned}$$

where $G(t_2) = g_1(t_2, 0) - 1.9g_2(t_2, 0)$. The rest of components of the iteration formula are obtained in the same manner. The responses of $u(t_1, t_2)$ are shown in Figure 2.

Now, we have demonstrated a method for finding an approximate solution of nonlinear partial dynamic equations on $\overline{q_1^{\mathbb{N}}} \times \overline{q_2^{\mathbb{N}}}$. The proposed tool could also be applied to other nonlinear q -partial dynamic equations.

In future studies, we intend to derive the multiplication rule of two generalized polynomials and extend the application of the variational iteration method to nonlinear partial dynamic equations on other time scales.

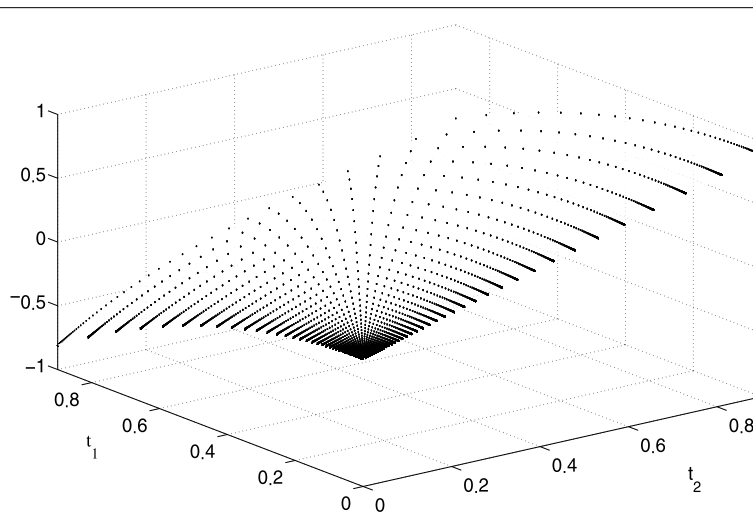


Figure 2 Response for (9) with 2 VIM iterations.

6 Conclusion and future direction

In this paper, we have propose a method to find the exact solution of the linear partial dynamic equation on time scales and to find an approximate solution of the nonlinear q -partial dynamic equations. Moreover, this method is applied to provide an approximate solution of the q -Berger equations and the q -Fisher equations.

To extend the method to other time scales, it is important to derive a multiplication rule of two generalized polynomials on the other time scales. On the other hand, approximate solutions as well as their properties of the nonlinear partial dynamic equations, such as Benjamin-Ono equations and the Benjamin-Bona-Mahony equations, are not found on $\overline{q^{\mathbb{N}}}$ yet. In the future studies, we would intend to derive the multiplication rule of two generalized polynomials or to provide an approximation of other nonlinear q -partial dynamic equations by using the proposing method.

Appendix: Basic ideas of the variational iteration method

To clarify the ideas of the variational iteration method, we consider the following nonlinear equation:

$$Lu(t) + Nu(t) = g(t),$$

where L is a linear operator, N is a nonlinear operator and g is an inhomogeneous term. According to the variational iteration method, we can construct a correction functional as follows:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda \{Lu_n(s) + N\tilde{u}_n(s) - g(s)\} ds,$$

where λ is a general Lagrange multiplier, u_0 is an initial approximation which must be chosen suitably and \tilde{u}_n is considered as a restricted variation, that is, $\delta\tilde{u}_n = 0$. To find the optimal value of λ , we make the above correction functional stationary with respect to u_n , noticing that $\delta u_n(0) = 0$, and have

$$\delta u_{n+1}(t) = \delta u_n(t) + \delta \int_0^t \lambda Lu_n(s) ds = 0.$$

Having obtained the optimal Lagrange multiplier, the successive approximations u_n , $n \geq 0$, of the solution u are determined upon the initial function u_0 . Therefore, the exact solution is obtained at the limit of the resulting successive approximations.

Competing interests

The author declares that they have no competing interests.

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