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Solvability of Neumann boundary value problem for fractional p-Laplacian equation

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Abstract

We consider the existence of solutions for a Neumann boundary value problem for the fractional p-Laplacian equation. Under certain nonlinear growth conditions of the nonlinearity, we obtain a new result on the existence of solutions by using the continuation theorem of coincidence degree theory.

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Keywords: Neumann boundary value problem; fractional differential equation; p-Laplacian operator; continuation theorem

1 Introduction

The purpose of this paper is to establish the existence of solutions for the following Neumann boundary value problem (NBVP for short) for a fractional p-Laplacian equation:

$$\begin{cases} D_{0+}^{\beta}\phi_p(D_{0+}^{\alpha}x(t)) = g(t,x(t)), & t \in [0,T], \\ D_{0+}^{\alpha}x(0) = D_{0+}^{\alpha}x(T) = 0, \end{cases} \tag{1.1}$$

where $0 < \alpha, \beta \le 1$, D_{0+}^{α} is a Caputo fractional derivative, $\phi_p(s) = |s|^{p-2}s$ $(p > 1)$, $T > 0$ is a given constant and $g : [0,T] \times \mathbb{R} \to \mathbb{R}$ is continuous. Obviously, ϕ_p is invertible and its inverse operator is ϕ_q, where $q > 1$ is a constant such that $1/p + 1/q = 1$.

The fractional calculus is a generalization of the ordinary differentiation and integration on an arbitrary order that can be noninteger. Fractional differential equations appear in a number of fields such as physics, polymer rheology, regular variation in thermodynamics, biophysics, blood flow phenomena, aerodynamics, electro-dynamics of a complex medium, viscoelasticity, Bode analysis of feedback amplifiers, capacitor theory, electrical circuits, electro-analytical chemistry, biology, control theory, fitting of experimental data, *etc.* (see [1–4]). In recent years, because of the intensive development of the fractional calculus theory itself and its applications, fractional differential equations have been of great interest. For example, Agarwal *et al.* (see [5]) considered a two-point boundary value problem at nonresonance, and Bai (see [6]) considered a m-point boundary value problem at resonance. For more papers on fractional boundary value problems, see [7–15] and the references therein.

In [7], by using the coincidence degree theory for Fredholm operators, the authors studied the existence of solutions for the following NBVP:

$$\begin{cases} D_{0+}^{\beta}\phi_p(D_{0+}^{\alpha}x(t)) = f(t,x(t),D_{0+}^{\alpha}x(t)), & t \in [0,1], \\ D_{0+}^{\alpha}x(0) = D_{0+}^{\alpha}x(1) = 0. \end{cases}$$

Notice that $D_{0^+}^{\beta}\phi_p(D_{0^+}^{\alpha})$ is a nonlinear operator, so it is not a Fredholm operator. Hence, there is a bug in the proof of the main result.

2 Preliminaries

In this section, for convenience of the reader, we will present here some necessary basic knowledge and definitions as regards the fractional calculus theory, which can be found, for instance, in [2, 4].

Definition 2.1 The Riemann-Liouville fractional integral operator of order $\alpha > 0$ of a function $u : (0, +\infty) \to \mathbb{R}$ is given by

$$I_{0^+}^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)\,ds,$$

provided that the right-side integral is pointwise defined on $(0, +\infty)$.

Definition 2.2 The Caputo fractional derivative of order $\alpha > 0$ of a continuous function $u : (0, +\infty) \to \mathbb{R}$ is given by

$$\begin{aligned} D_{0^+}^{\alpha}u(t) &= I_{0^+}^{n-\alpha}\frac{d^n u(t)}{dt^n} \\ &= \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}u^{(n)}(s)\,ds, \end{aligned}$$

where n is the smallest integer greater than or equal to α, provided that the right-side integral is pointwise defined on $(0, +\infty)$.

Lemma 2.1 (see [1]) *Let $\alpha > 0$. Assume that $u, D_{0^+}^{\alpha}u \in L([0,T],\mathbb{R})$. Then the following equality holds*:

$$I_{0^+}^{\alpha}D_{0^+}^{\alpha}u(t) = u(t) + c_0 + c_1 t + \cdots + c_{n-1}t^{n-1},$$

where $c_i \in \mathbb{R}$, $i = 0, 1, \ldots, n-1$, here n is the smallest integer greater than or equal to α.

Lemma 2.2 (see [16]) *For any $u, v \geq 0$, then*

$$\begin{aligned} &\phi_p(u+v) \leq \phi_p(u) + \phi_p(v), \quad \textit{if } p < 2; \\ &\phi_p(u+v) \leq 2^{p-2}\bigl(\phi_p(u) + \phi_p(v)\bigr), \quad \textit{if } p \geq 2. \end{aligned}$$

Now we briefly recall some notations and an abstract existence result, which can be found in [17].

Let X, Y be real Banach spaces, $L : \operatorname{dom} L \subset X \to Y$ be a Fredholm operator with index zero, and $P : X \to X$, $Q : Y \to Y$ be projectors such that

$$\begin{aligned} &\operatorname{Im} P = \operatorname{Ker} L, \qquad \operatorname{Ker} Q = \operatorname{Im} L, \\ &X = \operatorname{Ker} L \oplus \operatorname{Ker} P, \qquad Y = \operatorname{Im} L \oplus \operatorname{Im} Q. \end{aligned}$$

It follows that

$$L|_{\operatorname{dom} L\cap \operatorname{Ker} P} : \operatorname{dom} L \cap \operatorname{Ker} P \to \operatorname{Im} L$$

is invertible. We denote the inverse by K_P.

If Ω is an open bounded subset of X such that $\operatorname{dom} L \cap \bar{\Omega} \neq \varnothing$, then the map $N : X \to Y$ will be called L-compact on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $K_P(I - Q)N : \bar{\Omega} \to X$ is compact.

Lemma 2.3 (see [17]) *Let X and Y be two Banach spaces, $L : \operatorname{dom} L \subset X \to Y$ be a Fredholm operator with index zero, $\Omega \subset X$ be an open bounded set, and $N : \bar{\Omega} \to Y$ be L-compact on $\bar{\Omega}$. Suppose that all of the following conditions hold*:

(1) $Lx \neq \lambda Nx$, $\forall x \in \partial\Omega \cap \operatorname{dom} L$, $\lambda \in (0,1)$;

(2) $QNx \neq 0$, $\forall x \in \partial\Omega \cap \operatorname{Ker} L$;

(3) $\deg(JQN, \Omega \cap \operatorname{Ker} L, 0) \neq 0$, *where $J : \operatorname{Im} Q \to \operatorname{Ker} L$ is an isomorphism map.*

Then the equation $Lx = Nx$ has at least one solution on $\bar{\Omega} \cap \operatorname{dom} L$.

3 Main result

In this section, we will give the main result on the existence of solutions for NBVP (1.1).

Theorem 3.1 *Let $g : [0, T] \times \mathbb{R} \to \mathbb{R}$ be continuous. Assume that*

(C_1) *there exists a constant $d > 0$ such that*

$$(-1)^i ug(t,u) > 0 \quad (i = 1,2), \forall t \in [0,T], |u| > d;$$

(C_2) *there exist nonnegative functions $a, b \in C[0,T]$ such that*

$$\bigl|g(t,u)\bigr| \le a(t)|u|^{p-1} + b(t), \quad \forall t \in [0,T], u \in \mathbb{R}.$$

Then NBVP (1.1) *has at least one solution, provided that*

$$\begin{aligned} \gamma_1 &:= \frac{2^{p-1} T^{\beta+\alpha p-\alpha} \|a\|_0}{\Gamma(\beta+1)(\Gamma(\alpha+1))^{p-1}} < 1, \quad \textit{if } p < 2; \\ \gamma_2 &:= \frac{2^{2p-3} T^{\beta+\alpha p-\alpha} \|a\|_0}{\Gamma(\beta+1)(\Gamma(\alpha+1))^{p-1}} < 1, \quad \textit{if } p \ge 2. \end{aligned} \tag{3.1}$$

For making use of the continuation theorem to study the existence of solutions for NBVP (1.1), we consider the following system:

$$\begin{cases} D_{0^+}^{\alpha} x_1(t) = \phi_q(x_2(t)), \\ D_{0^+}^{\beta} x_2(t) = g(t, x_1(t)), \\ x_2(0) = x_2(T) = 0. \end{cases} \tag{3.2}$$

Clearly, if $x(\cdot) = (x_1(\cdot), x_2(\cdot))^{\mathrm{T}}$ is a solution of NBVP (3.2), then $x_1(\cdot)$ must be a solution of NBVP (1.1). Hence, to prove that NBVP (1.1) has solutions, it suffices to show that NBVP (3.2) has solutions.

In this paper, we take $X = \{x = (x_1, x_2)^{\mathrm{T}} | x_1, x_2 \in C[0,T]\}$ with the norm $\|x\| = \max\{\|x_1\|_0, \|x_2\|_0\}$, where $\|x_i\|_0 = \max_{t\in[0,T]} |x_i(t)|$ $(i = 1,2)$. By means of the linear functional analysis theory, we can prove X is a Banach space.

Define the operator $L : \operatorname{dom} L \subset X \to X$ by

$$Lx = \begin{pmatrix} D_{0^+}^{\alpha} x_1 \\ D_{0^+}^{\beta} x_2 \end{pmatrix}, \tag{3.3}$$

where

$$\operatorname{dom} L = \bigl\{x \in X | D_{0^+}^{\alpha} x_1, D_{0^+}^{\beta} x_2 \in C[0,T], x_2(0) = x_2(T) = 0\bigr\}.$$

Let $N : X \to X$ be the Nemytskii operator

$$Nx(t) = \begin{pmatrix} \phi_q(x_2(t)) \\ g(t, x_1(t)) \end{pmatrix}, \quad \forall t \in [0,T]. \tag{3.4}$$

Then NBVP (3.2) is equivalent to the operator equation as follows:

$$Lx = Nx, \quad x \in \operatorname{dom} L.$$

Next we will give some lemmas which are useful in the proof of Theorem 3.1.

Lemma 3.1 *Let L be defined by* (3.3), *then*

$$\operatorname{Ker} L = \bigl\{x \in X | x(t) = c \in \mathbb{R}^2, \forall t \in [0,T]\bigr\}, \tag{3.5}$$

$$\operatorname{Im} L = \left\{ y \in X \,\middle|\, y_1(0) = 0, \int_0^T (T-s)^{\beta-1} y_2(s)\, ds = 0 \right\}. \tag{3.6}$$

Proof Obviously, from Lemma 2.1, (3.5) holds.

If $y \in \operatorname{Im} L$, then there exists $x \in \operatorname{dom} L$ such that $y = Lx$. That is, $y_1(t) = D_{0^+}^{\alpha} x_1(t)$, $y_2(t) = D_{0^+}^{\beta} x_2(t)$. By Lemma 2.1, we have

$$x_2(t) = c + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} y_2(s)\, ds, \quad c \in \mathbb{R}.$$

From the boundary value conditions $D_{0^+}^{\alpha} x_1(0) = x_2(0) = x_2(T) = 0$, we obtain

$$y_1(0) = 0, \qquad \int_0^T (T-s)^{\beta-1} y_2(s)\, ds = 0. \tag{3.7}$$

So we get (3.6).

On the other hand, suppose $y \in X$ which satisfies (3.7). Let $x_1(t) = I_{0^+}^{\alpha} y_1(t)$, $x_2(t) = I_{0^+}^{\beta} y_2(t)$. Clearly $x_2(0) = x_2(T) = 0$. Hence $x = (x_1, x_2)^{\mathrm{T}} \in \operatorname{dom} L$ and $Lx = y$. Thus $y \in \operatorname{Im} L$. The proof is completed. □

Lemma 3.2 *Let L be defined by* (3.3), *then L is a Fredholm operator of index zero. The linear projectors $P : X \to X$ and $Q : X \to X$ can be defined as*

$$Px(t) = x(0), \quad \forall t \in [0,T],$$

$$Qy(t) = \begin{pmatrix} y_1(0) \\ \frac{\beta}{T^\beta} \int_0^T (T-s)^{\beta-1} y_2(s)\, ds \end{pmatrix} := \begin{pmatrix} (Qy)_1(t) \\ (Qy)_2(t) \end{pmatrix}, \quad \forall t \in [0,T].$$

Furthermore, the operator $K_P : \operatorname{Im} L \to \operatorname{dom} L \cap \operatorname{Ker} P$ *can be written by*

$$K_P y = \begin{pmatrix} I_{0^+}^{\alpha} y_1 \\ I_{0^+}^{\beta} y_2 \end{pmatrix}.$$

Proof For any $y \in X$, we have

$$\begin{aligned} Q^2 y(t) &= Q \begin{pmatrix} y_1(0) \\ (Qy)_2(t) \end{pmatrix} \\ &= \begin{pmatrix} y_1(0) \\ (Qy)_2(t) \cdot \frac{\beta}{T^\beta} \int_0^T (T-s)^{\beta-1}\,ds \end{pmatrix} = Qy(t). \end{aligned}$$

Let $y^* = y - Qy$, then we get $y_1^*(0) = 0$ and

$$\begin{aligned} &\int_0^T (T-s)^{\beta-1} y_2^*(s)\,ds \\ &\quad = \int_0^T (T-s)^{\beta-1} y_2(s)\,ds - \int_0^T (T-s)^{\beta-1} (Qy)_2(s)\,ds \\ &\quad = \frac{T^\beta}{\beta} \bigl((Qy)_2(t) - \bigl(Q^2 y\bigr)_2(t) \bigr) = 0. \end{aligned}$$

So $y^* \in \operatorname{Im} L$. Thus $X = \operatorname{Im} L + \operatorname{Im} Q$. Since $\operatorname{Im} L \cap \operatorname{Im} Q = \{0\}$, we have $X = \operatorname{Im} L \oplus \operatorname{Im} Q$. Hence

$$\dim \operatorname{Ker} L = \dim \operatorname{Im} Q = \operatorname{codim} \operatorname{Im} L = 2.$$

This means that L is a Fredholm operator of index zero.

For $y \in \operatorname{Im} L$, from the definition of K_P, we have

$$L K_P y = \begin{pmatrix} D_{0^+}^{\alpha} I_{0^+}^{\alpha} y_1 \\ D_{0^+}^{\beta} I_{0^+}^{\beta} y_2 \end{pmatrix} = y.$$

On the other hand, for $x \in \operatorname{dom} L \cap \operatorname{Ker} P$, we get $x_1(0) = x_2(0) = 0$. By Lemma 2.1, we obtain

$$K_P L x = \begin{pmatrix} x_1 - x_1(0) \\ x_2 - x_2(0) \end{pmatrix} = x.$$

So we know that $K_P = (L_{\operatorname{dom} L \cap \operatorname{Ker} P})^{-1}$. The proof is completed. □

Lemma 3.3 *Let* N *be defined by* (3.4). *Assume* $\Omega \subset X$ *is an open bounded subset such that* $\operatorname{dom} L \cap \bar{\Omega} \neq \varnothing$, *then* N *is* L*-compact on* $\bar{\Omega}$.

Proof By the continuity of ϕ_q and g, we find that $QN(\bar{\Omega})$ and $K_P(I-Q)N(\bar{\Omega})$ are bounded. Moreover, there exists a constant $A > 0$ such that $\|(I-Q)Nx\| \leq A$, $\forall x \in \bar{\Omega}$, $t \in [0,T]$. Hence, in view of the Arzelà-Ascoli theorem, we need only to prove that $K_P(I-Q)N(\bar{\Omega}) \subset X$ is equicontinuous.

For $0 \le t_1 < t_2 \le T$, $x \in \bar{\Omega}$, we have

$$\begin{aligned} &K_P(I-Q)Nx(t_2) - K_P(I-Q)Nx(t_1) \\ &\quad = \begin{pmatrix} I_{0^+}^{\alpha}((I-Q)Nx)_1(t_2) - I_{0^+}^{\alpha}((I-Q)Nx)_1(t_1) \\ I_{0^+}^{\beta}((I-Q)Nx)_2(t_2) - I_{0^+}^{\beta}((I-Q)Nx)_2(t_1) \end{pmatrix}. \end{aligned}$$

From $\|(I-Q)Nx\| \le A$, $\forall x \in \bar{\Omega}$, $t \in [0,T]$, we can see that

$$\begin{aligned} &\left| I_{0^+}^{\alpha}\big((I-Q)Nx\big)_1(t_2) - I_{0^+}^{\alpha}\big((I-Q)Nx\big)_1(t_1)\right| \\ &\quad = \frac{1}{\Gamma(\alpha)}\left| \int_0^{t_2} (t_2-s)^{\alpha-1}\big((I-Q)Nx\big)_1(s)\,ds \right. \\ &\qquad \left. - \int_0^{t_1} (t_1-s)^{\alpha-1}\big((I-Q)Nx\big)_1(s)\,ds \right| \\ &\quad \le \frac{A}{\Gamma(\alpha)}\left\{ \int_0^{t_1} \big[(t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}\big]\,ds + \int_{t_1}^{t_2} (t_2-s)^{\alpha-1}\,ds \right\} \\ &\quad = \frac{A}{\Gamma(\alpha+1)}\big[t_1^{\alpha} - t_2^{\alpha} + 2(t_2-t_1)^{\alpha}\big]. \end{aligned}$$

Since t^{α} is uniformly continuous on $[0,T]$, we can obtain that $(K_P(I-Q)N(\bar{\Omega}))_1 \subset C[0,T]$ is equicontinuous. A similar proof can show that $(K_P(I-Q)N(\bar{\Omega}))_2 \subset C[0,T]$ is also equicontinuous. Hence, we find that $K_P(I-Q)N : \bar{\Omega} \to X$ is compact. The proof is completed. □

Lemma 3.4 *Suppose* (C_1), (C_2) *hold, then the set*

$$\Omega_1 = \big\{ x \in \operatorname{dom} L \,|\, Lx = \lambda Nx, \lambda \in (0,1) \big\}$$

is bounded.

Proof For $x \in \Omega_1$, we have $Nx \in \operatorname{Im} L$. Thus, from (3.6), we obtain

$$\int_0^T (T-s)^{\beta-1} g\big(s, x_1(s)\big)\,ds = 0.$$

Then, by the integral mean value theorem, there exists a constant $\xi \in (0,T)$ such that $g(\xi, x_1(\xi)) = 0$. So, from (C_1), we get $|x_1(\xi)| \le d$. By Lemma 2.1, we have

$$x_1(t) = x_1(\xi) - I_{0^+}^{\alpha} D_{0^+}^{\alpha} x_1(\xi) + I_{0^+}^{\alpha} D_{0^+}^{\alpha} x_1(t),$$

which together with

$$\begin{aligned} \left| I_{0^+}^{\alpha} D_{0^+}^{\alpha} x_1(t) \right| &= \frac{1}{\Gamma(\alpha)} \left| \int_0^t (t-s)^{\alpha-1} D_{0^+}^{\alpha} x_1(s)\,ds \right| \\ &\le \frac{1}{\Gamma(\alpha)} \left\| D_{0^+}^{\alpha} x_1 \right\|_0 \cdot \frac{1}{\alpha} t^{\alpha} \\ &\le \frac{T^{\alpha}}{\Gamma(\alpha+1)} \left\| D_{0^+}^{\alpha} x_1 \right\|_0, \quad \forall t \in [0,T], \end{aligned}$$

and $|x_1(\xi)| \leq d$ yields

$$\|x_1\|_0 \leq d + \frac{2T^\alpha}{\Gamma(\alpha+1)}\left\|D_{0^+}^\alpha x_1\right\|_0. \tag{3.8}$$

By $Lx = \lambda Nx$, we have

$$\begin{cases} D_{0^+}^\alpha x_1(t) = \lambda\phi_q(x_2(t)), \\ D_{0^+}^\beta x_2(t) = \lambda g(t, x_1(t)). \end{cases} \tag{3.9}$$

From the first equation of (3.9), we get $x_2(t) = \lambda^{1-p}\phi_p(D_{0^+}^\alpha x_1(t))$. Then, by substituting it to the second equation of (3.9), we get

$$D_{0^+}^\beta \phi_p\bigl(D_{0^+}^\alpha x_1(t)\bigr) = \lambda^p g(t, x_1) := \lambda^p N_g x_1(t).$$

Thus, from Lemma 2.1 and the boundary value condition $x_2(0) = 0$, we obtain

$$\phi_p\bigl(D_{0^+}^\alpha x_1(t)\bigr) = \lambda^p I_{0^+}^\beta N_g x_1(t).$$

So, from (C_2), we have

$$\begin{aligned} \bigl|\phi_p\bigl(D_{0^+}^\alpha x_1(t)\bigr)\bigr| &\leq \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\bigl|g\bigl(s, x_1(s)\bigr)\bigr|\,ds \\ &\leq \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\bigl(a(s)\bigl|x_1(s)\bigr|^{p-1} + b(s)\bigr)\,ds \\ &\leq \frac{T^\beta}{\Gamma(\beta+1)}\bigl(\|a\|_0\|x_1\|_0^{p-1} + \|b\|_0\bigr), \quad \forall t \in [0, T], \end{aligned}$$

which together with $|\phi_p(D_{0^+}^\alpha x_1(t))| = |D_{0^+}^\alpha x_1(t)|^{p-1}$ and (3.8) yields

$$\left\|D_{0^+}^\alpha x_1\right\|_0^{p-1} \leq \frac{T^\beta}{\Gamma(\beta+1)}\left[\|b\|_0 + \|a\|_0\left(d + \frac{2T^\alpha}{\Gamma(\alpha+1)}\left\|D_{0^+}^\alpha x_1\right\|_0\right)^{p-1}\right].$$

If $p < 2$, by Lemma 2.2, we get

$$\begin{aligned} \left\|D_{0^+}^\alpha x_1\right\|_0^{p-1} &\leq \frac{T^\beta}{\Gamma(\beta+1)}\left[\|b\|_0 + \|a\|_0\left(d^{p-1} + \frac{(2T^\alpha)^{p-1}}{(\Gamma(\alpha+1))^{p-1}}\left\|D_{0^+}^\alpha x_1\right\|_0^{p-1}\right)\right] \\ &= A_1 + \frac{2^{p-1}T^{\beta+\alpha p-\alpha}\|a\|_0}{\Gamma(\beta+1)(\Gamma(\alpha+1))^{p-1}}\left\|D_{0^+}^\alpha x_1\right\|_0^{p-1}, \end{aligned}$$

where $A_1 = \frac{T^\beta}{\Gamma(\beta+1)}(\|b\|_0 + d^{p-1}\|a\|_0)$. Then, from (3.1), we have

$$\left\|D_{0^+}^\alpha x_1\right\|_0 \leq \left(\frac{A_1}{1-\gamma_1}\right)^{q-1} := B_1.$$

Thus, from (3.8), we get

$$\|x_1\|_0 \leq d + \frac{2T^\alpha}{\Gamma(\alpha+1)}B_1. \tag{3.10}$$

If $p \geq 2$, similar to the above argument, we let $A_2 = \frac{T^\beta}{\Gamma(\beta+1)}(\|b\|_0 + 2^{p-2}d^{p-1}\|a\|_0)$, we obtain

$$\|x_1\|_0 \leq d + \frac{2T^\alpha}{\Gamma(\alpha+1)}B_2, \tag{3.11}$$

where $B_2 = (\frac{A_2}{1-\gamma_2})^{q-1}$. Hence, combining (3.10) with (3.11), we have

$$\|x_1\|_0 \leq \max\left\{d + \frac{2T^\alpha}{\Gamma(\alpha+1)}B_1, d + \frac{2T^\alpha}{\Gamma(\alpha+1)}B_2\right\} := B. \tag{3.12}$$

From the second equation of (3.9), Lemma 2.1, and $x_2(0) = 0$, we have

$$x_2(t) = \lambda I_{0^+}^\beta N_g x_1(t).$$

So we have

$$\|x_2\|_0 \leq \frac{T^\beta}{\Gamma(\beta+1)}G_B,$$

where $G_B = \max\{|g(t,x)||t \in [0,T], |x| \leq B\}$. Thus, from (3.12), we obtain

$$\|x\| = \max\{\|x_1\|_0, \|x_2\|_0\} \leq \max\left\{B, \frac{T^\beta}{\Gamma(\beta+1)}G_B\right\} := M.$$

Hence, Ω_1 is bounded. The proof is completed. □

Lemma 3.5 *Suppose* (C_1) *holds, then the set*

$$\Omega_2 = \{x \in \operatorname{Ker} L | QNx = 0\}$$

is bounded.

Proof For $x \in \Omega_2$, we have $x_1(t) = c_1$, $x_2(t) = c_2$, $\forall t \in [0,T]$, $c_1, c_2 \in \mathbb{R}$, and

$$\phi_q(c_2) = 0, \tag{3.13}$$

$$\int_0^T (T-s)^{\beta-1} g(s, c_1)\,ds = 0. \tag{3.14}$$

From (3.13), we get $c_2 = 0$. From (3.14) and (C_1), we get $|c_1| \leq d$. Thus, we have

$$\|x\| = \max\{|c_1|, |c_2|\} \leq d.$$

Hence, Ω_2 is bounded. The proof is completed. □

Lemma 3.6 *Suppose* (C_1) *holds, then the set*

$$\Omega_3 = \{x \in \operatorname{Ker} L | \mu x + (1-\mu)JQNx = 0, \mu \in [0,1]\}$$

is bounded, where $J : \operatorname{Im} Q \to \operatorname{Ker} L$ *defined by*

$$J(x_1, x_2)^{\mathrm{T}} = \bigl((-1)^i x_2, x_1\bigr)^{\mathrm{T}}$$

is an isomorphism map.

Proof For $x \in \Omega_3$, we have $x_1(t) = c_1$, $x_2(t) = c_2$, $\forall t \in [0, T]$, $c_1, c_2 \in \mathbb{R}$, and

$$\mu c_1 + (-1)^i(1 - \mu)\frac{\beta}{T^\beta}\int_0^T (T - s)^{\beta-1} g(s, c_1)\, ds = 0, \tag{3.15}$$

$$\mu c_2 + (1 - \mu)\phi_q(c_2) = 0. \tag{3.16}$$

From (3.16), we get $c_2 = 0$ because c_2 and $\phi_q(c_2)$ have the same sign. From (3.15), if $\mu = 0$, we get $|c_1| \le d$ because of (C_1). If $\mu \in (0, 1]$, we can also get $|c_1| \le d$. In fact, if $|c_1| > d$, in view of (C_1), one has

$$\mu c_1^2 + (1 - \mu)\frac{\beta}{T^\beta}\int_0^T (T - s)^{\beta-1}(-1)^i c_1 g(s, c_1)\, ds > 0,$$

which contradicts to (3.15). So $\|x\| \le d$. Hence, Ω_3 is bounded. The proof is completed. □

Proof of Theorem 3.1 Set

$$\Omega = \bigl\{x \in X \,|\, \|x\| < \max\{M, d\} + 1\bigr\}.$$

Obviously $(\Omega_1 \cup \Omega_2 \cup \Omega_3) \subset \Omega$. It follows from Lemmas 3.2 and 3.3 that L (defined by (3.3)) is a Fredholm operator of index zero and N (defined by (3.4)) is L-compact on $\bar{\Omega}$. Moreover, by Lemmas 3.4 and 3.5, the conditions (1) and (2) of Lemma 2.3 are satisfied. Hence, it remains to verify the condition (3) of Lemma 2.3. Define the operator $H : \bar{\Omega} \times [0, 1] \to X$ by

$$H(x, \mu) = \mu x + (1 - \mu)JQNx.$$

Then, from Lemma 3.6, we have

$$H(x, \mu) \neq 0, \quad \forall (x, \mu) \in (\partial\Omega \cap \operatorname{Ker} L) \times [0, 1].$$

Thus, by the homotopy property of the degree, we have

$$\begin{aligned}
\deg(JQN, \Omega \cap \operatorname{Ker} L, \theta) &= \deg\bigl(H(\cdot, 0), \Omega \cap \operatorname{Ker} L, \theta\bigr) \\
&= \deg\bigl(H(\cdot, 1), \Omega \cap \operatorname{Ker} L, \theta\bigr) \\
&= \deg(I, \Omega \cap \operatorname{Ker} L, \theta) \neq 0,
\end{aligned}$$

where θ is the zero element of X. So the condition (3) of Lemma 2.3 is satisfied.

Consequently, by Lemma 2.3, the operator equation $Lx = Nx$ has at least one solution $x(\cdot) = (x_1(\cdot), x_2(\cdot))^{\mathrm{T}}$ on $\bar{\Omega} \cap \operatorname{dom} L$. Namely, NBVP (1.1) has at least one solution $x_1(\cdot)$. The proof is completed. □

Competing interests
The author declares that he has no competing interests.

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References

1. Kilbas, AA, Srivastava, HM, Trujillo, JJ: Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam (2006)
2. Podlubny, I: Fractional Differential Equations. Academic Press, San Diego (1999)
3. Sabatier, J, Agrawal, OP, Machado, JAT (eds.): Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering. Springer, Dordrecht (2007)
4. Samko, SG, Kilbas, AA, Marichev, OI: Fractional Integrals and Derivatives: Theory and Applications. Gordon & Breach, Yverdon (1993)
5. Agarwal, RP, O'Regan, D, Stanek, S: Positive solutions for Dirichlet problems of singular nonlinear fractional differential equations. J. Math. Anal. Appl. **371**, 57-68 (2010)
6. Bai, Z: On solutions of some fractional *m*-point boundary value problems at resonance. Electron. J. Qual. Theory Differ. Equ. **2010**, 37 (2010)
7. Chen, T, Liu, W, Hu, Z: A boundary value problem for fractional differential equation with *p*-Laplacian operator at resonance. Nonlinear Anal. TMA **75**, 3210-3217 (2012)
8. Chen, T, Liu, W: An anti-periodic boundary value problem for the fractional differential equation with a *p*-Laplacian operator. Appl. Math. Lett. **25**, 1671-1675 (2012)
9. Benchohra, M, Hamani, S, Ntouyas, SK: Boundary value problems for differential equations with fractional order and nonlocal conditions. Nonlinear Anal. TMA **71**, 2391-2396 (2009)
10. Bai, Z, Lü, H: Positive solutions for boundary value problem of nonlinear fractional differential equation. J. Math. Anal. Appl. **311**, 495-505 (2005)
11. Darwish, MA, Ntouyas, SK: On initial and boundary value problems for fractional order mixed type functional differential inclusions. Comput. Math. Appl. **59**, 1253-1265 (2010)
12. El-Shahed, M, Nieto, JJ: Nontrivial solutions for a nonlinear multi-point boundary value problem of fractional order. Comput. Math. Appl. **59**, 3438-3443 (2010)
13. Jiang, W: The existence of solutions to boundary value problems of fractional differential equations at resonance. Nonlinear Anal. TMA **74**, 1987-1994 (2011)
14. Kosmatov, N: A boundary value problem of fractional order at resonance. Electron. J. Differ. Equ. **2010**, 135 (2010)
15. Su, X: Boundary value problem for a coupled system of nonlinear fractional differential equations. Appl. Math. Lett. **22**, 64-69 (2009)
16. Lian, H: Boundary Value Problems for Nonlinear Ordinary Differential Equations on Infinite Intervals. Doctoral thesis (2007)
17. Gaines, R, Mawhin, J: Coincidence Degree and Nonlinear Differential Equations. Springer, Berlin (1977)