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An algorithm for treating asymptotically strict pseudocontractions and monotone operators

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Abstract

In this paper, an algorithm for treating asymptotically κ -strict pseudocontractions and monotone operators is proposed. Convergence analysis of the algorithm is investigated in the framework of Hilbert spaces.

Keywords: asymptotically κ -strict pseudocontraction; inverse-strongly monotone mapping; maximal monotone operator; fixed point

1 Introduction-preliminaries

In this paper, we are concerned with the problem of finding a common element in the intersection $F(T) \cap (A + B)^{-1}(0)$, where $F(T)$ denotes the fixed point set of the mapping T and $(A + B)^{-1}(0)$ denotes the zero point set of the sum of the operator A and the operator B .

The motivation for the common element problem is mainly due to its possible applications to mathematical modeling of concrete complex problems. The common element problems include mini-max problems, complementarity problems, equilibrium problems, common fixed point problems and variational inequalities as special cases; see, for example, [1–35] and the references therein.

Throughout the article, we always assume that H is a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$, respectively. Let C be a nonempty closed convex subset of H , and let Proj_C be the metric projection from H onto C .

Let $A : C \rightarrow H$ be a mapping. $A^{-1}(0)$ stands for the zero point set of A ; that is, $A^{-1}(0) := \{x \in C : Ax = 0\}$. Recall that A is said to be monotone iff

$$\langle Ax - Ay, x - y \rangle \geq 0, \quad \forall x, y \in C.$$

A is said to be α -strongly monotone iff there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|x - y\|^2, \quad \forall x, y \in C.$$

A is said to be α -inverse-strongly monotone iff there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C.$$

It is not hard to see that α -inverse-strongly monotone mappings are Lipschitz continuous. Indeed, we have

$$\alpha \|Ax - Ay\|^2 \leq \langle Ax - Ay, x - y \rangle \leq \|Ax - Ay\| \|x - y\|.$$

This shows that $\|Ax - Ay\| \leq \frac{1}{\alpha} \|x - y\|$.

Recall that the classical variational inequality, denoted by $VI(C, A)$, is to find $u \in C$ such that

$$\langle Au, v - u \rangle \geq 0, \quad \forall v \in C. \tag{1.1}$$

One can see that the variational inequality (1.1) is equivalent to a fixed point problem of the mapping $\text{Proj}_C(I - rA)$, where I is the identity and r is some positive real number. The element $u \in C$ is a solution of the variational inequality (1.1) iff $u \in C$ satisfies the equation $u = P_C(u - rAu)$. This alternative equivalent formulation has played a significant role in the studies of variational inequalities and related optimization problems.

A multivalued operator $B : H \rightarrow 2^H$ with the domain $D(B) = \{x \in H : Bx \neq \emptyset\}$ and the range $R(B) = \{Bx : x \in D(B)\}$ is said to be monotone if for $x_1 \in D(B)$, $x_2 \in D(B)$, $y_1 \in Bx_1$ and $y_2 \in Bx_2$, we have $\langle x_1 - x_2, y_1 - y_2 \rangle \geq 0$. A monotone operator B is said to be maximal if its graph $G(B) = \{(x, y) : y \in Bx\}$ is not properly contained in the graph of any other monotone operator. Let I denote the identity operator on H and $B : H \rightarrow 2^H$ be a maximal monotone operator. Then we can define, for each $r > 0$, a nonexpansive single-valued mapping $J_r : H \rightarrow H$ by $J_r = (I + rB)^{-1}$. It is called the resolvent of B . We know that $B^{-1}0 = F(J_r)$ for all $r > 0$ and J_r is firmly nonexpansive.

Let $T : C \rightarrow C$ be a mapping. In this paper, we use $F(T)$ to denote the fixed point set of T ; that is, $F(T) := \{x \in C : x = Tx\}$. Recall that T is said to be nonexpansive iff

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

T is said to be asymptotically nonexpansive iff there exists a sequence $\{k_n\} \subset [1, \infty)$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|, \quad \forall x, y \in C, n \geq 1.$$

T is said to be a κ -strict pseudocontraction iff there exists a constant $\kappa \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa \|(x - Tx) - (y - Ty)\|^2, \quad \forall x, y \in C.$$

Note that the class of κ -strict pseudocontractions strictly includes the class of nonexpansive mappings as a special case. That is, T is nonexpansive iff the coefficient $\kappa = 0$. T is said to be an asymptotically κ -strict pseudocontraction iff there exist a constant $\kappa \in [0, 1)$ and a sequence $\{k_n\}$ in $[1, \infty)$ such that

$$\|T^n x - T^n y\|^2 \leq k_n \|x - y\|^2 + \kappa \|(x - T^n x) - (y - T^n y)\|^2, \quad \forall x, y \in C, n \geq 1.$$

Note that the class of asymptotically κ -strict pseudocontractions strictly includes the class of asymptotically nonexpansive mappings as a special case. That is, T is asymptotically nonexpansive iff the coefficient $\kappa = 0$.

In [24], Kamimura and Takahashi investigated the problem of finding zero points of a maximal monotone operator based on the following iterative algorithm:

$$x_1 \in H, \quad x_{n+1} = \alpha_n x + (1 - \alpha_n)J_{r_n}x_n, \quad \forall n \geq 1, \tag{1.2}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$, $\{r_n\}$ is a positive real number sequence, $B : H \rightarrow 2^H$ is maximal monotone and $J_{r_n} = (I + r_n B)^{-1}$. It is proved that the sequence $\{x_n\}$ generated in (1.2) converges strongly to some $z \in B^{-1}(0)$ provided that the control sequence satisfies some restrictions. Further, using this result, they also investigated the case that $B = \partial f$, where $f : H \rightarrow (-\infty, \infty]$ is a proper lower semicontinuous convex function. Convergence theorems are established in the framework of real Hilbert spaces; for more details, see [24].

Recently, Takahashi *et al.* studied zero point problems of the sum of two monotone mappings and fixed point problems of a nonexpansive mapping based on the following iterative algorithm:

$$\begin{cases} x_1 \in C, \\ y_n = \alpha_n x + (1 - \alpha_n)J_{r_n}(x_n - r_n A x_n), \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)T y_n, \quad \forall n \geq 1, \end{cases} \tag{1.3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real number sequences in $(0, 1)$, $\{r_n\}$ is a positive sequence, $T : C \rightarrow C$ is a nonexpansive mapping and $A : C \rightarrow H$ is an inverse-strongly monotone mapping. It is proved that the sequence $\{x_n\}$ generated in (1.3) converges strongly to some $z \in (A + B)^{-1}(0) \cap F(S)$ provided that the control sequence satisfies some restrictions; for more details, see [2].

Motivated by the above results, we investigate fixed point problems of asymptotically strict pseudocontractions and zero point problems of the sum of two monotone mappings.

In order to state our main results, we need the following tools.

Recall that a space is said to satisfy Opial's condition [36] if, for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, where \rightharpoonup denotes the weak convergence, the inequality

$$\liminf_{n \rightarrow \infty} \|x_n - x\| < \liminf_{n \rightarrow \infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$. Indeed, the above inequality is equivalent to the following:

$$\limsup_{n \rightarrow \infty} \|x_n - x\| < \limsup_{n \rightarrow \infty} \|x_n - y\|.$$

Lemma 1.1 [2] *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X , and let β_n be a sequence in $[0, 1]$ with $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$. Suppose that $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and*

$$\limsup_{n \rightarrow \infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then $\lim_{n \rightarrow \infty} \|y_n - x_n\| = 0$.

Lemma 1.2 [37] *Let C be a nonempty, closed and convex subset of H , let $A : C \rightarrow H$ be a mapping, and let $B : H \rightrightarrows H$ be a maximal monotone operator. Then $F(J_r(I - rA)) = (A + B)^{-1}(0)$.*

Lemma 1.3 [38] *Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \delta_n,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

- (i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
- (ii) $\limsup_{n \rightarrow \infty} \delta_n / \gamma_n \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n \rightarrow \infty} \alpha_n = 0$.

Lemma 1.4 [39] *Let C be a nonempty, closed and convex subset of H . Let $T : C \rightarrow C$ be an asymptotically strict pseudocontraction. Then T is Lipschitz continuous and $I - T$ is demiclosed at zero.*

2 Main results

Theorem 2.1 *Let C be a nonempty closed convex subset of H . Let $T : C \rightarrow C$ be an asymptotically κ -strict pseudocontraction. Let $A : C \rightarrow H$ be an α -inverse-strongly monotone mapping, and let B be a maximal monotone operator on H . Assume that $F(T) \cap (A + B)^{-1}(0) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be real number sequences in $(0, 1)$. Let $J_{r_n} = (I + r_n B)^{-1}$, where $\{r_n\}$ is a positive real number sequence. Let $\{x_n\}$ be a sequence in C generated by: $x_1 \in C$ is chosen arbitrarily and*

$$\begin{cases} z_n = \text{Proj}_C(\alpha_n u + (1 - \alpha_n)J_{r_n}(x_n - r_n A x_n)), \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)(\gamma_n z_n + (1 - \gamma_n)T^n z_n), \quad \forall n \geq 1. \end{cases}$$

Assume that the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{r_n\}$ satisfy the following restrictions:

- (a) $0 < a \leq r_n \leq b < 2\alpha$, $\lim_{n \rightarrow \infty} |r_{n+1} - r_n| = 0$;
- (b) $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- (c) $0 < c \leq \beta_n \leq d < 1$;
- (d) $\kappa \leq \gamma_n \leq e < 1$, $\lim_{n \rightarrow \infty} |\gamma_{n+1} - \gamma_n| = 0$,

where a, b, c, d and e are some real numbers. If T is asymptotically regular, then the sequence $\{x_n\}$ converges strongly to some point \bar{x} , where $\bar{x} = P_{F(T) \cap (A+B)^{-1}(0)} u$.

Proof First, we show that the mapping $I - r_n A$ is nonexpansive. Indeed, we have

$$\begin{aligned} & \| (I - r_n A)x - (I - r_n A)y \|^2 \\ &= \| x - y \|^2 - 2r_n \langle x - y, Ax - Ay \rangle + r_n^2 \| Ax - Ay \|^2 \\ &\leq \| x - y \|^2 - r_n(2\alpha - r_n) \| Ax - Ay \|^2. \end{aligned}$$

It follows from Restriction (a) that $I - r_n A$ is nonexpansive. Put $y_n = \gamma_n z_n + (1 - \gamma_n)T^n z_n$ and fix $p \in F(T) \cap (A + B)^{-1}(0)$. It follows from Lemma 1.2 that

$$\begin{aligned} \| z_n - p \| &= \| \text{Proj}_C(\alpha_n u + (1 - \alpha_n)J_{r_n}(x_n - r_n A x_n)) - p \| \\ &\leq \alpha_n \| u - p \| + (1 - \alpha_n) \| J_{r_n}(x_n - r_n A x_n) - p \| \\ &\leq \alpha_n \| u - p \| + (1 - \alpha_n) \| x_n - p \|. \end{aligned} \tag{2.1}$$

In view of Restriction (d), we find that

$$\begin{aligned}
 \|y_n - p\|^2 &\leq \|\gamma_n(z_n - p) + (1 - \gamma_n)(T^n z_n - T^n p)\|^2 \\
 &= \gamma_n \|z_n - p\|^2 + (1 - \gamma_n) \|T^n z_n - T^n p\|^2 \\
 &\quad - \gamma_n(1 - \gamma_n) \|(z_n - p) - (T^n z_n - T^n p)\|^2 \\
 &\leq \gamma_n \|z_n - p\|^2 + (1 - \gamma_n) (\|z_n - p\|^2 + \kappa \|(z_n - p) - (T^n z_n - T^n p)\|^2) \\
 &\quad - \gamma_n(1 - \gamma_n) \|(z_n - p) - (T^n z_n - T^n p)\|^2 \\
 &= \|z_n - p\|^2 - (1 - \gamma_n)(\gamma_n - \kappa) \|(z_n - p) - (T^n z_n - T^n p)\|^2 \\
 &\leq \|z_n - p\|^2.
 \end{aligned} \tag{2.2}$$

Substituting (2.1) into (2.2), we obtain that

$$\begin{aligned}
 \|x_{n+1} - p\| &\leq \beta_n \|x_n - p\| + (1 - \beta_n) \|y_n - p\| \\
 &\leq \beta_n \|x_n - p\| + (1 - \beta_n) (\alpha_n \|u - p\| + (1 - \alpha_n) \|x_n - p\|) \\
 &\leq (1 - \alpha_n(1 - \beta_n)) \|x_n - p\| + \alpha_n(1 - \beta_n) \|u - p\|.
 \end{aligned}$$

Putting $M = \max\{\|x_1 - p\|, \|u - p\|\}$, we find that $\|x_n - p\| \leq M$ for all $n \geq 1$. Indeed, it is clear that $\|x_1 - p\| \leq M$. Suppose that $\|x_m - p\| \leq M$ for some positive integer m . It follows that

$$\begin{aligned}
 \|x_{m+1} - p\| &\leq (1 - \alpha_m(1 - \beta_m)) \|x_m - p\| + \alpha_m(1 - \beta_m) \|u - p\| \\
 &\leq (1 - \alpha_m(1 - \beta_m)) M + \alpha_m(1 - \beta_m) M \\
 &= M.
 \end{aligned}$$

This finds that $\{x_n\}$ is bounded. Putting $\rho_n = J_{r_n}(x_n - r_n A x_n)$, we find that

$$\begin{aligned}
 \|\rho_{n+1} - \rho_n\| &\leq \|J_{r_{n+1}}(x_{n+1} - r_{n+1} A x_{n+1}) - J_{r_{n+1}}(x_n - r_n A x_n)\| \\
 &\quad + \|J_{r_{n+1}}(x_n - r_n A x_n) - J_{r_n}(x_n - r_n A x_n)\| \\
 &\leq \|(x_{n+1} - r_{n+1} A x_{n+1}) - (x_n - r_n A x_n)\| \\
 &\quad + \|J_{r_{n+1}}(x_n - r_n A x_n) - J_{r_n}(x_n - r_n A x_n)\| \\
 &\leq \|x_{n+1} - x_n\| + |r_{n+1} - r_n| \|A x_n\| \\
 &\quad + \|J_{r_{n+1}}(x_n - r_n A x_n) - J_{r_n}(x_n - r_n A x_n)\|.
 \end{aligned} \tag{2.3}$$

On the other hand, we have

$$\begin{aligned}
 \|z_{n+1} - z_n\| &\leq \|(\alpha_{n+1} u + (1 - \alpha_{n+1}) \rho_{n+1}) - (\alpha_n u + (1 - \alpha_n) \rho_n)\| \\
 &\leq (1 - \alpha_{n+1}) \|\rho_{n+1} - \rho_n\| + |\alpha_{n+1} - \alpha_n| \|u - p\|.
 \end{aligned} \tag{2.4}$$

Substituting (2.3) into (2.4), we find that

$$\begin{aligned} \|z_{n+1} - z_n\| &\leq \|(\alpha_{n+1}u + (1 - \alpha_{n+1})\rho_{n+1}) - (\alpha_n u + (1 - \alpha_n)\rho_n)\| \\ &\leq (1 - \alpha_{n+1})\|x_{n+1} - x_n\| + |r_{n+1} - r_n|\|Ax_n\| + |\alpha_{n+1} - \alpha_n|\|\rho_n - u\| \\ &\quad + (1 - \alpha_{n+1})\|J_{r_{n+1}}(x_n - r_n Ax_n) - J_{r_n}(x_n - r_n Ax_n)\|. \end{aligned} \tag{2.5}$$

Put $\xi_n = x_n - r_n Ax_n$. Since B is monotone, we find that

$$\left\langle J_{r_{n+1}}\xi_n - J_{r_n}\xi_n, \frac{\xi_n - J_{r_{n+1}}\xi_n}{r_{n+1}} - \frac{\xi_n - J_{r_n}\xi_n}{r_n} \right\rangle \geq 0.$$

It follows that $\langle J_{r_{n+1}}\xi_n - J_{r_n}\xi_n, (1 - \frac{r_{n+1}}{r_n})(\xi_n - J_{r_n}\xi_n) \rangle \geq \|J_{r_{n+1}}\xi_n - J_{r_n}\xi_n\|^2$. This yields that $|r_{n+1} - r_n|\|\xi_n - J_{r_n}\xi_n\| \geq r_n\|J_{r_{n+1}}\xi_n - J_{r_n}\xi_n\|$. This combines with (2.5) to yield that

$$\begin{aligned} \|z_{n+1} - z_n\| &\leq (1 - \alpha_{n+1})\|x_{n+1} - x_n\| + |r_{n+1} - r_n|\|Ax_n\| \\ &\quad + |\alpha_{n+1} - \alpha_n|\|\rho_n - u\| + \frac{|r_{n+1} - r_n|}{r_n}\|\xi_n - J_{r_n}\xi_n\|. \end{aligned} \tag{2.6}$$

On the other hand, we have

$$\begin{aligned} \|y_{n+1} - y_n\| &\leq \gamma_{n+1}\|z_{n+1} - z_n\| + |\gamma_{n+1} - \gamma_n|\|z_n - T^n z_n\| \\ &\quad + (1 - \gamma_{n+1})\|T^{n+1}z_{n+1} - T^n z_n\|. \end{aligned} \tag{2.7}$$

Substituting (2.6) into (2.7), we find that

$$\begin{aligned} \|y_{n+1} - y_n\| - \|x_{n+1} - x_n\| &\leq |r_{n+1} - r_n|\|Ax_n\| + |\alpha_{n+1} - \alpha_n|\|\rho_n - u\| \\ &\quad + \frac{|r_{n+1} - r_n|}{r_n}\|\xi_n - J_{r_n}\xi_n\| + |\gamma_{n+1} - \gamma_n|\|z_n - T^n z_n\| \\ &\quad + (1 - \gamma_{n+1})\|T^{n+1}z_{n+1} - T^n z_n\|. \end{aligned}$$

It follows from Restrictions (a), (c) and (d) that

$$\limsup_{n \rightarrow \infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

It follows from Lemma 1.1 that $\lim_{n \rightarrow \infty} \|y_n - x_n\| = 0$. Since $x_{n+1} - x_n = (1 - \beta_n)(y_n - x_n)$, we find that $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$. Notice that

$$\begin{aligned} &\|J_{r_n}(I - r_n A)x_n - J_{r_n}(I - r_n A)p\|^2 \\ &= \|(x_n - p) - r_n(Ax_n - Ap)\|^2 \\ &= \|x_n - p\|^2 - 2r_n\langle x_n - p, Ax_n - Ap \rangle + r_n^2\|Ax_n - Ap\|^2 \\ &\leq \|x_n - p\|^2 - r_n(2\alpha - r_n)\|Ax_n - Ap\|^2. \end{aligned} \tag{2.8}$$

Since the norm is convex, we see from (2.2) and (2.8) that

$$\begin{aligned} \|x_{n+1} - p\|^2 &\leq \beta_n \|x_n - p\|^2 + (1 - \beta_n) \|y_n - p\|^2 \\ &\leq \beta_n \|x_n - p\|^2 + (1 - \beta_n) \|z_n - p\|^2 \\ &\leq \beta_n \|x_n - p\|^2 + (1 - \beta_n) \|\alpha_n(u - p) + (1 - \alpha_n)(J_{r_n}(x_n - r_nAx_n) - p)\|^2 \\ &\leq \beta_n \|x_n - p\|^2 + \alpha_n(1 - \beta_n) \|u - p\|^2 \\ &\quad + (1 - \alpha_n)(1 - \beta_n) \|J_{r_n}(x_n - r_nAx_n) - p\|^2 \\ &\leq \|x_n - p\|^2 + \alpha_n \|u - p\|^2 - r_n(2\alpha - r_n)(1 - \alpha_n)(1 - \beta_n) \|Ax_n - Ap\|^2. \end{aligned} \tag{2.9}$$

This yields that

$$\begin{aligned} r_n(2\alpha - r_n)(1 - \alpha_n)(1 - \beta_n) \|Ax_n - Ap\|^2 \\ \leq \|x_n - p\|^2 + \alpha_n \|u - p\|^2 - \|x_{n+1} - p\|^2 \\ \leq (\|x_n - p\| + \|x_{n+1} - p\|) \|x_{n+1} - x_n\| + \alpha_n \|u - p\|^2. \end{aligned}$$

In view of Restrictions (a), (b) and (c), we obtain that

$$\lim_{n \rightarrow \infty} \|Ax_n - Ap\| = 0. \tag{2.10}$$

Notice that

$$\begin{aligned} \|\rho_n - p\|^2 &= \|J_{r_n}(x_n - r_nAx_n) - J_{r_n}(p - r_nAp)\|^2 \\ &\leq \langle (x_n - r_nAx_n) - (p - r_nAp), \rho_n - p \rangle \\ &= \frac{1}{2} (\|(x_n - r_nAx_n) - (p - r_nAp)\|^2 + \|\rho_n - p\|^2 \\ &\quad - \|(x_n - r_nAx_n) - (p - r_nAp) - (\rho_n - p)\|^2) \\ &\leq \frac{1}{2} (\|x_n - p\|^2 + \|\rho_n - p\|^2 - \|x_n - \rho_n - r_n(Ax_n - Ap)\|^2) \\ &\leq \frac{1}{2} (\|x_n - p\|^2 + \|\rho_n - p\|^2 - \|x_n - \rho_n\|^2 - r_n^2 \|Ax_n - Ap\|^2 \\ &\quad + 2r_n \|x_n - \rho_n\| \|Ax_n - Ap\|) \\ &\leq \frac{1}{2} (\|x_n - p\|^2 + \|\rho_n - p\|^2 - \|x_n - \rho_n\|^2 + 2r_n \|x_n - \rho_n\| \|Ax_n - Ap\|). \end{aligned}$$

It follows that

$$\|\rho_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - \rho_n\|^2 + 2r_n \|x_n - \rho_n\| \|Ax_n - Ap\|. \tag{2.11}$$

This yields that

$$\begin{aligned} \|z_n - p\|^2 &\leq \alpha_n \|u - p\|^2 + (1 - \alpha_n) \|\rho_n - p\|^2 \\ &\leq \alpha_n \|u - p\|^2 + \|x_n - p\|^2 - (1 - \alpha_n) \|x_n - \rho_n\|^2 + 2r_n \|x_n - \rho_n\| \|Ax_n - Ap\|. \end{aligned}$$

It follows from (2.2) that

$$\begin{aligned} \|x_{n+1} - p\|^2 &\leq \beta_n \|x_n - p\|^2 + (1 - \beta_n) \|y_n - p\|^2 \\ &\leq \beta_n \|x_n - p\|^2 + (1 - \beta_n) \|z_n - p\|^2 \\ &\leq \|x_n - p\|^2 + \alpha_n \|u - p\|^2 - (1 - \alpha_n)(1 - \beta_n) \|x_n - \rho_n\|^2 \\ &\quad + 2r_n(1 - \beta_n) \|x_n - \rho_n\| \|Ax_n - Ap\|. \end{aligned}$$

We therefore obtain that

$$\begin{aligned} &(1 - \alpha_n)(1 - \beta_n) \|x_n - \rho_n\|^2 \\ &\leq \|x_n - p\|^2 + \alpha_n \|u - p\|^2 - \|x_{n+1} - p\|^2 \\ &\quad + 2r_n(1 - \beta_n) \|x_n - \rho_n\| \|Ax_n - Ap\| \\ &\leq (\|x_n - p\| + \|x_{n+1} - p\|) \|x_n - x_{n+1}\| + \alpha_n \|u - p\|^2 \\ &\quad + 2r_n(1 - \beta_n) \|x_n - \rho_n\| \|Ax_n - Ap\|. \end{aligned}$$

In view of Restrictions (a), (b) and (c), we find from (2.10) that

$$\lim_{n \rightarrow \infty} \|x_n - \rho_n\| = 0. \tag{2.12}$$

Next, we show that $\limsup_{n \rightarrow \infty} \langle u - \bar{x}, \rho_n - \bar{x} \rangle \leq 0$, where $\bar{x} = P_{F(T) \cap (A+B)^{-1}(0)} u$. To show it, we can choose a subsequence $\{\rho_{n_i}\}$ of $\{\rho_n\}$ such that

$$\limsup_{n \rightarrow \infty} \langle u - \bar{x}, \rho_n - \bar{x} \rangle = \lim_{i \rightarrow \infty} \langle u - \bar{x}, \rho_{n_i} - \bar{x} \rangle.$$

Since ρ_{n_i} is bounded, we can choose a subsequence $\{\rho_{n_{i_j}}\}$ of $\{\rho_{n_i}\}$ which converges weakly to some point x . We may assume, without loss of generality, that $\rho_{n_{i_j}}$ converges weakly to x . Since $\rho_n = J_{r_n}(x_n - r_n Ax_n)$, we find that $\frac{x_n - \rho_n}{r_n} - Ax_n \in B\rho_n$. Since B is monotone, we get, for any $(\mu, v) \in B$, that $\langle \rho_n - \mu, \frac{x_n - \rho_n}{r_n} - Ax_n - v \rangle \geq 0$. Replacing n by n_i and letting $i \rightarrow \infty$, we obtain from (2.12) that $\langle x - \mu, -Ax - v \rangle \geq 0$. This means $-Ax \in Bx$, that is, $0 \in (A + B)(x)$. Hence we get $x \in (A + B)^{-1}(0)$. Next, we show that $x \in F(T)$. Notice that

$$\begin{aligned} \|z_n - x_n\| &\leq \|\text{Proj}_C(\alpha_n u + (1 - \alpha_n)J_{r_n}(x_n - r_n Ax_n)) - x_n\| \\ &\leq \alpha_n \|u - x_n\| + (1 - \alpha_n) \|J_{r_n}(x_n - r_n Ax_n) - x_n\|. \end{aligned}$$

In view of Restriction (a), we find from (2.12) that $\lim_{n \rightarrow \infty} \|z_n - x_n\| = 0$. Note that

$$\begin{aligned} &\|(\gamma_n x_n + (1 - \gamma_n)T^n x_n) - x_n\| \\ &\leq \|(\gamma_n x_n + (1 - \gamma_n)T^n x_n) - (\gamma_n z_n + (1 - \gamma_n)T^n z_n)\| \\ &\quad + \|(\gamma_n z_n + (1 - \gamma_n)T^n z_n) - x_n\| \\ &\leq \gamma_n \|x_n - z_n\| + (1 - \gamma_n) \|T^n x_n - T^n z_n\| + \|(\gamma_n z_n + (1 - \gamma_n)T^n z_n) - x_n\| \\ &\leq L \|x_n - z_n\| + \|y_n - x_n\|. \end{aligned}$$

It follows from (2.8) that

$$\lim_{n \rightarrow \infty} \|(\gamma_n x_n + (1 - \gamma_n)T^n x_n) - x_n\| = 0. \tag{2.13}$$

Note that

$$\begin{aligned} \|T^n x_n - x_n\| &\leq \|T^n x_n - (\gamma_n x_n + (1 - \gamma_n)T^n x_n)\| \\ &\quad + \|(\gamma_n x_n + (1 - \gamma_n)T^n x_n) - x_n\| \\ &\leq \gamma_n \|T^n x_n - x_n\| + \|(\gamma_n x_n + (1 - \gamma_n)T^n x_n) - x_n\|. \end{aligned}$$

It follows that $(1 - \gamma_n)\|T^n x_n - x_n\| \leq \|(\gamma_n x_n + (1 - \gamma_n)T^n x_n) - x_n\|$. This implies from Restriction (d) and (2.13) that $\lim_{n \rightarrow \infty} \|T^n x_n - x_n\| = 0$. Since T is uniformly L -Lipschitz continuous, we can obtain that $\lim_{n \rightarrow \infty} \|Tx_n - x_n\| = 0$. In view of Lemma 1.4, we find that $x \in F(T)$. This implies that

$$\limsup_{n \rightarrow \infty} \langle u - \bar{x}, \rho_n - \bar{x} \rangle = \langle u - \bar{x}, x - \bar{x} \rangle \leq 0.$$

On the other hand, we have

$$\begin{aligned} \|x_{n+1} - \bar{x}\|^2 &\leq \beta_n \|x_n - \bar{x}\|^2 + (1 - \beta_n) \|\gamma_n - \bar{x}\|^2 \\ &\leq \beta_n \|x_n - x\|^2 + (1 - \beta_n) \|z_n - \bar{x}\|^2 \\ &\leq \beta_n \|x_n - x\|^2 + (1 - \beta_n) \|\alpha_n(u_n - \bar{x}) + (1 - \alpha_n)(\rho_n - \bar{x})\|^2 \\ &\leq \beta_n \|x_n - x\|^2 + (1 - \alpha_n)^2 (1 - \beta_n) \|\rho_n - x\|^2 \\ &\quad + 2\alpha_n(1 - \beta_n) \langle u - \bar{x}, \rho_n - x \rangle \\ &\leq (1 - \alpha_n(1 - \beta_n)) \|x_n - \bar{x}\|^2 + 2\alpha_n(1 - \beta_n) \langle u - \bar{x}, \rho_n - \bar{x} \rangle. \end{aligned}$$

From Lemma 1.3, we find that $\lim_{n \rightarrow \infty} \|x_n - \bar{x}\| = 0$. This completes the proof. □

If T is asymptotically nonexpansive, then we find the following result.

Corollary 2.2 *Let C be a nonempty closed convex subset of H . Let $T : C \rightarrow C$ be an asymptotically nonexpansive mapping. Let $A : C \rightarrow H$ be an α -inverse-strongly monotone mapping, and let B be a maximal monotone operator on H . Assume that $F(T) \cap (A + B)^{-1}(0) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real number sequences in $(0, 1)$. Let $J_{r_n} = (I + r_n B)^{-1}$, where $\{r_n\}$ is a positive real number sequence. Let $\{x_n\}$ be a sequence in C generated by: $x_1 \in C$ is chosen arbitrarily and*

$$\begin{cases} z_n = P_C(\alpha_n u + (1 - \alpha_n)J_{r_n}(x_n - r_n A x_n)), \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)T^n z_n, \quad \forall n \geq 1. \end{cases}$$

Assume that the sequences $\{\alpha_n\}$, $\{\gamma_n\}$ and $\{r_n\}$ satisfy the following restrictions:

- (a) $0 < a \leq r_n \leq b < 2\alpha$, $\lim_{n \rightarrow \infty} |r_{n+1} - r_n| = 0$;
- (b) $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- (c) $0 < c \leq \beta_n \leq d < 1$,

where a, b, c and d are some real numbers. If T is asymptotically regular, then the sequence $\{x_n\}$ converges strongly to some point \bar{x} , where $\bar{x} = P_{F(T) \cap (A+B)^{-1}(0)}u$.

3 Applications

In this section, we shall consider equilibrium problems and variational inequalities.

Let F be a bifunction of $C \times C$ into \mathbb{R} , where \mathbb{R} denotes the set of real numbers. Recall the following equilibrium problem:

$$\text{Find } x \in C \text{ such that } F(x, y) \geq 0, \quad \forall y \in C.$$

In this work, we use $EP(F)$ to denote the solution set of the equilibrium problem.

To study the equilibrium problems, we may assume that F satisfies the following conditions:

- (A1) $F(x, x) = 0$ for all $x \in C$;
- (A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;
- (A3) for each $x, y, z \in C$,

$$\limsup_{t \downarrow 0} F(tz + (1-t)x, y) \leq F(x, y);$$

- (A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and weakly lower semi-continuous.

Putting $F(x, y) = \langle Ax, y - x \rangle$ for every $x, y \in C$, we see that the equilibrium problem is reduced to the variational inequality (1.1).

The following lemma can be found in [40].

Lemma 3.1 *Let C be a nonempty closed convex subset of H , and let $F : C \times C \rightarrow \mathbb{R}$ be a bifunction satisfying (A1)-(A4). Then, for any $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \quad \forall y \in C.$$

Further, define

$$T_r x = \left\{ z \in C : F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \forall y \in C \right\} \tag{3.1}$$

for all $r > 0$ and $x \in H$. Then the following hold:

- (a) T_r is single-valued;
- (b) T_r is firmly nonexpansive, i.e., for any $x, y \in H$,

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle;$$

- (c) $F(T_r) = EP(F)$;
- (d) $EP(F)$ is closed and convex.

Lemma 3.2 [2] *Let C be a nonempty closed convex subset of a real Hilbert space H , let F be a bifunction from $C \times C$ to \mathbb{R} which satisfies (A1)-(A4), and let A_F be a multivalued*

mapping of H into itself defined by

$$A_F x = \begin{cases} \{z \in H : F(x, y) \geq \langle y - x, z \rangle, \forall y \in C\}, & x \in C, \\ \emptyset, & x \notin C. \end{cases} \quad (3.2)$$

Then A_F is a maximal monotone operator with the domain $D(A_F) \subset C$, $EP(F) = A_F^{-1}(0)$ and

$$T_r x = (I + rA_F)^{-1}x, \quad \forall x \in H, r > 0,$$

where T_r is defined as in (3.1).

The following result is not derived based on Theorem 2.1 and Lemma 3.2.

Theorem 3.3 *Let C be a nonempty closed convex subset of H . Let $T : C \rightarrow C$ be an asymptotically κ -strict pseudocontraction. Let F be a bifunction from $C \times C$ to \mathbb{R} which satisfies (A1)-(A4). Assume that $F(T) \cap EP(F) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be real number sequences in $(0, 1)$. Let $\{x_n\}$ be a sequence in C generated by: $x_1 \in C$ is chosen arbitrarily and*

$$\begin{cases} w_n \in C \text{ such that } F_B(w_n, u) + \frac{1}{r_n} \langle u - w_n, w_n - x_n \rangle \geq 0, & \forall u \in C, \\ z_n = P_C(\alpha_n u + (1 - \alpha_n)w_n), \\ y_n = \gamma_n z_n + (1 - \gamma_n)T^n z_n, \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)y_n, & \forall n \geq 1. \end{cases}$$

Assume that the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{r_n\}$ satisfy the following restrictions:

- (a) $0 < a \leq r_n \leq b < 2\alpha$, $\lim_{n \rightarrow \infty} |r_{n+1} - r_n| = 0$;
- (b) $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- (c) $0 < c \leq \beta_n \leq d < 1$;
- (d) $\kappa \leq \gamma_n \leq e < 1$, $\lim_{n \rightarrow \infty} |\gamma_{n+1} - \gamma_n| = 0$,

where a , b , c , d and e are some real numbers. If T is asymptotically regular, then the sequence $\{x_n\}$ converges strongly to some point \bar{x} , where $\bar{x} = P_{F(T) \cap EP(F)}u$.

If T is asymptotically nonexpansive, then Theorem 3.3 is reduced to the following.

Corollary 3.4 *Let C be a nonempty closed convex subset of H . Let $T : C \rightarrow C$ be an asymptotically nonexpansive mapping. Let F be a bifunction from $C \times C$ to \mathbb{R} which satisfies (A1)-(A4). Assume that $F(T) \cap EP(F) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real number sequences in $(0, 1)$. Let $\{x_n\}$ be a sequence in C generated by: $x_1 \in C$ is chosen arbitrarily and*

$$\begin{cases} w_n \in C \text{ such that } F_B(w_n, u) + \frac{1}{r_n} \langle u - w_n, w_n - x_n \rangle \geq 0, & \forall u \in C, \\ z_n = P_C(\alpha_n u + (1 - \alpha_n)w_n), \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)T^n z_n, & \forall n \geq 1. \end{cases}$$

Assume that the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{r_n\}$ satisfy the following restrictions:

- (a) $0 < a \leq r_n \leq b < 2\alpha$, $\lim_{n \rightarrow \infty} |r_{n+1} - r_n| = 0$;
- (b) $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- (c) $0 < c \leq \beta_n \leq d < 1$,

where a, b, c and d are some real numbers. If T is asymptotically regular, then the sequence $\{x_n\}$ converges strongly to some point \bar{x} , where $\bar{x} = P_{F(T) \cap EP(F)}u$.

Let H be a Hilbert space and $f : H \rightarrow (-\infty, +\infty]$ be a proper convex lower semicontinuous function. Then the subdifferential ∂f of f is defined as follows:

$$\partial f(x) = \{y \in H : f(z) \geq f(x) + \langle z - x, y \rangle, z \in H\}, \quad \forall x \in H.$$

From Rockafellar [41], we find that ∂f is maximal monotone. It is easy to verify that $0 \in \partial f(x)$ if and only if $f(x) = \min_{y \in H} f(y)$. Let I_C be the indicator function of C , i.e.,

$$I_C(x) = \begin{cases} 0, & x \in C, \\ +\infty, & x \notin C. \end{cases}$$

Since I_C is a proper lower semicontinuous convex function on H , we see that the subdifferential ∂I_C of I_C is a maximal monotone operator.

Lemma 3.5 [2] *Let C be a nonempty closed convex subset of a real Hilbert space H , Proj_C the metric projection from H onto C , ∂I_C the subdifferential of I_C , where I_C is defined above and $J_\lambda = (I + \lambda \partial I_C)^{-1}$. Then*

$$y = J_\lambda x \iff y = \text{Proj}_C x, \quad x \in H, y \in C.$$

Now, we consider a variation inequality problem.

Theorem 3.6 *Let C be a nonempty closed convex subset of H . Let $T : C \rightarrow C$ be an asymptotically κ -strict pseudocontraction. Let $A : C \rightarrow H$ be an α -inverse-strongly monotone mapping. Assume that $F(T) \cap VI(C, A) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be real number sequences in $(0, 1)$. Let $\{x_n\}$ be a sequence in C generated by: $x_1 \in C$ is chosen arbitrarily and*

$$\begin{cases} z_n = P_C(\alpha_n u + (1 - \alpha_n)P_C(x_n - r_n A x_n)), \\ y_n = \gamma_n z_n + (1 - \gamma_n)T^n z_n, \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)y_n, \quad \forall n \geq 1. \end{cases}$$

Assume that the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{r_n\}$ satisfy the following restrictions:

- (a) $0 < a \leq r_n \leq b < 2\alpha$, $\lim_{n \rightarrow \infty} |r_{n+1} - r_n| = 0$;
- (b) $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- (c) $0 < c \leq \beta_n \leq d < 1$;
- (d) $\kappa \leq \gamma_n \leq e < 1$, $\lim_{n \rightarrow \infty} |\gamma_{n+1} - \gamma_n| = 0$,

where a, b, c, d and e are some real numbers. If T is asymptotically regular, then the sequence $\{x_n\}$ converges strongly to some point \bar{x} , where $\bar{x} = P_{F(T) \cap VI(C, A)}u$.

Proof Put $Bx = \partial I_C$. Next, we show that $VI(C, A) = (A + \partial I_C)^{-1}(0)$. Notice that

$$\begin{aligned} x \in (A + \partial I_C)^{-1}(0) &\iff 0 \in Ax + \partial I_C x \\ &\iff -Ax \in \partial I_C x \end{aligned}$$

$$\iff \langle Ax, y - x \rangle \geq 0$$

$$\iff x \in VI(C, A).$$

From Lemma 3.5, we can conclude the desired conclusion immediately. \square

Competing interests

The author declares that they have no competing interests.

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