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Blow-up of solutions to a class of Kirchhoff equations with strong damping and nonlinear dissipation

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Abstract
The initial boundary value problem of a class of Kirchhoff equations with strong damping and nonlinear dissipation is considered. By modifying Vitillaro's argument, we prove a blow-up result for solutions with positive and negative initial energy respectively.

Keywords: Kirchhoff equation; blow-up; strong damping; nonlinear dissipation

1 Introduction

In this paper, we consider the initial boundary value problem of the following nonlinear wave equations of Kirchhoff type:

$$u_{tt} - \omega\Delta u_t - M\big(\|\nabla u\|^2\big)\Delta u + h(u_t) = f(u), \quad x \in \Omega, t > 0, \tag{1.1}$$

$$u(x,t) = 0, \quad x \in \partial\Omega, t > 0, \tag{1.2}$$

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \quad x \in \Omega, \tag{1.3}$$

where Ω is a bounded domain in R^n, $n \geq 1$, with smooth boundary $\partial\Omega$, so that the divergence theorem can be applied, $M(s) = a + bs^r$, $h(s) = |s|^{m-2}s$, and $f(u) = |s|^{p-2}s$. Here $\omega > 0$, $a > 0$, $b > 0$, $r > 0$, $m \geq 2$ and $p > 2$ are positive constants.

When $M = 1$, equation (1.1) becomes a semilinear hyperbolic problem

$$u_{tt} - \Delta u - \omega\Delta u_t + h(u_t) = f(u), \tag{1.4}$$

and many authors have studied the existence and uniqueness of global solution, the blow-up of the solution (see [1–6] and the references therein).

When M is not a constant function, equation (1.1) without the damping and source terms is often called a Kirchhoff-type wave equation; it has first been introduced by Kirchhoff [7] in order to describe the nonlinear vibrations of an elastic string. When $\omega = 0$ or $h(u_t) = 0$, the nonexistence of the global solutions of Kirchhoff equations was investigated by many authors (see [8–24] and the references therein). The work of Ono [10, 11] dealt with equation (1.1) with $\omega = 0$ and $f(u) = |u|^{p-2}u$. When $h(u_t) = -\Delta u_t$ or u_t, Ono showed that the local solutions blow up in finite time with $E(0) \leq 0$ by applying the concavity method. Ono also combined the so-called potential well method and concavity method to show blow-up properties with $E(0) > 0$. When $h(u_t) = |u_t|^{m-2}u_t$, $m > 2$, Ono proved that the local solution is not global when $p > \max\{2r+2, m\}$ and $E(0) < 0$. Wu [13] extended the result of [11, 12] in the case of $h(u_t) = -\Delta u_t$ or u_t by the energy method and gave some estimates for the life span of solutions. Wu also extended the result of [10] to general $M(s)$ and to the condition that $E(0) \geq 0$ for nonlinear dissipative term $h(u_t) = |u_t|^{m-2}u_t$ by Vitillaro's argument [2]. For more blow-up results of problem (1.1)-(1.3) with $\omega = 0$, $h(u_t) = |u_t|^{m-2}u_t$ and $f(u) = |u|^{p-2}u$ see [14–21].

However, a natural question is whether nonlinear sources can cause finite time blow-up for solutions to problem (1.1)-(1.3) when introducing both the presence of the nonlinear weak damping term $h(u_t) = |u_t|^{m-2}u_t$ and the linear strong damping term Δu_t (*i.e.* $\omega \neq 0$). This question has been addressed for the wave equation (1.4) by Gazzola and Squassina [3] and Yu [4] (see also Graber and Said-Houari [25] for a strongly damped wave equation with dynamic boundary conditions). From the physics point of view, the strong damping term Δu_t and the nonlinear dissipative damping term $h(u_t)$ play a dissipative or inhibitive part in the energy accumulation in the configurations, which dissipates energy and drives the system toward stability, while the nonlinear source term $f(u)$ models an external force that amplifies the energy and drives the system to possible solutions that blow up in finite time. It is well known that if $\omega = 0$, $h(u_t) = |u_t|^{m-2}u_t$, $f(u) = |u|^{p-2}u$, the solutions of (1.4) with any initial data continue to exist globally 'in time' if $m \geq p$ and blow up in finite time if $p > m$ and the initial energy is sufficiently negative or certain positive initial energy (see [1–6] and the references therein). However, introducing both a nonlinear weak damping term $h(u_t)$ and a linear strong damping term Δu_t makes the problem very interesting but difficult as well. Indeed, a strong action of dissipative terms could make the existence of global solutions easier, since they play the role of stabilizing terms and their smoothing effect makes the blow-up more difficult [21, 26]. Introducing a strong damping term Δu_t makes the problem different from the one mentioned in [1]. The most frequently used technique in the proof of blow-up named 'concavity argument' is no longer applied, and the techniques in the papers mentioned above also cannot be used directly due to the term Δu_t. Thereby, at present, less results are at present time known for the wave equation with a strong damping term, and there still exist many other unsolved problems; see Gazzola and Squassina [3] for the case $m = 2$ (see also [4–6, 25] and the references therein).

Recently, Autuori *et al.* [26] studied the blow-up at infinity of polyharmonic Kirchhoff systems with nonlinear damping $h(u_t)$ and strongly damping of Kelvin-Voigt type. Chen and Liu [27] studied the local, global existence and exponential decay result of the following equation:

$$u_{tt} - M\left(\|\nabla u\|^2\right)\Delta u - \Delta u_t + \int_0^t g(t-s)\Delta u(s)\,ds + h(u_t) = f(u), \tag{1.5}$$

and they also proved that the energy will grow at least as an exponential function of time when the weak damping term is nonlinear and will blow up when the weak damping term is linear. But they did not find the result of the blow-up solution when the weak damping term is nonlinear.

Motivated by these papers, the purpose of this paper is to investigate the nonexistence result of global solutions of the problem (1.1)-(1.3) with both terms Δu_t and $h(u_t)$. More precisely, we shall show global nonexistence results of the problem (1.1)-(1.3) by adopting and modifying the method of [2, 17, 26] and combining with potential well theory. We will construct a function $L(t)$ (see Section 3) which is different from that in [2, 5, 17, 26]. The method can also be extended to equation (1.1) with the general function $M(s)$, $h(s)$ and $g(s)$ as in [26], and it can also be extended to equation (1.5) as in [27]. The plan of this article is as follows. In Section 2, some notations, assumptions and preliminaries are introduced and the main results of this article are shown in Section 3.

2 Preliminaries

In this section, we give some assumptions and preliminary results in order to state the main results of this article. Throughout this article, the following notations are used for precise statements: $L^p(\Omega)$ $(1 < p < \infty)$ denotes the usual space of all L^p-functions on Ω with norm $\|u\|_{L^p(\Omega)} = \|u\|_p$ and the inner product $(u, v) = \int_\Omega uv\,dx$. For simplicity, we denote $\|u\|_{L^2(\Omega)} = \|u\|$. The constants C used in this paper are positive generic constants, which may be different in various occurrences. For simplicity, we take $\omega = a = b = 1$.

First, we present the following assumptions.

(A) $p > \max\{2(r+1), m\}$ and $1 < m < p \leq \frac{2(n-1)}{n-2}$ if $n \geq 3$, $1 < m < p \leq \infty$ if $n = 1, 2$.

Next, we present the following local existence theorem, which can be founded in [27].

Theorem 2.1 ([27]) *Suppose that* (A) *hold, and that* $u_0, u_1 \in H^2 \cap H_0^1$, *then the problem* (1.1)-(1.3) *admits a unique solution*

$$u \in C_w^0\big([0,T]; H^2 \cap H_0^1\big) \cap C^0\big([0,T]; H_0^1\big) \cap C_w^1\big([0,T]; H_0^1\big) \cap C^1\big([0,T]; L^2\big),$$

and $u_t \in L^2([0,T]; H_0^1) \cap L^m([0,T] \times \Omega)$, *where the subscript* w *means weak continuity with respect to* t.

Now, for the problem (1.1)-(1.3) we introduce the following function:

$$J(t) = J(u) = \frac{1}{2}\|\nabla u\|^2 + \frac{1}{2(r+1)}\|\nabla u\|^{2(r+1)} - \frac{1}{p}\|u\|_p^p, \tag{2.1}$$

and define the energy of the problem (1.1)-(1.3) by

$$E(t) = E(u) = \frac{1}{2}\|u_t\|^2 + J(u). \tag{2.2}$$

Then we have the following results.

Lemma 2.2 ([27]) *$E(t)$ is a non-increasing function on* $[0, \infty)$ *and*

$$E'(t) = -\|u_t\|_m^m - \|\nabla u_t\|^2 \leq 0. \tag{2.3}$$

We denote $\lambda_1 = B^{-\frac{p}{p-2(r+1)}}$ and $E_1 = (\frac{1}{2(r+1)} - \frac{1}{p})\lambda_1^{2(r+1)}$, where B is the Poincaré constant. From the Poincaré inequality, we get

$$\begin{aligned} E(t) &\geq \frac{1}{2}\|\nabla u\|^2 + \frac{1}{2(r+1)}\|\nabla u\|^{2(r+1)} - \frac{B^p}{p}\|\nabla u\|^p \\ &> \frac{1}{2(r+1)}\left[\|\nabla u\|^2 + \|\nabla u\|^{2(r+1)}\right] - \frac{B^p}{p}\left[\|\nabla u\|^2 + \|\nabla u\|^{2(r+1)}\right]^{\frac{p}{2(r+1)}} \\ &> G\bigl(\lambda(t)\bigr), \end{aligned} \tag{2.4}$$

for $t \geq 0$, $G(\lambda(t)) = \frac{1}{2(r+1)}\lambda^{2(r+1)}(t) - \frac{B^p}{p}\lambda^p(t)$, and $\lambda(t) = [\|\nabla u\|^2 + \|\nabla u\|^{2(r+1)}]^{\frac{1}{2(r+1)}}$. It is easy to verify that $G(\lambda)$ has a maximum at $\lambda_1 = B^{-\frac{p}{p-2(r+1)}}$ and the maximum value is $E_1 = (\frac{1}{2(r+1)} - \frac{1}{p})\lambda_1^{2(r+1)}$. We see that $G(\lambda)$ increases in $(0, \lambda_1)$, and it decreases in (λ_1, ∞), and $G(\lambda) \to -\infty$ as $\lambda \to +\infty$.

A similar argument from [2, 13, 15, 27] gives the following result.

Lemma 2.3 *Assume that* (A) *holds,* $u_0, u_1 \in H^2 \cap H_0^1$ *and let* u *be a solution of the problem* (1.1)-(1.3) *with initial data satisfying* $E(0) < E_1$ *and* $\lambda_0 = [\|\nabla u_0\|^2 + \|\nabla u_0\|^{2(r+1)}]^{\frac{1}{2(r+1)}} \geq \lambda_1$. *Then there exists a constant* $\lambda_2 > \lambda_1$, *such that*

$$\|\nabla u\|^2 + \|\nabla u\|^{2(r+1)} > \lambda_2^{2(r+1)}, \quad \forall t \in [0, T). \tag{2.5}$$

Proof Since $E(0) < E_1$ and $G(\lambda)$ is a continuous function, there exist λ_2' and λ_2 with $\lambda_2' < \lambda_1 < \lambda_2$ such that $G(\lambda_2') = G(\lambda_2) = E(0)$, by (2.4), implies

$$G\bigl(\lambda(0)\bigr) \leq E(0) = G(\lambda_2). \tag{2.6}$$

From the assumption, the properties of $G(\lambda)$ and (2.6), we conclude

$$\lambda(0) \geq \lambda_2. \tag{2.7}$$

If it does not hold, then there exists $t_0 > 0$ such that $\lambda(t_0) = [\|\nabla u(t_0)\|^2 + \|\nabla u(t_0)\|^{2(r+1)}]^{\frac{1}{2(r+1)}} < \lambda_2$. If $\lambda_2' < \lambda(t_0) < \lambda_2$, according to (2.3) and the properties of $G(\lambda)$, we know that $G(\lambda(t_0)) > E(0) \geq E(t_0)$, which contradicts (2.4). If $\lambda(t_0) < \lambda_2'$, then $\lambda(t_0) < \lambda_2' < \lambda_2$. Setting $h(t) = \lambda(t) - \frac{\lambda_2 + \lambda_2'}{2}$, it is clear that $h(t)$ is a continuous function, $h(t_0) < 0$ and $h(0) > 0$ ((2.7)). Hence, there exists $t_1 \in (0, t_0)$ such that $h(t_1) = 0$, which means that $\lambda(t_1) = \frac{\lambda_2 + \lambda_2'}{2}$, implying $G(\lambda(t_1)) > E(0) \geq E(t_1)$, which contradicts (2.4). Then we conclude the result. □

3 Main results

Now, we give our main results.

Theorem 3.1 *Assuming that* (A) *holds and* $u_0, u_1 \in H^2 \cap H_0^1$, *then any solution* u *of the problem* (1.1)-(1.3) *with initial data satisfying* $E(0) < E_1$ *and* $\|\nabla u_0\|^2 + \|\nabla u_0\|^{2(r+1)} \geq \lambda_1^{2(r+1)}$ *will blow up in finite time.*

Proof We set

$$H(t) = E_2 - E(t), \quad \textit{for } t \geq 0, \tag{3.1}$$

where $E_2 \in (E(0), E_1)$. From (2.3) and (3.1), we get

$$H'(t) = -E'(t) = \|u_t\|_m^m + \|\nabla u_t\|^2 > 0; \tag{3.2}$$

then $H(t)$ is an increasing function and

$$H(t) \geq H(0) = E_2 - E(0) > 0. \tag{3.3}$$

On the other hand, by Lemma 2.3, we have

$$\begin{aligned} H(t) &< E_2 - \frac{1}{2}\left[\|u_t\|^2 + \|\nabla u\|^2 + \frac{1}{r+1}\|\nabla u\|^{2(r+1)}\right] + \frac{1}{p}\|u\|_p^p \\ &< E_1 - \frac{1}{2(r+1)}\left[\|\nabla u\|^2 + \|\nabla u\|^{2(r+1)}\right] + \frac{1}{p}\|u\|_p^p \\ &\leq E_1 - \frac{1}{2(r+1)}\lambda_1^{2(r+1)} + \frac{1}{p}\|u\|_p^p \\ &= -\frac{1}{p}\lambda_1^{2(r+1)} + \frac{1}{p}\|u\|_p^p \leq \frac{1}{p}\|u\|_p^p. \end{aligned} \tag{3.4}$$

Hence, combining (3.3) and (3.4) with the embedding $H_0^1 \hookrightarrow L^p$, we have

$$0 < H(0) \leq H(t) \leq \frac{1}{p}\|u\|_p^p \leq \frac{B^p}{p}\|\nabla u\|^p. \tag{3.5}$$

We set

$$G(t) = (u, u_t) + \frac{1}{2}\|\nabla u\|^2,$$

and then we define

$$L(t) = H^{k(1-\alpha)}(t) + \epsilon G(t), \tag{3.6}$$

where $\alpha, k, \epsilon > 0$ are small enough to be chosen later. By the definition of the solution, we have

$$G'(t) = \|u_t\|^2 - \|\nabla u\|^2 - \|\nabla u\|^{2(r+1)} - \int_\Omega |u_t|^{m-2}u_t u\,dx + \|u\|_p^p. \tag{3.7}$$

Adding the term $p(H(t) - E_2 + E(t))$ and using the definition of $E(t)$ in (2.2), then (3.7) becomes

$$\begin{aligned} G'(t) \geq{}& \|u_t\|^2 - \|\nabla u\|^2 - \|\nabla u\|^{2(r+1)} - \int_\Omega |u_t|^{m-2}u_t u\,dx \\ &+ \frac{p}{2}\left[\|u_t\|^2 + \|\nabla u\|^2 + \frac{1}{r+1}\|\nabla u\|^{2(r+1)}\right] + pH(t) - pE_2 \end{aligned}$$

$$
\begin{aligned}
&= \frac{p+2}{2}\|u_t\|^2 + \frac{p-2}{2}\big\|\nabla u(t)\big\|^2 + \frac{p-2(r+1)}{2(r+1)}\|\nabla u\|^{2(r+1)} \\
&\quad - \int_\Omega |u_t|^{m-2}u_t u\,dx + pH(t) - pE_2.
\end{aligned}
\tag{3.8}
$$

By $r > 0$ and Lemma 2.3 again, we have

$$
\begin{aligned}
&\frac{p-2}{2}\big\|\nabla u(t)\big\|^2 + \frac{p-2(r+1)}{2(r+1)}\|\nabla u\|^{2(r+1)} - pE_2 \\
&\geq \frac{p-2(r+1)}{2(r+1)}\big[\big\|\nabla u(t)\big\|^2 + \|\nabla u\|^{2(r+1)}\big] - pE_2 \\
&\geq \frac{p-2(r+1)}{2(r+1)}\frac{\lambda_2^{2(r+1)} - \lambda_1^{2(r+1)}}{\lambda_2^{2(r+1)}}\big[\big\|\nabla u(t)\big\|^2 + \|\nabla u\|^{2(r+1)}\big] \\
&\quad + \frac{p-2(r+1)}{2(r+1)}\frac{\lambda_1^{2(r+1)}[\|\nabla u(t)\|^2 + \|\nabla u\|^{2(r+1)}]}{\lambda_2^{2(r+1)}} - pE_2 \\
&\geq \frac{p-2(r+1)}{2(r+1)}\frac{\lambda_2^{2(r+1)} - \lambda_1^{2(r+1)}}{\lambda_2^{2(r+1)}}\big[\big\|\nabla u(t)\big\|^2 \\
&\quad + \|\nabla u\|^{2(r+1)}\big] + \frac{p-2(r+1)}{2(r+1)}\lambda_1^{2(r+1)} - pE_2.
\end{aligned}
\tag{3.9}
$$

From the fact that $p > 2(r+1)$, Lemma 2.3 and $E_2 < E_1$, we see that

$$
\begin{aligned}
&\frac{p-2(r+1)}{2(r+1)}\frac{\lambda_2^{2(r+1)} - \lambda_1^{2(r+1)}}{\lambda_2^{2(r+1)}} > 0, \\
&\frac{p-2(r+1)}{2(r+1)}\lambda_1^{2(r+1)} - pE_2 > \frac{p-2(r+1)}{2(r+1)}\lambda_1^{2(r+1)} - pE_1 = 0.
\end{aligned}
\tag{3.10}
$$

It follows from (3.8), (3.9) and (3.10) that

$$
\begin{aligned}
G'(t) &\geq \frac{p+2}{2}\|u_t\|^2 + \frac{p-2(r+1)}{2(r+1)}\frac{\lambda_2^{2(r+1)} - \lambda_1^{2(r+1)}}{\lambda_2^{2(r+1)}}\big[\big\|\nabla u(t)\big\|^2 + \|\nabla u\|^{2(r+1)}\big] \\
&\quad - \int_\Omega |u_t|^{m-2}u_t u\,dt + pH(t).
\end{aligned}
\tag{3.11}
$$

From the Hölder inequality, $p > m$ and (3.5), we have

$$
\begin{aligned}
\left|\int_\Omega |u_t|^{m-2}u_t u\,dt\right| &\leq \int_\Omega |u_t|^{m-1}|u|\,dx \\
&\leq \|u\|_m\|u_t\|_m^{m-1} \\
&\leq C\|u\|_p^{1-\frac{p}{m}}\|u\|_p^{\frac{p}{m}}\|u_t\|_m^{m-1} \\
&\leq C\|u\|_p^{\frac{p}{m}}H^{\frac{1}{p}-\frac{1}{m}}(t)\|u_t\|_m^{m-1}.
\end{aligned}
\tag{3.12}
$$

From (3.5), Young's inequality and the fact that $\|u_t\|_m^m \leq H'(t)$, we get

$$
\left|\int_\Omega |u_t|^{m-2}u_t u\,dt\right| \leq C\big[\epsilon_1^m\|u\|_p^p + \epsilon_1^{-\frac{m}{m-1}}H'(t)\big]H^{-\alpha_1}(t),
\tag{3.13}
$$

where $\alpha_1 = \frac{1}{m} - \frac{1}{p} > 0$, $\epsilon_1 > 0$. Now, we take α and k satisfying

$$\begin{aligned} &0 < \alpha < \min\left\{\alpha_1, \frac{1}{2} - \frac{1}{p}, \frac{p-m}{p(m-1)}\right\}, \\ &\max\left\{\frac{1}{2}, 1-\alpha_1, \frac{1}{r+1}, \frac{p+2}{2p}\right\} < k(1-\alpha) < 1, \end{aligned} \tag{3.14}$$

and then we have

$$1 - k(1-\alpha) - \alpha_1 < 0, \qquad 1 < \frac{1}{k(1-\alpha)} < r+1, \qquad k(1-\alpha) > \frac{p+2}{2p}. \tag{3.15}$$

Furthermore, from (3.13) and (3.5), we have

$$\begin{aligned} &\left|\int_\Omega |u_t|^{m-2} u_t u \, dt\right| \\ &\quad \le C\left[\epsilon_1^m H^{-\alpha_1}(0)\|u\|_p^p + \epsilon_1^{-\frac{m}{m-1}} H^{1-k(1-\alpha)-\alpha_1}(0) H^{k(1-\alpha)-1}(t) H'(t)\right]. \end{aligned} \tag{3.16}$$

By differentiating (3.6), we see from (3.11) and (3.16) that

$$\begin{aligned} L'(t) \ge{}& \left[k(1-\alpha) - \epsilon C \epsilon_1^{-\frac{m}{m-1}} H^{1-k(1-\alpha)-\alpha_1}(0)\right] H^{k(1-\alpha)-1}(t) H'(t) \\ &+ \epsilon \frac{p+2}{2}\|u_t\|^2 + \epsilon p H(t) - \epsilon C \epsilon_1^m H^{-\alpha_1}(0)\|u\|_p^p \\ &+ \epsilon \frac{p-2(r+1)}{2(r+1)} \frac{\lambda_2^{2(r+1)} - \lambda_1^{2(r+1)}}{\lambda_2^{2(r+1)}} \left[\left\|\nabla u(t)\right\|^2 + \|\nabla u\|^{2(r+1)}\right]. \end{aligned} \tag{3.17}$$

Letting $\delta = \frac{1}{2}\min\{\frac{p+2}{2}, \frac{p}{2}, \frac{p-2(r+1)}{2(r+1)} \frac{\lambda_2^{2(r+1)} - \lambda_1^{2(r+1)}}{\lambda_2^{2(r+1)}}\}$ and decomposing $\epsilon p H(t)$ in (3.17) by $\epsilon p H(t) = 2\delta\epsilon H(t) + (p - 2\delta)\epsilon H(t)$, we find from (3.3) and (3.17) that

$$\begin{aligned} L'(t) \ge{}& \left[k(1-\alpha) - \epsilon C \epsilon_1^{-\frac{m}{m-1}} H^{1-k(1-\alpha)-\alpha_1}(0)\right] H^{k(1-\alpha)-1}(t) H'(t) \\ &+ \epsilon\left[\frac{2\delta}{p} - C\epsilon_1^m H^{-\alpha_1}(0)\right]\|u\|_p^p + \epsilon\left[\frac{p+2}{2} - \delta\right]\|u_t\|^2 \\ &+ \epsilon\left[\frac{p-2(r+1)}{2(r+1)} \frac{\lambda_2^{2(r+1)} - \lambda_1^{2(r+1)}}{\lambda_2^{2(r+1)}} - \delta\right]\left[\left\|\nabla u(t)\right\|^2 + \|\nabla u\|^{2(r+1)}\right] \\ &+ (p - 2\delta)\epsilon H(t). \end{aligned} \tag{3.18}$$

Choosing $\epsilon_1 > 0$ small enough so that $\epsilon_1^m < \frac{\delta}{pC} H^{\alpha_1}(0)$ and $0 < \epsilon < \frac{k(1-\alpha)}{2C} H^{-(1-k(1-\alpha)-\alpha_1)}(0)\epsilon_1^{\frac{m}{m-1}}$, we have from (3.18)

$$L'(t) \ge C\epsilon\left(\|u\|_p^p + \|u_t\|^2 + H(t) + \left\|\nabla u(t)\right\|^2 + \|\nabla u\|^{2(r+1)}\right), \tag{3.19}$$

for a positive constant C. Therefore, $L(t)$ is a nondecreasing function. Letting ϵ in (3.6) be small enough, we get $L(0) > 0$. Consequently, we obtain $L(t) \ge L(0) > 0$ for $t \ge 0$.

We claim the inequality

$$L'(t) \ge C L(t)^{\frac{1}{k(1-\alpha)}}. \tag{3.20}$$

For the proof of (3.20), we consider two alternatives:

(i) If there exists a $t > 0$ so that $G(t) < 0$, then

$$L(t)^{\frac{1}{k(1-\alpha)}} = \left[H^{k(1-\alpha)}(t) + \epsilon G(t)\right]^{\frac{1}{k(1-\alpha)}} \leq H(t). \tag{3.21}$$

Thus (3.20) follows from (3.21).

(ii) If there exists a $t > 0$ so that $G(t) \geq 0$, since $1 < \frac{1}{k(1-\alpha)} < 1 + r$ by (3.15), then we deduce from (3.6), the Young inequality, the Hölder inequality and the embedding $L^p \hookrightarrow L^2$ that

$$\begin{aligned} L(t)^{\frac{1}{k(1-\alpha)}} &\leq \left[H^{k(1-\alpha)}(t) + \|u\|^{\tau} + \|u_t\|^{s} + \frac{1}{2}\|\nabla u\|^2\right]^{\frac{1}{k(1-\alpha)}} \\ &\leq C\left[H(t) + \|u\|_p^{\frac{\tau}{k(1-\alpha)}} + \|u_t\|^{\frac{s}{k(1-\alpha)}} + \|\nabla u\|^{\frac{2}{k(1-\alpha)}}\right], \end{aligned} \tag{3.22}$$

for $\frac{1}{\tau} + \frac{1}{s} = 1$, $\tau > 0$, $s > 0$. If wee take $s = 2k(1-\alpha)$, then $s > 1$ by (3.14), and $\frac{s}{k(1-\alpha)} = 2$. By (3.15), we have $\frac{\tau}{k(1-\alpha)} = \frac{2}{2k(1-\alpha)-1} < p$, $\frac{2}{k(1-\alpha)} < 2(r+1)$. Furthermore, we get

$$\|u_t\|^{\frac{s}{k(1-\alpha)}} = \|u_t\|^2, \qquad \|u\|_p^{\frac{\tau}{k(1-\alpha)}} = \|u\|_p^{\frac{2}{2k(1-\alpha)-1}}. \tag{3.23}$$

Thus from (3.22), (3.23) and (3.5), we have

$$\begin{aligned} L(t)^{\frac{1}{k(1-\alpha)}} &\leq C\left[H(t) + \|u_t\|^2 + \|u\|_p^{\frac{2}{2k(1-\alpha)-1}-p}\|u\|_p^p + \|\nabla u\|^{\frac{2}{k(1-\alpha)}-2(r+1)}\|\nabla u\|^{2(r+1)}\right] \\ &\leq C\Bigg[H(t) + \|u_t\|^2 + \big(pH(0)\big)^{\frac{1}{p}(\frac{2}{2k(1-\alpha)-1}-p)}\|u\|_p^p \\ &\quad + \left(\frac{p}{B^p}H(0)\right)^{\frac{1}{p}(\frac{2}{k(1-\alpha)}-2(r+1))}\|\nabla u\|^{2(r+1)}\Bigg] \\ &\leq C\left[H(t) + \|u_t\|^2 + \|u\|_p^p + \|\nabla u\|^{2(r+1)}\right]. \end{aligned} \tag{3.24}$$

This inequality together with (3.19) implies (3.20).

Then, by integrating both sides of (3.20) over $[0,t]$, it follows that there exists a $T_0 > 0$ so that

$$\lim_{t\to T_0^-} L(t) = \lim_{t\to T_0^-}\left(H^{k(1-\alpha)}(t) + \epsilon G(t)\right) = \infty. \tag{3.25}$$

This combined with (3.24), (3.21) and (3.5) gives

$$\lim_{t\to T_0^-}\left(\|u\|_p^p + \|\nabla u\|^2 + \|\nabla u\|^{2(r+1)} + \|u_t\|^2\right) = \infty.$$

This theorem is proved. □

Theorem 3.2 *Assuming that* $u_0 \in H^2 \cap H_0^1$, $u_1 \in H_0^1$, *and* $p > \max\{2(r+1), m\}$, $E(0) < 0$, *then the local solution of the problem* (1.1)-(1.3) *blows up in finite time.*

Proof Setting $H(t) = -E(t)$ instead of $H(t)$ in (3.1) and then applying the same arguments as in Theorem 3.1, we get the desired result. □

Remark 3.3 We point out that the method can also be extended to equation (1.1) with the general function $M(s)$, $h(s)$ and $g(s)$ as in [26], and it can also be extended to equation (1.5) as in [27].

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The work presented here was carried out in collaboration among all authors. HW found the motivation of this paper and suggested the outline of the proofs. QY and DJ provided many good ideas for completing this paper. All authors have contributed to, read and approved the manuscript.

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