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# Numerical solution of a stochastic population growth model in a closed system

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## Abstract

In this paper, we introduce a stochastic population model in a closed system. This model is a nonlinear stochastic integro-differential equation. At first, we solve this problem *via* the stochastic  $\theta$ -method. Then we solve it by using the Bernstein polynomials and collocation method. This method reduces integro-differential equation to a system of nonlinear algebraic equations. The results demonstrate applicability and accuracy of this method.

## 1 Introduction

Several phenomena in life and sciences, especially in mechanics, engineering and, since recently, in finance, have been found to depend on random excitations. It therefore seems natural that a current trend in describing and studying these phenomena is focused on the use of stochastic mathematical models rather than deterministic ones.

Having in mind that in many cases random excitations are of the Gaussian white noise type, which is mathematically described as a formal derivative of the Brownian motion, all such phenomena are mathematically modeled and essentially represented by complex stochastic differential or integro-differential equations of the Itô type. In mathematical literature, many population models have been considered, from deterministic and stochastic population models, where the population size is represented by a discrete random variable, to very complex continuous stochastic models [1–3].

This article deals with a mathematical model of the accumulated effect of toxins on a population living in a closed system [4]. We obtain a stochastic model of it. Then we apply numerical methods to solve the problem. At first, we convert it to a stochastic differential equation and solve with the stochastic  $\theta$ -method. Then we convert the problem to a stochastic integral equation (SIE) and introduce Bernstein polynomials for solving the SIE.

Bernstein polynomials are differentiable and integrable piecewise polynomials. In recent years, these polynomials have been used for solving differential and integral equations [5–8]. We use them to solve a nonlinear stochastic integro-differential equation (SIDE) that arises in a population growth model in a closed system. By using Bernstein polynomials and their derivatives along with the collocation method, SIDE is converted to nonlinear algebraic equations.

In Section 2, we review deterministic population growth in a closed system. Section 3 introduces the stochastic population growth in a closed system. Section 4 solves the problem by using the stochastic  $\theta$ -method. In Section 5, we introduce Bernstein polynomials

and convert SIDE to a nonlinear algebraic system. Finally, the conclusion is given in Section 6.

## 2 Preliminaries

**Definition 2.1** (Brownian motion process) A real-valued stochastic process  $B(t)$ ,  $t \in [0, T]$  is called Brownian motion, if it satisfies the following properties

- (i) the process has independent increments for  $0 \leq t_0 \leq t_1 \leq \dots \leq t_n \leq T$ ,
- (ii) for all  $t \geq 0$ ,  $h > 0$ ,  $B(t+h) - B(t)$  is normally distributed with mean 0 and variance  $h$ ,
- (iii) function  $t \rightarrow B(t)$  is continuous a.s.

**Definition 2.2** Suppose  $0 \leq s \leq T$ , let  $D = D(s, T)$  be the class of functions

$$f(t, \omega) : [0, \infty] \times \Omega \rightarrow R$$

that satisfy

- (i) The function  $(t, \omega) \rightarrow f(t, \omega)$  is  $\beta \times F$  measurable, where  $\beta$  is the Borel algebra on  $[0, \infty)$  and  $F$  is the  $\sigma$ -algebra on  $\Omega$ .
- (ii)  $f$  is adapted to  $F_t$ , where  $F_t$  is the  $\sigma$ -algebra generated by the random variables  $B(s)$ ;  $s \leq t$  and adapted means that  $f$  is determined by  $B(s)$ ;  $s \leq t$ .
- (iii)  $E[\int_s^T f(t, \omega)^2 dt] < \infty$ .

**Definition 2.3** (The Itô integral) Let  $f \in D(s, T)$ , then the Itô integral of  $f$  is defined by

$$\int_s^T f(t, \omega) dB(t)(\omega) = \lim_{n \rightarrow \infty} \int_s^T \phi_n(t, \omega) dB(t)(\omega),$$

where,  $\{\phi_n\}$  is a sequence of elementary functions such that

$$E \left[ \int_s^T (f - \phi)^2 dt \right] \rightarrow 0 \quad \text{a.s. } n \rightarrow \infty.$$

See [9].

**Lemma 2.1** Let  $f(t)$  be a regular adapted process such that with probability one  $\int_0^T f^2(t) dt < \infty$ . Then the Itô integral  $\int_0^T f(t) dB(t)$  is defined and can be approximated by

$$\sum_{i=0}^{n-1} f(t_i)(B(t_{i+1}) - B(t_i)) \quad \text{for all } \omega,$$

where,  $\{t_i\}$  is a partition of  $[0, T]$  with  $\delta_n = \max(t_{i+1} - t_i) \rightarrow 0$  as  $n \rightarrow \infty$ .

*Proof* See [10]. □

## 3 Deterministic population growth in a closed system

One possible model for the size of such a population is a model balancing a standard logistic growth with the effect of the build up of toxins, which are detrimental to the population

size. This model may be expressed mathematically as the following Volterra integral equation [11]:

$$\frac{du}{dt} = au - bu^2 - cu \int_0^t u(s) ds, \quad u(0) = u_0,$$

where  $a > 0$  is the birth rate coefficient,  $b > 0$  is the crowding coefficient and the last term contains the integral indicating the ‘total metabolism’ or a total amount of toxins produced since time zero. Since the system is closed, the presence of the toxins term always causes the population level to fall to zero in the long run. Several analytical and numerical methods have been proposed to solve the classical population growth model [12–14].

Before introducing a stochastic model, we apply scale time and population by using non-dimensional variables  $t = \frac{c}{b}\tilde{t}$  and defining  $x = \frac{b}{a}u$  and by defining  $k = \frac{ab}{c}$ , we obtain

$$\frac{dx(t)}{dt} = kx(t) \left( 1 - x(t) - \int_0^t x(s) ds \right), \quad x(0) = x_0. \tag{1}$$

This model is a first-order integro-differential equation. In [11], the author considered two cases  $k = \frac{ab}{c}$  small and  $k = \frac{ab}{c}$  large. He showed that for the case  $k \gg 1$ , where population is weakly sensitive to toxins, a rapid rise occurs along the logistic curve that will reach a peak and then is followed by a slow exponential decay. And for small  $k$ , where populations is strongly sensitive to toxins, the solutions are proportional to  $\text{sech}^2(t)$ .

#### 4 Stochastic population growth in a closed system

In stochastic form, the coefficient  $k$  is not completely definite and depends on some random environment effects. We may replace this coefficient by an average value plus a random function term

$$k = r + \text{error},$$

or

$$k = r + \alpha W(t),$$

where  $W(t) = \frac{dB(t)}{dt}$  is a one-dimensional white noise process and  $B(t)$  is a one-dimensional Brownian motion and  $\alpha$  is a nonrandom coefficient that shows the intensity of noise at time  $t$ . So, the stochastic form of (1) is given by

$$\frac{dx(t)}{dt} = \left( r + \alpha \frac{dB(t)}{dt} \right) x(t) \left( 1 - x(t) - \int_0^t x(s) ds \right), \quad t \geq 0,$$

thus, we can write

$$dx(t) = rx(t) \left( 1 - x(t) - \int_0^t x(s) ds \right) dt + \alpha x(t) \left( 1 - x(t) - \int_0^t x(s) ds \right) dB(t), \quad t \geq 0, \tag{2}$$

with  $x(0) = x_0$ , and  $B = \{B(t), t \geq 0\}$  is a standard Brownian motion defined on a probability space  $\{\Omega, F, \{F_t\}_{t \geq 0}, P\}$  with a filtration  $\{F_t\}_{t \geq 0}$  that is right continuous.

### 5 Numerical solution of SIDE

To find a solution for Eq. (2), we can write it as

$$dx(t) = f(t, x(t)) dt + g(t, x(t)) dB(t), \tag{3}$$

with

$$f(t, x(t)) = rx(t) \left( 1 - x(t) - \int_0^t x(s) ds \right) \tag{4}$$

and

$$g(t, x(t)) = \alpha x(t) \left( 1 - x(t) - \int_0^t x(s) ds \right). \tag{5}$$

By using the stochastic  $\theta$ -method, we get

$$x_{n+1} = x_n + h \left[ (1 - \theta)f(t_n, x_n) + \theta f(t_{n+1}, x_{n+1}) \right] + g(t_n, x_n) \Delta B_n, \tag{6}$$

where  $\Delta B_n = B(t_n) - B(t_{n-1})$ ,  $h = t_n - t_{n-1}$ . Here  $x_n$  denotes the numerical approximation of  $x(t_n)$ . Also, we use the following approximation to compute the integral term:

$$\int_0^{t_n} x(s) ds \simeq \frac{h}{2} \left( x_0 + 2 \sum_{j=0}^{n-1} x_j + x_n \right). \tag{7}$$

By substituting (4), (5) and (7) into (6), the model converts to quadratic for  $x_{n+1}$  and can be solved by the quadratic equation.

The results can be seen in Figures 1 and 2 with  $\theta = 0.5$ .

### 6 Bernstein polynomials and function approximation

The general form of the Bernstein polynomials of the  $n$ th degree over the interval  $[a, b]$  is defined by

$$\beta_{i,n}(t) = \binom{n}{i} \frac{(t-a)^i (b-t)^{n-i}}{(b-a)^n}, \quad a \leq t \leq b, i = 0, 1, \dots, n. \tag{8}$$

Note that each of these  $(n + 1)$  polynomials have degree  $n$  and satisfy the following properties

- (i)  $\beta_{i,n}(t) = 0$  if  $i < 0$  or  $i > n$ .
- (ii)  $\sum_0^n \beta_{i,n}(t) = 1$ .
- (iii)  $\beta_{i,n}(a) = \beta_{i,n}(b) = 0, 1 \leq i \leq n - 1$ .

The Bernstein approximation of  $f : [0, 1] \rightarrow R$  is given by

$$\beta_n(f(t)) = \sum_{i=0}^n \binom{n}{i} f\left(\frac{i}{n}\right) t^i (1-t)^{n-i} = \sum_{i=0}^n c_i \beta_{i,n}, \tag{9}$$

where

$$\beta_{i,n} = \binom{n}{i} t^i (1-t)^{n-i}.$$

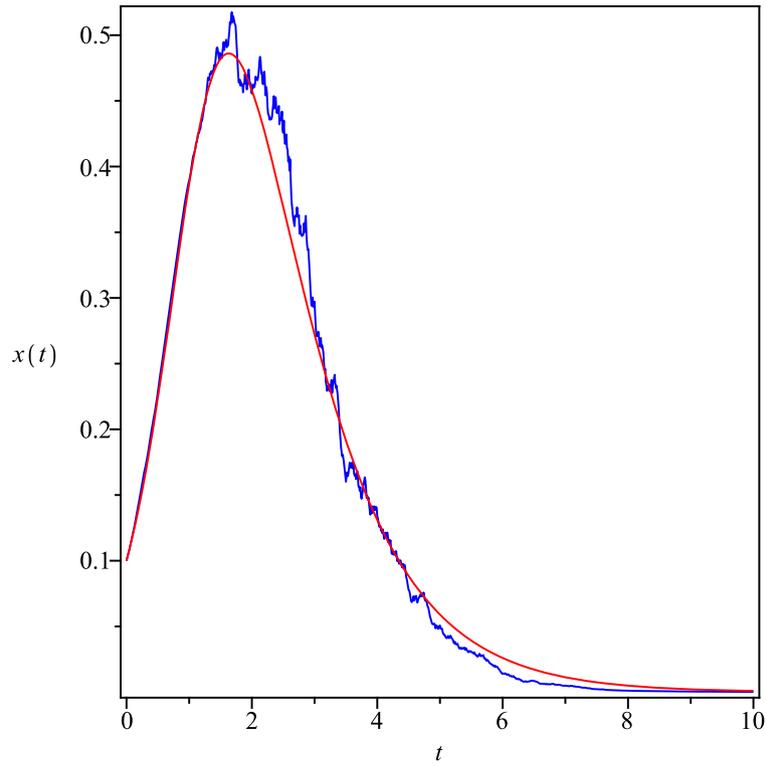


Figure 1 Results of  $\theta$ -method calculations with  $r = 2$ ,  $\alpha = 0$  and  $\alpha = 0.2$ .

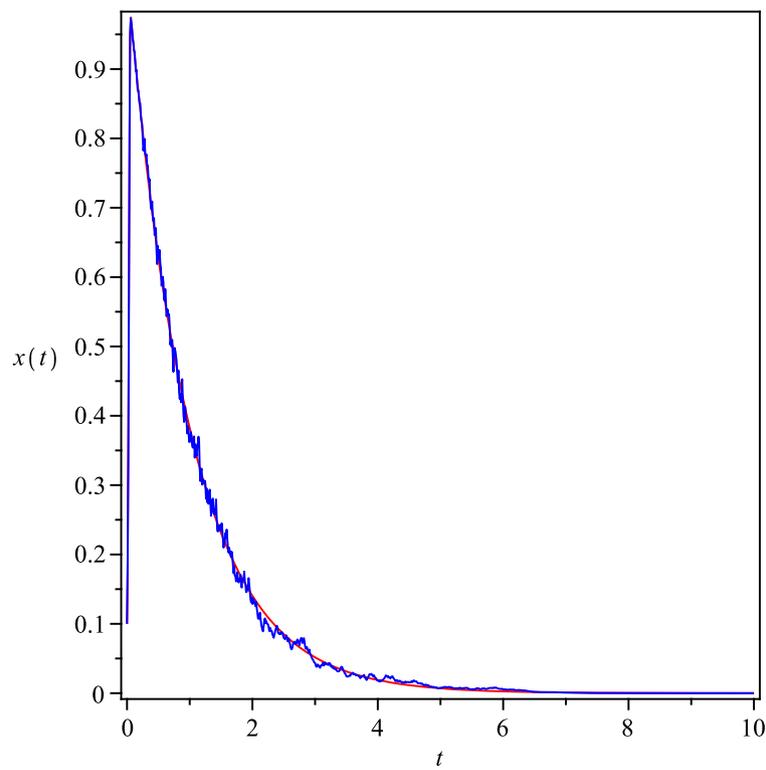


Figure 2 Results of  $\theta$ -method calculations with  $r = 100$ ,  $\alpha = 0$  and  $\alpha = 10$ .

**Theorem 6.1** For all function  $f$  in  $C[0,1]$ , the sequence  $\{\beta_n(f); n = 1, 2, \dots\}$  converges uniformly to  $f$ .

*Proof* See [15]. □

One of the many remarkable properties of the Bernestein approximation is that the derivatives of  $\beta_n(f)$  of any order converge to the corresponding derivatives of  $f$  [16].

If  $f \in C_k[0,1]$  for any  $k \geq 0$ , then

$$\lim_{n \rightarrow \infty} (\beta_n(f))^{(k)} = f^{(k)} \quad \text{uniformly on } [0,1].$$

### 7 The numerical method based on Bernstein polynomials

In this section, Bernstein polynomial basis is used to find a solution for the integral form of Eq. (2) as

$$\begin{aligned} x(t) = & x_0 + r \int_0^t \left[ x(s) \left( 1 - x(s) - \int_0^s x(z) dz \right) \right] ds \\ & + \alpha \int_0^t \left[ x(s) \left( 1 - x(s) - \int_0^s x(z) dz \right) \right] dB(s), \\ & t \geq 0. \end{aligned} \tag{10}$$

Let

$$y(t) = \int_0^t x(s) ds, \tag{11}$$

then

$$\frac{dy(t)}{dt} = x(t), \quad y(0) = 0. \tag{12}$$

To approximate the solution, replace (11) and (12) into (10) and rewrite it as

$$\begin{aligned} y'(t) = & x_0 + r \int_0^t \left[ y'(s) - (y'(s))^2 - y'(s)y(s) \right] ds \\ & + \alpha \int_0^t \left[ y'(s) - (y'(s))^2 - y'(s)y(s) \right] dB(s). \end{aligned} \tag{13}$$

Function  $y(t)$  can be approximated as follows:

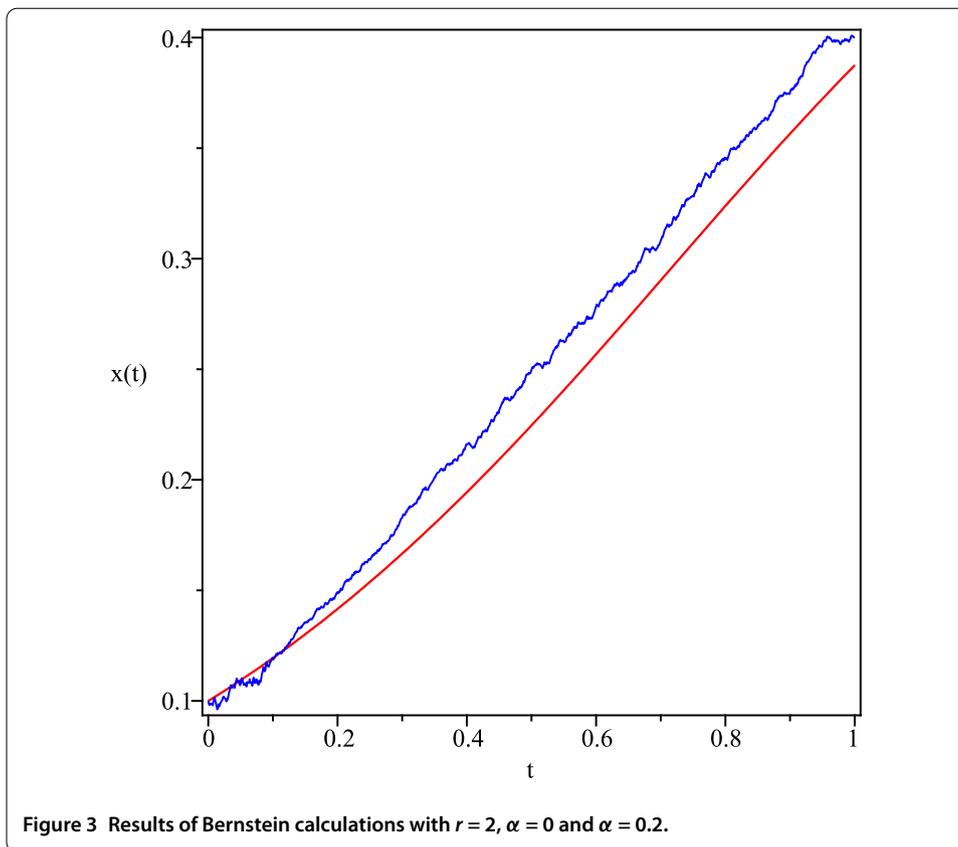
$$y(t) = \sum_{i=0}^n c_i \beta_{i,n} = C^T \Phi(t), \quad n \geq 1, 0 \leq t \leq 1, \tag{14}$$

where,  $C$  and  $\Phi(t)$  are  $(n + 1) \times 1$  vectors given by

$$C = [c_0, c_1, \dots, c_n]^T, \quad \Phi(t) = [\beta_{0,n}, \dots, \beta_{n,n}]^T,$$

so, we can write

$$\frac{dy(t)}{dt} = C^T \frac{d\Phi(t)}{dt}. \tag{15}$$



By substituting (14) and (15) into (13), we have

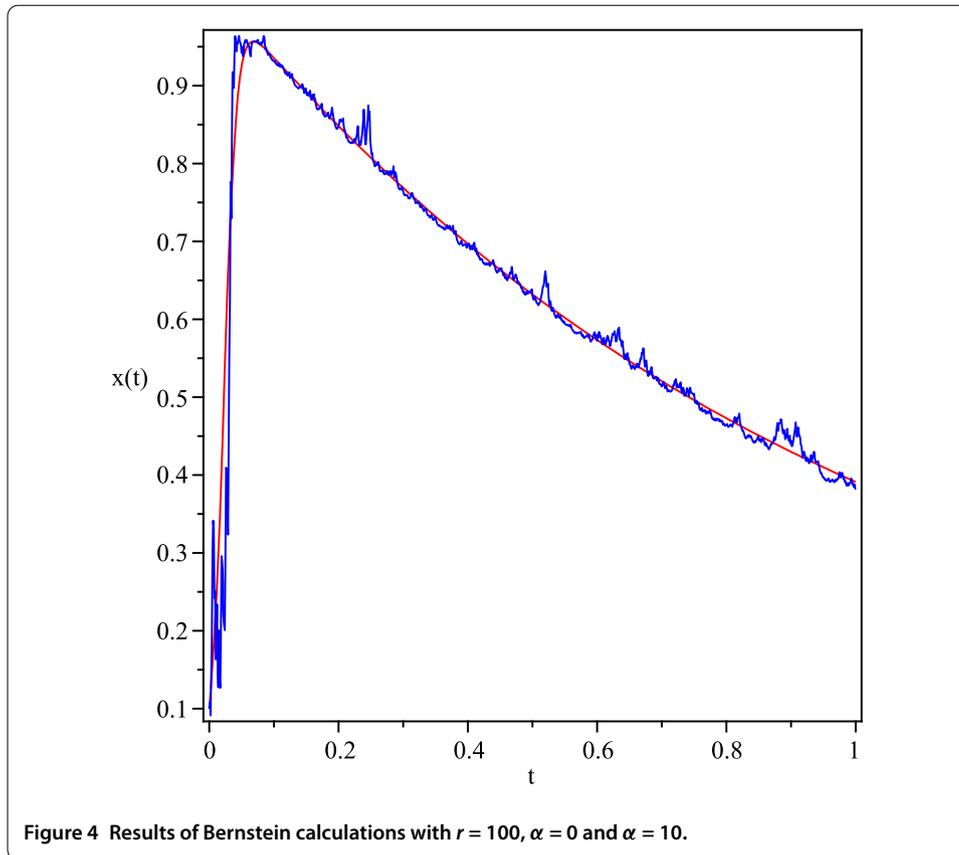
$$\begin{aligned}
 C^T \Phi'(t) &= x_0 + r \int_0^t [C^T \Phi'(s) - (C^T \Phi'(s))^2 - (C^T \Phi'(s))(C^T \Phi(s))] ds \\
 &+ \alpha \int_0^t [C^T \Phi'(s) - (C^T \Phi'(s))^2 - (C^T \Phi'(s))(C^T \Phi(s))] dB(s). \tag{16}
 \end{aligned}$$

The collocation method with  $t_j = \frac{2j-1}{2(n+1)}$ ,  $j = 1, \dots, n + 1$ , is used for determination of the unknown vector  $C$  as follows:

$$\begin{aligned}
 C^T \Phi'(t_j) &= x_0 + r \int_0^{t_j} [C^T \Phi'(s) - (C^T \Phi'(s))^2 - (C^T \Phi'(s))(C^T \Phi(s))] ds \\
 &+ \alpha \int_0^{t_j} [C^T \Phi'(s) - (C^T \Phi'(s))^2 - (C^T \Phi'(s))(C^T \Phi(s))] dB(s), \\
 j &= 1, \dots, n + 1. \tag{17}
 \end{aligned}$$

We use Lemma 2.1 to calculate Itô integrals. By solving the nonlinear system (17), we find the unknown coefficient. Then we get the approximate solution  $y(t)$  and  $x(t)$ .

The figures show the results of a numerical solution generated by the stochastic  $\theta$ -method with  $\theta = 0.5$  and the Bernstein approximation with  $n = 12$  for two cases of  $r$ . The results show a rapid rise along the logistic curve and then a fast exponential decay to zero for big  $r$ .



They also illustrate a comparison between the numerical solutions of the deterministic and the stochastic models.

## 8 Conclusion

This paper suggests a stochastic model for the population growth of a species in a closed system. We applied two numerical methods to solving a nonlinear integro-differential equation and showed the results in Figures 1-4. For big values of  $r$ , the problem becomes very stiff and requires very small steps in the numerical methods. In such a case, it is practical to implement the Bernstein collocation method which is effective and easy to use.

### Competing interests

The authors declare that they have no competing interests.

### Authors' contributions

All authors contributed extensively to the work presented in this paper.

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