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The existence of solutions for nonlinear fractional multipoint boundary value problems at resonance

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Abstract

A class of nonlinear fractional multipoint boundary value problems at resonance is considered in this article. The existence results are obtained by the method of the coincidence degree theory of Mawhin. An example is given to illustrate the results.
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1 Introduction

The subject of fractional calculus has gained considerable popularity during the past decades, due mainly to its frequent appearance in a variety of different areas such as physics, aerodynamics, polymer rheology, etc. (see [1–3]). Many methods have been introduced for solving fractional differential equations (FDEs for short in the remaining), such as the Laplace transform method, the iteration method, the Fourier transform method, etc. (see [4]).

Recently, there have been many works related to the existence of solutions for multipoint boundary value problems (BVPs for short in the remaining) at nonresonance of FDEs (see [5–11]). Motivated by the above articles and recent studies on FDEs (see [12–19]), we consider the existence of solutions for a nonlinear fractional multipoint BVPs at resonance in this article.

In [16], Zhang and Bai considered the following fractional three-point boundary value problems at resonance:

$$\begin{cases} D_{0+}^{\alpha} u(t) = f\left(t, u(t), D_{0+}^{\alpha-(n-1)} u(t), \ldots, D_{0+}^{\alpha-1} u(t)\right) + e(t), & 0 < t < 1, \\ I_{0+}^{n-\alpha} u(0) = D_{0+}^{\alpha-(n-1)} u(0) = \cdots = D_{0+}^{\alpha-2} u(0) = 0, & u(1) = \sigma u(\eta), \end{cases}$$

where $n > 2$ is a natural number; $n - 1 < \alpha \le n$ is a real number; D_{0+}^{α} and I_{0+}^{α} are the standard Riemann-Liouville derivative and integral respectively; $f : [0,1] \times \mathbf{R}^n \mapsto \mathbf{R}$ is continuous; $e(t) \in L[0,1]$; $\sigma \in (0, +\infty)$, $\eta \in (0,1)$ are given constants such that $\sigma \eta^{\alpha-1} = 1$. In their article, they made the operator $Lu = D_{0+}^{\alpha} u$ and got $\dim \operatorname{Ker} L = 1$. In [17], Bai discussed fractional m-point boundary value problems at resonance with the case of $\dim \operatorname{Ker} L = 1$.

In 2010, Bai and Jiang studied the fractional differential equation of boundary value problems at resonance with the case of $\dim \operatorname{Ker} L = 2$ respectively (see [18, 19]), and we

can see that they obtained the results by the assumption that a specific algebraic expression is not equal to zero; for example,

$$R = \frac{1}{\alpha}\eta^{\alpha}\frac{\Gamma(\alpha)\Gamma(\alpha-1)}{\Gamma(2\alpha-1)}\left[1 - \sum_{i=1}^{m}\alpha_i\eta_i^{2\alpha-2}\right] - \frac{1}{\alpha-1}\eta^{\alpha-1}\frac{\Gamma^2(\alpha)}{\Gamma(2\alpha)}\left[1 - \sum_{i=1}^{m}\alpha_i\eta_i^{2\alpha-1}\right] \neq 0$$

is referred to as a condition in [18]. We will show that the assumption like above $R \neq 0$ is not necessary.

In this article, we will use the coincidence degree theory to study the existence of solutions for a nonlinear FDEs at resonance which is given by

$$D_{0+}^{\alpha}u(t) = f\big(t, u(t), D_{0+}^{\alpha-1}u(t), D_{0+}^{\alpha-2}u(t)\big), \quad 0 < t < 1, \tag{1.1}$$

with boundary conditions

$$I_{0+}^{3-\alpha}u(t)|_{t=0} = 0, \qquad D_{0+}^{\alpha-1}u(1) = \sum_{i=1}^{m}a_i D_{0+}^{\alpha-1}u(\xi_i),$$

$$D_{0+}^{\alpha-2}u(1) = \sum_{i=1}^{m}b_i D_{0+}^{\alpha-2}u(\eta_i), \tag{1.2}$$

where $2 < \alpha \le 3$; $0 < \xi_1 < \cdots < \xi_m < 1$; $0 < \eta_1 < \cdots < \eta_m < 1$; $a_i, b_i \in \mathbf{R}$; $f : [0,1] \times \mathbf{R}^3 \mapsto \mathbf{R}$ with satisfying Carathéodory conditions; D_{0+}^{α} and I_{0+}^{α} are the standard Riemann-Liouville fractional derivative and fractional integral respectively.

BVPs (1.1)-(1.2) being at resonance means that the associated linear homogeneous equation $D_{0+}^{\alpha}u(t) = 0$ with boundary conditions (1.2) has $u(t) = at^{\alpha-1} + bt^{\alpha-2}$ as a nontrivial solution, where $0 < t < 1$, $a, b \in \mathbf{R}$.

We will always suppose that the following conditions hold:

$$\sum_{i=1}^{m}a_i = 1, \qquad \sum_{i=1}^{m}b_i\eta_i = 1, \qquad \sum_{i=1}^{m}b_i = 1. \tag{C}$$

The rest of this article is organized as follows: In Section 2, we give some definitions, lemmas and notations. In Section 3, we establish theorems of existence result for BVPs (1.1)-(1.2). In Section 4, we give an example to illustrate our result.

2 Preliminaries

We present here some necessary basic knowledge and definitions of the fractional calculus theory, which can be found in [1–3].

Definition 2.1 The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $y : (0,\infty) \mapsto \mathbf{R}$ is given by

$$I_{0+}^{\alpha}y(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)\,\mathrm{d}s,$$

where $\Gamma(\cdot)$ is the Gamma function, provided the right side is pointwise defined on $(0,\infty)$.

Definition 2.2 The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $y : (0, \infty) \mapsto \mathbf{R}$ is given by

$$D_{0^+}^\alpha y(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t} \right)^n \int_0^t \frac{y(s)}{(t-s)^{\alpha-n+1}} \, \mathrm{d}s,$$

where $n = [\alpha] + 1$, provided the right side is pointwise defined on $(0, \infty)$.

Definition 2.3 ([18]) We say that the map $f : [0,1] \times \mathbf{R}^n \mapsto \mathbf{R}$ satisfies Carathéodory conditions with respect to $L[0,1]$ if the following conditions are satisfied:

  (i) for each $z \in \mathbf{R}^n$, the mapping $t \mapsto f(t, z)$ is Lebesgue measurable;
  (ii) for almost every $t \in [0,1]$, the mapping $t \mapsto f(t, z)$ is continuous on $\mathbf{R}^n$;
  (iii) for each $E > 0$, there exists a $\rho_E \in L[0,1]$ such that, for a.e. $t \in [0,1]$ and every $|u| \le E$, we have $f(t, u) \le \rho_E(t)$.

Lemma 2.4 ([2]) *Assume $y(t) \in C[0,1] \cap L[0,1]$, $0 \le \beta \le \alpha$, then $D_{0^+}^\beta I_{0^+}^\alpha y(t) = I_{0^+}^{\alpha-\beta} y(t)$. And, for all $\alpha \ge 0$, $\beta > -1$, we have that*

$$I_{0^+}^\alpha t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} t^{\alpha+\beta}, \qquad D_{0^+}^\alpha t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}.$$

Lemma 2.5 ([2]) *Let $\alpha > 0$, $n = [\alpha] + 1$ and assume that $y, D_{0^+}^\alpha y \in L(0,1)$, then the following equality holds almost everywhere on $[0,1]$,*

$$\left(I_{0^+}^\alpha D_{0^+}^\alpha y \right)(t) = y(t) - \sum_{i=1}^n \frac{((I_{0^+}^{n-\alpha} y)(t))^{(n-i)}|_{t=0}}{\Gamma(\alpha-i+1)} t^{\alpha-i}.$$

Now, we briefly recall some notations and an abstract existence result, which can be found in [20]. Let Y, Z be real Banach spaces, $L : \mathrm{dom}\, L \subset Y \mapsto Z$ be a Fredholm map of index zero, and $P : Y \mapsto Y$, $Q : Z \mapsto Z$ be continuous projectors such that

$$\mathrm{Im}\, P = \mathrm{Ker}\, L, \qquad \mathrm{Ker}\, Q = \mathrm{Im}\, L, \qquad Y = \mathrm{Ker}\, L \oplus \mathrm{Ker}\, P, \qquad Z = \mathrm{Im}\, Q \oplus \mathrm{Im}\, L.$$

It follows that $L|_{\mathrm{dom}\, L \cap \mathrm{Ker}\, P} : \mathrm{dom}\, L \cap \mathrm{Ker}\, P \mapsto \mathrm{Im}\, L$ is invertible. We denote the inverse by K_p. If Ω is an open bounded subset of Y such that $\mathrm{dom}\, L \cap \Omega \ne \varnothing$, the map $N : Y \mapsto Z$ will be called L-compact on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $K_p(I - Q)N : \bar{\Omega} \mapsto Y$ is compact.

Lemma 2.6 ([20]) *Let L be a Fredholm operator of index zero and N be L-compact on $\bar{\Omega}$. The equation $Lx = Nx$ has at least one solution in $\mathrm{dom}\, L \cap \bar{\Omega}$ if the following conditions are satisfied:*

  (i) *$Lx \ne \lambda Nx$ for each $(x, \lambda) \in [\mathrm{dom}\, L \setminus \mathrm{Ker}\, L \cap \partial\Omega] \times [0,1]$;*
  (ii) *$Nx \notin \mathrm{Im}\, L$ for each $x \in \mathrm{Ker}\, L \cap \partial\Omega$;*
  (iii) *$\deg(JQN|_{\mathrm{Ker}\, L}, \mathrm{Ker}\, L \cap \Omega, 0) \ne 0$,*

where $Q : Z \mapsto Z$ is a projection such that $\mathrm{Ker}\, Q = \mathrm{Im}\, L$ and $J : \mathrm{Im}\, Q \mapsto \mathrm{Ker}\, L$ is a any isomorphism.

In this article, we use the Banach space $C[0,1]$ with the norm $\|u\|_\infty = \max_{t \in [0,1]} |u(t)|$.

Lemma 2.7 ([16]) *Given $\mu > 0$ and $N = [\mu] + 1$, for any $x \in C[0,1]$, $c_i \in \mathbf{R}$ $(i = 1, 2, \ldots, N-1)$, we can define a Banach space*

$$C^\mu[0,1] = \left\{ u(t) | u(t) = I_{0^+}^\mu x(t) + c_1 t^{\mu-1} + c_2 t^{\mu-2} + \cdots + c_{N-1} t^{\mu-(N-1)}, t \in (0,1) \right\},$$

with the norm defined by $\|u\|_{C^\mu} = \|D_{0^+}^\mu u\|_\infty + \cdots + \|D_{0^+}^{\mu-(N-1)} u\|_\infty + \|u\|_\infty$.

Lemma 2.8 ([16]) *$E \subset C^\mu[0,1]$ is a sequentially compact set if and only if E is uniformly bounded and equicontinuous. Here, a uniform bound means that there exists a constant $M > 0$ with each $u \in E$, such that*

$$\|u\|_{C^\mu} = \left\|D_{0^+}^\mu u\right\|_\infty + \cdots + \left\|D_{0^+}^{\mu-(N-1)} u\right\|_\infty + \|u\|_\infty < M,$$

and equicontinuation means that there exists a $\delta > 0$ with $|t_1 - t_2| < \delta$ for any $t_1, t_2 \in [0,1]$, $u \in E$ and $\varepsilon > 0$, such that

$$\left|u(t_1) - u(t_2)\right| < \varepsilon, \qquad \left|D_{0^+}^{\alpha-i} u(t_1) - D_{0^+}^{\alpha-i} u(t_2)\right| < \varepsilon \quad (i = 1, 2, \ldots, N-1).$$

In this article, let $Z = L[0,1]$ with the norm $\|y\|_1 = \int_0^1 |y(s)| \, ds$ and $Y = C^{\alpha-1}[0,1]$ with the norm $\|u\|_Y = \|D_{0^+}^{\alpha-1} u\|_\infty + \|D_{0^+}^{\alpha-2} u\|_\infty + \|u\|_\infty$. Define the operator $L : \operatorname{dom} L \cap Y \mapsto Z$ by

$$Lu = D_{0^+}^\alpha u, \tag{2.1}$$

where $\operatorname{dom} L = \{u \in C^{\alpha-1}[0,1] | D_{0^+}^\alpha u \in Z, u \text{ satisfies } (1.2)\}$. Define the operator $N : Y \mapsto Z$ by

$$Nu(t) = f\left(t, u(t), D_{0^+}^{\alpha-1} u(t), D_{0^+}^{\alpha-2} u(t)\right), \quad \forall t \in [0,1]. \tag{2.2}$$

Thus, BVP (1.1) can be written as $Lu = Nu$ for each $u \in \operatorname{dom} L$.

3 Main results

First, let us introduce the following notations for convenience, with setting $p \in \{1, 2, \ldots, m-1\}$ and $q \in \mathbf{Z}^+$ with $q \geq p + 1$,

$$\Lambda_1 := \frac{1}{p(p+1)} \left(1 - \sum_{i=1}^m b_i \eta_i^{p+1}\right), \qquad \Lambda_2 := \frac{1}{q(q-1)} \left(1 - \sum_{i=1}^m b_i \eta_i^q\right),$$

$$\Lambda_3 := \frac{1}{p} \left(1 - \sum_{i=1}^m a_i \xi_i^p\right), \qquad \Lambda_4 := \frac{1}{q-1} \left(1 - \sum_{i=1}^m a_i \xi_i^{q-1}\right),$$

$$\Lambda := \Lambda_1 \Lambda_4 - \Lambda_2 \Lambda_3, \qquad \mu := 3 + \frac{1}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha-1)},$$

$$\omega := 2 + \frac{1}{\Gamma(\alpha)}, \qquad \rho := 5 + \frac{2}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha-1)}.$$

Then, let us make some assumptions which will be used throughout the article.

(H1) There exist functions $h(t), r(t), s(t), w(t), e(t) \in L[0,1]$ and a constant $\theta \in [0,1)$ such that for all $(x, y, z) \in \mathbf{R}^3$, $t \in [0,1]$,

$$\left| f(t, x, y, z) \right| \le h(t)|x| + r(t)|y| + s(t)|z| + w(t)|z|^\theta + e(t);$$

(H2) For any $u \in \mathrm{dom}\, L$, $t \in [0,1]$, there exists a constant $A > 0$ such that if $D_{0^+}^{\alpha-1} u(t) > A$, then either

$$\Lambda_2 T_1 Nu - \Lambda_4 T_2 Nu < 0, \quad \text{or} \quad \Lambda_2 T_1 Nu - \Lambda_4 T_2 Nu > 0;$$

(H3) For any $u \in \mathrm{dom}\, L$, $t \in [0,1]$, there exists a constant $B > 0$ such that if $D_{0^+}^{\alpha-2} u(t) > B$, then either

$$\Lambda_1 T_1 Nu - \Lambda_3 T_2 Nu > 0, \quad \text{or} \quad \Lambda_1 T_1 Nu - \Lambda_3 T_2 Nu < 0.$$

Theorem 3.1 *If conditions (C), (H1)-(H3) hold, then BVPs (1.1)-(1.2) have at least one solution provided that $\rho(\|h\|_1 + \|r\|_1 + \|s\|_1) < 1$.*

In order to obtain our main result, we first present and prove Lemmas 3.2-3.8. Now, let us define operators $T_j : Z \mapsto Z$ ($j = 1, 2$) as follows:

$$T_1 x(t) = \int_0^1 x(s)\,\mathrm{d}s - \sum_{i=1}^m a_i \int_0^{\xi_i} x(s)\,\mathrm{d}s, \quad \forall t \in [0,1],$$

$$T_2 x(t) = \int_0^1 (1-s)x(s)\,\mathrm{d}s - \sum_{i=1}^m b_i \int_0^{\eta_i} (\eta_i - s)x(s)\,\mathrm{d}s, \quad \forall t \in [0,1].$$

Lemma 3.2 *If condition (C) holds and L is defined by (2.1), then*

$$\mathrm{Ker}\, L = \left\{ at^{\alpha-1} + bt^{\alpha-2} \,|\, a, b \in \mathbf{R} \right\}, \qquad \mathrm{Im}\, L = \{ x \in Z \,|\, T_j x = 0, j = 1, 2 \}.$$

Proof By (2.1) and Lemma 2.5, $D_{0^+}^{\alpha} u(t) = 0$ has a solution

$$u(t) = \frac{D_{0^+}^{\alpha-1} u(t)|_{t=0}}{\Gamma(\alpha)} t^{\alpha-1} + \frac{D_{0^+}^{\alpha-2} u(t)|_{t=0}}{\Gamma(\alpha-1)} t^{\alpha-2} + \frac{I_{0^+}^{3-\alpha} u(t)|_{t=0}}{\Gamma(\alpha-2)} t^{\alpha-3}.$$

Combining with the condition (1.2), we get $\mathrm{Ker}\, L = \{ at^{\alpha-1} + bt^{\alpha-2} \,|\, a, b \in \mathbf{R} \} \cong \mathbf{R}^2$.

Suppose $\forall x \in \mathrm{Im}\, L$, then there exists $u \in \mathrm{dom}\, L$ such that $x = Lu$, i.e., $u \in Y$, $x = D_{0^+}^{\alpha} u$. By Lemma 2.5, we have

$$I_{0^+}^{\alpha} x(t) = u(t) - \frac{D_{0^+}^{\alpha-1} u(t)|_{t=0}}{\Gamma(\alpha)} t^{\alpha-1} - \frac{D_{0^+}^{\alpha-2} u(t)|_{t=0}}{\Gamma(\alpha-1)} t^{\alpha-2} - \frac{I_{0^+}^{3-\alpha} u(t)|_{t=0}}{\Gamma(\alpha-2)} t^{\alpha-3}.$$

Then in view of condition (C), (1.2) and Lemma 2.4, x satisfies

$$\begin{cases} \displaystyle\int_0^1 x(s)\,\mathrm{d}s - \sum_{i=1}^m a_i \int_0^{\xi_i} x(s)\,\mathrm{d}s = 0, \\ \displaystyle\int_0^1 (1-s)x(s)\,\mathrm{d}s - \sum_{i=1}^m b_i \int_0^{\eta_i} (\eta_i - s)x(s)\,\mathrm{d}s = 0. \end{cases} \tag{3.1}$$

On the other hand, suppose $\forall x \in Z$ and it satisfies (3.1), let $u(t) = I_{0+}^{\alpha} x(t)$, then $u \in \operatorname{dom} L$, $D_{0+}^{\alpha} u(t) = x(t)$, i.e., $x \in \operatorname{Im} L$. Therefore, we obtain that

$$\operatorname{Im} L = \{x \in Z \mid T_j x = 0, j = 1, 2\}. \qquad \square$$

Lemma 3.3 *If condition (C) holds, then there exist two constants $q \in \mathbf{Z}^{+}$ and $p \in \{1, 2, \ldots, m-1\}$ with $q \geq p + 1$ such that $\Lambda \neq 0$.*

Proof From $\sum_{i=1}^{m} a_i = 1$, we obtain that for any nonnegative integer l, there exists $k_l - 1 \in \{lm + 1, \ldots, (l+1)m\}$ such that $\sum_{i=1}^{m} a_i \xi_i^{k_l - 1} \neq 1$. If else, we obtain that $\sum_{i=1}^{m} a_i \xi_i^{k_l - 1} = 1$, $k_l - 1 = 0, lm + 1, \ldots, (l+1)m$.

If $l = 0$, we have

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \xi_1 & \xi_2 & \cdots & \xi_m \\ \vdots & \ddots & & \vdots \\ \xi_1^m & \xi_2^m & \cdots & \xi_m^m \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

It is equal to

$$\begin{pmatrix} 1 - \xi_1 & 1 - \xi_2 & \cdots & 1 - \xi_m \\ \xi_1(1 - \xi_1) & \xi_2(1 - \xi_2) & \cdots & \xi_m(1 - \xi_m) \\ \vdots & & \ddots & \vdots \\ \xi_1^{m-1}(1 - \xi_1) & \xi_2^{m-1}(1 - \xi_2) & \cdots & \xi_m^{m-1}(1 - \xi_m) \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Since the determinant of coefficients is not equal to zero, we have that $a_i = 0$ ($i = 1, 2, \ldots, m$), which is a contradiction to condition (C).

If $l \in \mathbf{Z}^{+}$, we get

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \xi_1^{lm+1} & \xi_2^{lm+1} & \cdots & \xi_m^{lm+1} \\ \vdots & \ddots & \vdots \\ \xi_1^{lm+m} & \xi_2^{lm+m} & \cdots & \xi_m^{lm+m} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Similarly, we can deduce that the determinant of coefficients is not equal to zero, so we have that $a_i = 0$ ($i = 1, 2, \ldots, m$), which is a contradiction to condition (C). Thus, there exists $k_l - 1 \in \{lm + 1, \ldots, (l+1)m\}$ such that $\sum_{i=1}^{m} a_i \xi_i^{k_l - 1} \neq 1$.

Similarly, from $\sum_{i=1}^{m} b_i = \sum_{i=1}^{m} b_i \eta_i = 1$, we have that there exists a constant $p \in \{1, 2, \ldots, m - 1\}$ such that

$$\sum_{i=1}^{m} b_i \eta_i^{p+1} \neq 1. \tag{3.2}$$

Let

$$S = \left\{ (k_l - 1) \in \mathbf{Z}^{+} \,\middle|\, \frac{(p+1)(1 - \sum_{i=1}^{m} b_i \eta_i^{k_l})(1 - \sum_{i=1}^{m} a_i \xi_i^{p})}{k_l(1 - \sum_{i=1}^{m} a_i \xi_i^{k_l - 1})} = 1 - \sum_{i=1}^{m} b_i \eta_i^{p+1} \right\},$$

we shall prove that S is a finite set. If else, there exists a strict increasing sequence $\{k_{ln}\}_{n=1}^{\infty}$ such that

$$\frac{(p+1)(1-\sum_{i=1}^{m} b_i \eta_i^{k_{ln}})(1-\sum_{i=1}^{m} a_i \xi_i^{p})}{k_{ln}(1-\sum_{i=1}^{m} a_i \xi_i^{k_{ln}-1})} = 1 - \sum_{i=1}^{m} b_i \eta_i^{p+1}.$$

Since $\sum_{i=1}^{m} b_i \eta_i^{p+1} \neq 1$, we have $\sum_{i=1}^{m} a_i \xi_i^{p} \neq 1$. Thus,

$$1 - \sum_{i=1}^{m} b_i \eta_i^{p+1} = \lim_{k_{ln}\to+\infty} \frac{(p+1)(1-\sum_{i=1}^{m} b_i \eta_i^{k_{ln}})(1-\sum_{i=1}^{m} a_i \xi_i^{p})}{k_{ln}(1-\sum_{i=1}^{m} a_i \xi_i^{k_{ln}-1})} = 0,$$

which is a contradiction to (3.2). Therefore, there exists two constants $p \in \{1, 2, \ldots, m-1\}$ and $q \in \mathbf{Z}^+$ with $q \geq p+1$ such that $\Lambda \neq 0$. □

Lemma 3.4 *If the condition (C) holds and L is defined by (2.1), then L is a Fredholm operator of index zero. Define the linear operator $K_p : \operatorname{Im} L \mapsto \operatorname{dom} L \cap \operatorname{Ker} P$ with $K_p x = I_{0+}^{\alpha} x$, then it is the inverse of L. Furthermore, we have*

$$\|K_p x\|_Y \leq \omega \|x\|_1.$$

Proof For each $p \in \{1, 2, \ldots, m-1\}$ and $q \in \mathbf{Z}^+$ with $q \geq p+1$, define operator $Q : Z \mapsto Z$ by

$$Qx(t) = (Q_1 x(t)) t^{p-1} + (Q_2 x(t)) t^{q-2}, \quad \forall t \in [0,1], \tag{3.3}$$

where

$$Q_1 x(t) = \frac{1}{\Lambda}\left[-\Lambda_2 T_1 x(t) + \Lambda_4 T_2 x(t)\right], \qquad Q_2 x(t) = \frac{1}{\Lambda}\left[\Lambda_1 T_1 x(t) - \Lambda_3 T_2 x(t)\right]. \tag{3.4}$$

It is clear that $\dim \operatorname{Im} Q_1 = 2$. It follows from (3.4), the definition of T_1 and T_2 that

$$\begin{aligned}
Q_1\left((Q_1 x) t^{p-1}\right) &= \frac{1}{\Lambda}\left[-\Lambda_2 T_1\left((Q_1 x) t^{p-1}\right) + \Lambda_4 T_2\left((Q_1 x) t^{p-1}\right)\right] \\
&= \frac{1}{\Lambda}\left[-\Lambda_2 \Lambda_3 (Q_1 x) + \Lambda_1 \Lambda_4 (Q_1 x)\right] = Q_1 x,
\end{aligned} \tag{3.5}$$

similarly, we can derive that

$$Q_1\left((Q_2 x) t^{q-2}\right) = 0, \qquad Q_2\left((Q_1 x) t^{p-1}\right) = 0, \qquad Q_2\left((Q_2 x) t^{q-2}\right) = Q_2 x. \tag{3.6}$$

Hence, for each $x \in Z$ and $t \in [0,1]$, it follows from the (3.3)-(3.6) that

$$\begin{aligned}
Q^2 x &= Q_1\left[(Q_1 x) t^{p-1} + (Q_2 x) t^{q-2}\right] t^{p-1} + Q_2\left[(Q_1 x) t^{p-1} + (Q_2 x) t^{q-2}\right] t^{q-2} \\
&= (Q_1 x) t^{p-1} + (Q_2 x) t^{q-2} = Qx.
\end{aligned}$$

Furthermore, Q is a continuous linear projector.

For each $x \in \operatorname{Im} L$, we have $Qx = 0$, i.e., $x \in \operatorname{Ker} Q$. On the other hand, for each $x \in \operatorname{Ker} Q$, we have that

$$\begin{cases} -\Lambda_2 T_1 x + \Lambda_4 T_2 x = 0, \\ \Lambda_1 T_1 x - \Lambda_3 T_2 x = 0. \end{cases}$$

However, the determinant of coefficients is as follows

$$\begin{vmatrix} -\Lambda_2 & \Lambda_4 \\ \Lambda_1 & -\Lambda_3 \end{vmatrix} = -\Lambda \neq 0$$

then we have $T_j x = 0$ $(j = 1, 2)$, i.e., $x \in \operatorname{Im} L$. Thus, $\operatorname{Ker} Q = \operatorname{Im} L$.

Take any $x \in Z$ in the type $x = (x - Qx) + Qx$, obviously, $x - Qx \in \operatorname{Ker} Q = \operatorname{Im} L$ and $Qx \in \operatorname{Im} Q$, so $Z = \operatorname{Im} L + \operatorname{Im} Q$. For any $x \in \operatorname{Im} L \cap \operatorname{Im} Q$ with $x = at^{p-1} + bt^{q-2}$, by Lemma 3.2, we have

$$\int_0^1 \left(as^{p-1} + bs^{q-2} \right) \mathrm{d}s - \sum_{i=1}^m a_i \int_0^{\xi_i} \left(as^{p-1} + bs^{q-2} \right) \mathrm{d}s = 0,$$

$$\int_0^1 (1-s) \left(as^{p-1} + bs^{q-2} \right) \mathrm{d}s - \sum_{i=1}^m b_i \int_0^{\eta_i} (\eta_i - s) \left(as^{p-1} + bs^{q-2} \right) \mathrm{d}s = 0.$$

That is,

$$\begin{cases} a\Lambda_3 + b\Lambda_4 = 0, \\ a\Lambda_1 + b\Lambda_2 = 0, \end{cases}$$

but the determinant of coefficients is as follows

$$\begin{vmatrix} \Lambda_3 & \Lambda_4 \\ \Lambda_1 & \Lambda_2 \end{vmatrix} = -\Lambda \neq 0,$$

we can deduce that $a = b = 0$. Hence, $\operatorname{Im} L \cap \operatorname{Im} Q = 0$. Furthermore, we get $Z = \operatorname{Im} L \oplus \operatorname{Im} Q$. Therefore, $\dim \operatorname{Ker} L = \dim \operatorname{Im} Q = \operatorname{codim} \operatorname{Im} L = 2$, which means that L is a Fredholm operator of index zero.

Let operator $P : Y \mapsto Y$ and

$$Pu(t) = \frac{D_{0+}^{\alpha-1} u(t)|_{t=0}}{\Gamma(\alpha)} t^{\alpha-1} + \frac{D_{0+}^{\alpha-2} u(t)|_{t=0}}{\Gamma(\alpha-1)} t^{\alpha-2}, \quad \forall t \in [0,1]. \tag{3.7}$$

It is easy to calculate that $Pu(t) = P^2 u(t)$; furthermore, P is a continuous linear projector. Obviously

$$\operatorname{Ker} P = \left\{ u \in Y \,|\, D_{0+}^{\alpha-1} u(0) = D_{0+}^{\alpha-2} u(0) = 0 \right\}.$$

It is clear that $Y = \operatorname{Ker} L \oplus \operatorname{Ker} P$.

For any $x \in \operatorname{Im} L$, in view of the definition of operators K_P and L, we have $LK_P x = L I_{0+}^{\alpha} x = D_{0+}^{\alpha} I_{0+}^{\alpha} x = x$. On the other hand, if $u \in \operatorname{dom} L \cap \operatorname{Ker} P$, we have $D_{0+}^{\alpha-1} u(0) = D_{0+}^{\alpha-2} u(0) = 0$,

$u \in \operatorname{dom} L$. Therefore, by Lemma 2.5 and definitions of operators K_p and L, we know that $(K_p L)u = u$, which implies that $K_p = [L|_{\operatorname{dom} L \cap \operatorname{Ker} P}]^{-1}$. By the definition of K_p, we have

$$K_p x(t) = I_{0+}^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s)\, ds, \quad \forall x \in \operatorname{Im} L.$$

It follows from Lemma 2.4 that

$$D_{0+}^{\alpha-1}(K_p x)(t) = \int_0^t (t-s)x(s)\, ds, \qquad D_{0+}^{\alpha-2}(K_p x)(t) = \int_0^t x(s)\, ds.$$

Then, we have

$$\|K_p x\|_\infty \le \frac{1}{\Gamma(\alpha)} \|x\|_1, \qquad \left\|D_{0+}^{\alpha-1}(K_p x)\right\|_\infty \le \|x\|_1, \qquad \left\|D_{0+}^{\alpha-2}(K_p x)\right\|_\infty \le \|x\|_1.$$

By the definition of the norm in space Y, we get $\|K_p x\|_Y \le \omega \|x\|_1$. $\qquad\square$

Lemma 3.5 *Assume $\Omega \subset Y$ is an open bounded subset such that $\operatorname{dom} L \cap \bar\Omega \ne \varnothing$, and N is defined by (2.2), then N is L-compact on $\bar\Omega$.*

Proof In order to prove N is L-compact, we only need to prove that $QN(\bar\Omega)$ is bounded and $K_p(I-Q)N(u): \bar\Omega \mapsto Y$ is compact. Since the function f satisfies Carathéodory conditions and $u \in \bar\Omega$, for each $E > 0$, there exists a $\rho_E(t) \in L[0,1]$ such that, for a.e. $t \in [0,1]$ and every $|u| \le E$, we have $f \le \rho_E$. By the definition of operators Q and K_p on the interval $[0,1]$, it is easy to get that $QN(\bar\Omega)$ and $K_p(I-Q)N(\bar\Omega)$ are bounded. Thus, there exists a constant $r > 0$ with each $t \in [0,1]$, such that $|QNu(t)| \le r$.

For all $0 \le t_1 < t_2 \le 1$, $2 < \alpha \le 3$, $u \in \bar\Omega$, we have

$$
\begin{aligned}
&\left| K_p(I-Q)Nu(t_2) - K_p(I-Q)Nu(t_1) \right| \\
&\quad = \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_1} \left[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1} \right](I-Q)Nu(s)\, ds \right. \\
&\qquad \left. + \int_{t_1}^{t_2} (t_2-s)^{\alpha-1}(I-Q)Nu(s)\, ds \right| \\
&\quad \le \frac{1}{\Gamma(\alpha)} \left\{ \int_0^{t_1} \left[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1} \right]\left(r + |\rho_E(s)| \right) ds \right. \\
&\qquad \left. + \int_{t_1}^{t_2} (t_2-s)^{\alpha-1}\left(r + |\rho_E(s)| \right) ds \right\} \\
&\quad \le \frac{1}{\Gamma(\alpha)} \left\{ \int_0^{t_2} |\rho_E(s)|(t_2-s)^{\alpha-1}\, ds - \int_0^{t_1} |\rho_E(s)|(t_1-s)^{\alpha-1}\, ds \right\} + \frac{r}{\Gamma(\alpha+1)}\left(t_2^\alpha - t_1^\alpha \right),
\end{aligned}
$$

and

$$
\begin{aligned}
&\left| D^{\alpha-1}K_p(I-Q)Nu(t_2) - D^{\alpha-1}K_p(I-Q)Nu(t_1) \right| \\
&\quad = \left| \int_0^{t_2} (I-Q)Nu(s)\, ds - \int_0^{t_1} (I-Q)Nu(s)\, ds \right| \\
&\quad \le r(t_2 - t_1) + \int_{t_1}^{t_2} |\rho_E(s)|\, ds.
\end{aligned}
$$

Since t^α is uniformly continuous on $[0,1]$ and $\rho_E(t) \in L[0,1]$, so $K_p(I - Q)N(\bar{\Omega})$ and $D^{\alpha-1}K_p(I - Q)N(\bar{\Omega})$ are equicontinuous. By Lemma 2.8, we get that $K_p(I - Q)N : Y \mapsto Y$ is completely continuous. $\qquad\square$

Lemma 3.6 *Suppose (H1)-(H3) hold, then the set* $\Omega_1 = \{u \in \mathrm{dom}\, L \backslash \mathrm{Ker}\, L : Lu = \lambda Nu, \lambda \in [0,1]\}$ *is bounded.*

Proof Taking any $u \in \Omega_1$, then we have $Lu = \lambda Nu$, which yields $\lambda \neq 0$ and $Nu \in \mathrm{Im}\, L = \mathrm{Ker}\, Q$, i.e., $QNu = 0$ for all $t \in [0,1]$. It follows from (H2) and (H3) that there exists $t_0 \in [0,1]$ such that $|D_{0+}^{\alpha-1}u(t_0)| + |D_{0+}^{\alpha-2}u(t_0)| \leq A + B$. Then we can get that

$$D_{0+}^{\alpha-1}u(t) = D_{0+}^{\alpha-1}u(t_0) + \int_{t_0}^{t} D_{0+}^{\alpha}u(s)\,\mathrm{d}s,$$

$$D_{0+}^{\alpha-2}u(t) = D_{0+}^{\alpha-2}u(t_0) + \int_{t_0}^{t} D_{0+}^{\alpha-1}u(s)\,\mathrm{d}s.$$

Furthermore, we have that, with setting $M = A + B$,

$$
\begin{aligned}
\left|D_{0+}^{\alpha-1}u(0)\right| &\leq \left\|D_{0+}^{\alpha-1}u(t)\right\|_\infty \leq \left|D_{0+}^{\alpha-1}u(t_0)\right| + \left\|D_{0+}^{\alpha}u\right\|_1 \\
&\leq M + \|Lu\|_1 \leq M + \|Nu\|_1,
\end{aligned}
\tag{3.8}
$$

$$
\begin{aligned}
\left|D_{0+}^{\alpha-2}u(0)\right| &\leq \left\|D_{0+}^{\alpha-2}u(t)\right\|_\infty \leq \left|D_{0+}^{\alpha-2}u(t_0)\right| + \left\|D_{0+}^{\alpha-1}u\right\|_\infty \\
&\leq \left|D_{0+}^{\alpha-2}u(t_0)\right| + \left|D_{0+}^{\alpha-1}u(t_0)\right| + \left\|D_{0+}^{\alpha}u\right\|_1 \\
&\leq M + \|Lu\|_1 \leq M + \|Nu\|_1.
\end{aligned}
\tag{3.9}
$$

By (3.7)-(3.9) and Lemma 2.4, we have that

$$\|Pu\|_Y \leq \mu\big(M + \|Nu\|_1\big).$$

As before, for any $u \in \Omega_1$, we have $(I - P)(u) \in \mathrm{dom}\, L \cap \mathrm{Ker}\, P$ and $LP(u) = 0$. From Lemma 3.4 and for each $\lambda \in (0,1]$, we can get

$$\left\|(I - P)(u)\right\|_Y = \left\|K_p L(I - P)(u)\right\|_Y = \left\|K_p(Lu)\right\|_Y \leq \omega\|Nu\|_1.$$

Furthermore, we have

$$\|u\|_Y \leq \left\|(I - P)(u)\right\|_Y + \left\|P(u)\right\|_Y \leq \rho\|Nu\|_1 + \mu M.$$

By (H1) and the definition of N, we have

$$\|u\|_Y \leq \rho\Big[\|h\|_1\|u\|_\infty + \|r\|_1\left\|D_{0+}^{\alpha-1}u\right\|_\infty \|s\|_1\left\|D_{0+}^{\alpha-2}u\right\|_\infty + \|w\|_1\left\|D_{0+}^{\alpha-2}u\right\|_\infty^\theta + D\Big],$$

where $D = \|e\|_1 + \mu M/\rho$. Since $\max\{\|u\|_\infty, \|D_{0+}^{\alpha-1}u\|_\infty, \|D_{0+}^{\alpha-2}u\|_\infty\} \leq \|u\|_Y$ and $\rho(\|h\|_1 + \|r\|_1 + \|s\|_1) < 1$ hold true, we can get that

$$\|u\|_\infty \leq \frac{\rho}{1 - \rho\|h\|_1}\Big[\|r\|_1\left\|D_{0+}^{\alpha-1}u\right\|_\infty + \|s\|_1\left\|D_{0+}^{\alpha-2}u\right\|_\infty + \|w\|_1\left\|D_{0+}^{\alpha-2}u\right\|_\infty^\theta + D\Big],$$

which yield that

$$\left\| D_{0+}^{\alpha-1} u \right\|_\infty \le \frac{\rho}{1 - \rho \|h\|_1 - \rho \|r\|_1} \left[\|s\|_1 \left\| D_{0+}^{\alpha-2} u \right\|_\infty + \|w\|_1 \left\| D_{0+}^{\alpha-2} u \right\|_\infty^\theta + D \right].$$

Furthermore, from the previous inequalities, we know that

$$\left\| D_{0+}^{\alpha-2} u \right\|_\infty \le \frac{\rho}{1 - \rho \|h\|_1 - \rho \|r\|_1 - \rho \|s\|_1} \left(\|w\|_1 \left\| D_{0+}^{\alpha-2} u \right\|_\infty^\theta + D \right).$$

Since $\theta \in [0, 1)$, there exist constants $m_1, m_2, m_3 > 0$ such that

$$\left\| D_{0+}^{\alpha-2} u \right\|_\infty \le m_1, \qquad \left\| D_{0+}^{\alpha-1} u \right\|_\infty \le m_2, \qquad \|u\|_\infty \le m_3.$$

Therefore, Ω_1 is bounded. □

Lemma 3.7 *Suppose (H2) and (H3) hold, then the set $\Omega_2 = \{u \in \operatorname{Ker} L : Nu \in \operatorname{Im} L\}$ is bounded.*

Proof For any $u \in \Omega_2$ and $a, b \in \mathbf{R}$, then $u(t) = at^{\alpha-1} + bt^{\alpha-2}$ and $QNu = 0$. By (H2), we get that $|D_{0+}^{\alpha-1} u(t)| = |a\Gamma(\alpha)| \le A$, then we have $|a| \le A/\Gamma(\alpha)$. By (H3), we have that $|D_{0+}^{\alpha-2} u(t)| = |at\Gamma(\alpha) + b\Gamma(\alpha - 1)| \le B$, thus $|b| \le (B + A)/\Gamma(\alpha - 1)$. Therefore, Ω_2 is bounded. □

Lemma 3.8 *If the first parts of (H2) and (H3) hold, then the set $\Omega_3 = \{u \in \operatorname{Ker} L : \lambda J^{-1} u + (1 - \lambda) QNu = 0, \lambda \in [0, 1]\}$ is bounded.*

Proof Taking any $u \in \Omega_3$ and $a, b \in \mathbf{R}$, we have $u(t) = at^{\alpha-1} + bt^{\alpha-2}$. For all $t \in [0, 1]$, we define the isomorphism $J^{-1} : \operatorname{Ker} L \mapsto \operatorname{Im} Q$ by

$$J^{-1} \left(at^{\alpha-1} + bt^{\alpha-2} \right) = \frac{a^2}{\Lambda} t^{p-1} + \frac{b^2}{\Lambda} t^{q-2}.$$

By the definition of the set Ω_3, we can get that

$$\begin{cases} \lambda a^2 + (1 - \lambda) \left[-\Lambda_2 T_1 N \left(at^{\alpha-1} + bt^{\alpha-2} \right) + \Lambda_4 T_2 N \left(at^{\alpha-1} + bt^{\alpha-2} \right) \right] = 0, \\ \lambda b^2 + (1 - \lambda) \left[\Lambda_1 T_1 N \left(at^{\alpha-1} + bt^{\alpha-2} \right) - \Lambda_3 T_2 N \left(at^{\alpha-1} + bt^{\alpha-2} \right) \right] = 0. \end{cases} \tag{3.10}$$

If $\lambda = 0$, we have

$$T_1 N \left(at^{\alpha-1} + bt^{\alpha-2} \right) = 0, \qquad T_2 N \left(at^{\alpha-1} + bt^{\alpha-2} \right) = 0.$$

By the first parts of (H2) and (H3), similar to the proof of Lemma 3.7, then

$$|a| \le \frac{A}{\Gamma(\alpha)}, \qquad |b| \le \frac{B + A}{\Gamma(\alpha - 1)}.$$

Therefore, Ω_3 is bounded.

If $\lambda = 1$, we have $a = b = 0$.

If $0 < \lambda < 1$, we get that $|D^{\alpha-1}u(t)| \leq A$ and $|D^{\alpha-2}u(t)| \leq B$, similar to the proof of Lemma 3.7, Ω_3 is bounded. If else, we have that $\Lambda_2 T_1 Nu - \Lambda_4 T_2 Nu < 0$ and $\Lambda_1 T_1 Nu - \Lambda_3 T_2 Nu > 0$. It contradicts (3.10), thus Ω_3 is bounded. $\square$

Remark 3.9 If the other parts of (H2) and (H3) hold, then the set $\Omega_3 = \{u \in \operatorname{Ker} L : -\lambda J^{-1}u + (1 - \lambda)QNu = 0, \lambda \in [0, 1]\}$ is bounded.

Now with Lemmas 3.2-3.8 in hands, we can begin to prove our main result - Theorem 3.1.

Proof of Theorem 3.1 Assume that Ω is a bounded open set of Y with $\bigcup_{i=1}^{3} \bar{\Omega} \subset \Omega$. By Lemma 3.5, N is L-compact on $\bar{\Omega}$. Then by Lemmas 3.6 and 3.7, we have

(i) $Lu \neq \lambda Nu$ for every $(u, \lambda) \in [\operatorname{dom} L \setminus \operatorname{Ker} L \cap \partial \Omega] \times [0, 1]$;

(ii) $Nu \notin \operatorname{Im} L$ for every $u \in \operatorname{Ker} L \cap \partial \Omega$.

Finally, we will prove that (iii) of Lemma 2.6 is satisfied. We let I as the identity operator in the Banach space Y and $H(u, \lambda) = \pm \lambda J^{-1}(u) + (1 - \lambda)QN(u)$, according to Lemma 3.8 (or Remark 3.9) we know that for all $u \in \partial \Omega \cap \operatorname{Ker} L$, $H(u, \lambda) \neq 0$. By the homotopic property of degree, we have

$$\deg(JQN|_{\operatorname{Ker} L}, \operatorname{Ker} L \cap \Omega, 0) = \deg\big(H(\cdot, 0), \operatorname{Ker} L \cap \Omega, 0\big)$$

$$= \deg\big(H(\cdot, 1), \operatorname{Ker} L \cap \Omega, 0\big)$$

$$= \deg(\pm I, \operatorname{Ker} L \cap \Omega, 0) \neq 0,$$

so (iii) of Lemma 2.6 is satisfied.

Consequently, by Lemma 2.6, the equation $Lu = Nu$ has at least one solution in $\operatorname{dom} L \cap \bar{\Omega}$. Namely, BVPs (1.1)-(1.2) have at least one solution in the space Y. $\square$

According to Theorem 3.1, we have the following corollary.

Corollary 3.10 *Suppose that (H1) is replaced by the following condition,*

(H4) there exist functions $h(t), r(t), s(t), w(t), e(t) \in L[0, 1]$ and a constant $\theta \in [0, 1)$ such that for all $(x, y, z) \in \mathbf{R}^3$, $t \in [0, 1]$,

$$\big|f(t, x, y, z)\big| \leq h(t)|x| + r(t)|y| + s(t)|z| + w(t)|y|^{\theta} + e(t),$$

or

$$\big|f(t, x, y, z)\big| \leq h(t)|x| + r(t)|y| + s(t)|z| + w(t)|x|^{\theta} + e(t),$$

and the others in Theorem 3.1 are not changed, then BVPs (1.1)-(1.2) have at least one solution.

4 An example

Example Consider the following boundary value problem for all $t \in (0,1)$:

$$
\begin{cases}
D^{\frac{13}{5}} u(t) = \dfrac{7u(t)}{200} + \dfrac{9 D_{0+}^{\frac{8}{5}} u(t)}{500} + \dfrac{D_{0+}^{\frac{3}{5}} u(t)}{60} + \dfrac{3 \sin^{\frac{1}{7}} u(t)}{80} + \cos^2 t + 1, \\
I_{0+}^{\frac{2}{5}} u(0) = 0, \qquad D_{0+}^{\frac{8}{5}} u(1) = \dfrac{2}{5} D_{0+}^{\frac{8}{5}} u\left(\dfrac{1}{3}\right) + \dfrac{3}{5} D_{0+}^{\frac{8}{5}} u\left(\dfrac{1}{2}\right), \\
D_{0+}^{\frac{3}{5}} u(1) = -6 D_{0+}^{\frac{3}{5}} u\left(\dfrac{1}{8}\right) + 7 D_{0+}^{\frac{3}{5}} u\left(\dfrac{1}{4}\right).
\end{cases}
\tag{4.1}
$$

Let $\alpha = 13/5$, and $a_1 = 2/5$, $\xi_1 = 1/3$, $a_2 = 3/5$, $\xi_2 = 1/2$, $b_1 = -6$, $\eta_1 = 1/8$, $b_2 = 7$, $\eta_2 = 1/4$. We can get that the condition (C) holds, i.e., $\sum_{i=1}^2 a_i = 1$, $\sum_{i=1}^2 b_i \eta_i = 1$, $\sum_{i=1}^2 b_i = 1$. Moreover,

$$
f(t, x, y, z) = \frac{7}{200} x + \frac{9}{500} y + \frac{1}{60} z + \frac{3}{80} \sin^{1/7} x + \cos^2 t + 1.
$$

Thus, we have

$$
\left| f(t, x, y, z) \right| \le \frac{7}{200} |x| + \frac{9}{500} |y| + \frac{1}{60} |z| + \frac{3}{80} |x|^{1/7} + 2.
$$

Taking $h = 7/200$, $r = 9/500$, $s = 1/60$, $w = 3/80$, $e = 2$, $\Lambda_1 = (1 - \sum_{i=1}^2 b_i \eta_i^2)/2$, $\Lambda_2 = (1 - \sum_{i=1}^2 b_i \eta_i^2)/2$, $\Lambda_3 = 1 - \sum_{i=1}^2 a_i \xi_i$, $\Lambda_4 = 1 - \sum_{i=1}^2 a_i \xi_i$, $\mu = 3 + \Gamma^{-1}(13/5) + \Gamma^{-1}(8/5)$, $\omega = 2 + \Gamma^{-1}(13/5)$, $\rho = 5 + 2\Gamma^{-1}(13/5) + \Gamma^{-1}(8/5)$, we can calculate that (H1)-(H3) hold. Furthermore, we can get

$$
\rho \left(\|h\|_1 + \|r\|_1 + \|s\|_1 \right) = 0.52754.
$$

By Corollary 3.10, the BVP (4.1) has at least one solution in $C^{8/5}[0,1]$.

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