Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Strong convergence of new iterative algorithms for certain classes of asymptotically pseudocontractions

Micah Okwuchukwu Osilike*, Peter Uche Nwokoro and Eucharia Ezinwanne Chima

*Correspondence: osilike@yahoo.com; micah.osilike@unn.edu.ng
Department of Mathematics, University of Nigeria, Nsukka, Nigeria

Abstract

Let C be a nonempty closed convex subset of a real Hilbert space, and let $T: C \to C$ be an asymptotically k-strictly pseudocontractive mapping with $F(T) = \{x \in C : Tx = x\} \neq \emptyset$. Let $\{\alpha_n\}_{n=1}^\infty$ and $\{t_n\}_{n=1}^\infty$ be real sequences in $(0,1)$. Let $\{x_n\}_{n=1}^\infty$ be the sequence generated from an arbitrary $x_1 \in C$ by

$$\begin{cases} \nu_n = P_C((1-t_n)x_n), & n \geq 1, \\ x_{n+1} = (1-\alpha_n)\nu_n + \alpha_n T^n \nu_n, & n \geq 1, \end{cases}$$

where $P_C: H \to C$ is the metric projection. Under some appropriate mild conditions on $\{\alpha_n\}_{n=1}^\infty$ and $\{t_n\}_{n=1}^\infty$, we prove that $\{x_n\}_{n=1}^\infty$ converges strongly to a fixed point of T. Furthermore, if $T: C \to C$ is uniformly L-Lipschitzian and asymptotically pseudocontractive with $F(T) \neq \emptyset$, we first prove that $(I-T)$ is demiclosed at 0, and then prove that under some suitable conditions on the real sequences $\{\alpha_n\}_{n=1}^\infty$, $\{\beta_n\}_{n=1}^\infty$ and $\{t_n\}_{n=1}^\infty$ in $(0,1)$, the sequence $\{x_n\}_{n=1}^\infty$ generated from an arbitrary $x_1 \in C$ by

$$\begin{cases} \nu_n = P_C((1-t_n)x_n), & n \geq 1, \\ y_n = (1-\beta_n)\nu_n + \beta_n T^n \nu_n, & n \geq 1, \\ x_{n+1} = (1-\alpha_n)\nu_n + \alpha_n T^n y_n, & n \geq 1, \end{cases}$$

converges strongly to a fixed point of T. No compactness assumption is imposed on T or C and no further requirement is imposed on $F(T)$.

MSC: 47H09; 47J25; 65J15

Keywords: asymptotically pseudocontractive maps; fixed points; strong convergence; Hilbert spaces; iterative algorithm

1 Introduction

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H. A mapping $T: C \to C$ is said to be *L-Lipschitzian* if there exists $L \geq 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|, \quad \forall x, y \in C. \tag{1.1}$$

T is said to be a *contraction* if $L \in [0,1)$, and T is said to be *nonexpansive* if $L = 1$. T is said to be *asymptotically nonexpansive* (see, for example, [1]) if there exists a sequence

$\{k_n\}_{n=1}^{\infty} \subseteq [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \le k_n \|x - y\|, \quad \forall x, y \in C. \tag{1.2}$$

It is well known (see, for example, [1]) that the class of nonexpansive mappings is a proper subclass of the class of asymptotically nonexpansive mappings. T is said to be *asymptotically k-strictly pseudocontractive* (see, for example, [2]) if there exist $k \in [0,1)$ and a sequence $\{k_n\}_{n=1}^{\infty} \subseteq [1,\infty)$, $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\|^2 \le k_n \|x - y\|^2 + k\|(x - T^n x) - (y - T^n y)\|^2, \quad \forall x, y \in C. \tag{1.3}$$

T is said to be *asymptotically pseudocontractive* if there exists a sequence $\{k_n\}_{n=1}^{\infty} \subseteq [1,\infty)$, $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\|^2 \le k_n \|x - y\|^2 + \|(x - T^n x) - (y - T^n y)\|^2, \quad \forall x, y \in C. \tag{1.4}$$

It is well known that in real Hilbert spaces, the class of asymptotically nonexpansive maps is a proper subclass of the class of asymptotically k-strictly pseudocontractive maps. Furthermore, the class of asymptotically k-strictly pseudocontractive mappings is a proper subclass of the class of asymptotically pseudocontractive maps. T is said to be uniformly L-Lipschitzian if there exists $L \ge 0$ such that

$$\|T^n x - T^n y\| \le L\|x - y\|, \quad \forall x, y \in C.$$

T is said to be demiclosed at p if whenever $\{x_n\}_{n=1}^{\infty}$ is a sequence in C which converges weakly to $x^* \in C$ and $\{Tx_n\}_{n=1}^{\infty}$ converges strongly to p, then $Tx^* = p$. It is well known that if $T : C \to C$ is asymptotically k-strictly pseudocontractive, then T is uniformly L-Lipschitzian (see, for example, [3, 4]), and $(I - T)$ is demiclosed at 0 (see, for example, [5]). The modified Mann iteration scheme $\{x_n\}_{n=1}^{\infty}$ generated from an arbitrary $x_1 \in C$ by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \quad n \ge 1, \tag{1.5}$$

where the *control sequence* $\{\alpha_n\}_{n=1}^{\infty}$ is a real sequence in $(0,1)$ satisfying some appropriate conditions, has been used by several authors for the approximation of fixed points of asymptotically k-strictly pseudocontractive maps (see, for example, [2–10]). The iteration algorithm (1.5) is a modification of the well-known Mann iterative algorithm (see [11]) generated from an arbitrary $x_1 \in C$ by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n, \quad n \ge 1, \tag{1.6}$$

where the *control sequence* $\{\alpha_n\}_{n=1}^{\infty}$ is a real sequence in $(0,1)$ satisfying some appropriate conditions.

In real Hilbert spaces, it is known (see, for example, [3–5]) that if C is a nonempty closed convex subset of a real Hilbert space H, and $T : C \to C$ is an asymptotically k-strictly pseudocontractive mapping with a sequence $\{k_n\}_{n=1}^{\infty} \subseteq [1,\infty)$, $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, and a nonempty fixed point set $F(T)$, then the modified iteration sequence $\{x_n\}$ generated by

(1.5) is an approximate fixed point sequence (*i.e.*, $\|x_n - Tx_n\| \to 0$ as $n \to \infty$) if $\alpha_n \in [a,b] \subseteq (0,1-k)$. This together with the demiclosedness property of $(I-T)$ at 0 yields that $\{x_n\}$ converges weakly to a fixed point of T.

To obtain strong convergence of the modified Mann algorithm (1.5) to a fixed point of an asymptotically k-strictly pseudocontractive mapping, additional conditions are usually required on T and on the subset C (see, for example, [2–10]). Even for nonexpansive maps, additional conditions are required on T or C to obtain strong convergence using the Mann algorithm (1.6). In [12], Genel and Lindenstraus provided an example of a nonexpansive mapping defined on a bounded closed convex subset of a Hilbert space for which the Mann iteration does not converge to a fixed point of T. Recently Yao *et al.* [13] (see also [14, 15]) studied a modified Mann iteration algorithm $\{x_n\}$ generated from an arbitrary $x_1 \in H$ by

$$\begin{cases} v_n = (1-t_n)x_n, \\ x_{n+1} = (1-\alpha_n)v_n + \alpha_n T v_n, \end{cases} \tag{1.7}$$

where $\{t_n\}$ and $\{\alpha_n\}$ are real sequences in $(0,1)$ satisfying some appropriate conditions. They proved strong convergence of the modified algorithm to a fixed point of a nonexpansive mapping $T : H \to H$ when $F(T) \neq \emptyset$. Clearly, the modified Mann iteration algorithm reduces to the normal Mann iteration algorithm when $t_n \equiv 0$.

It is our purpose in this paper to modify algorithm (1.7) and prove that the modified algorithm converges strongly to a fixed point of an asymptotically k-strictly pseudocontractive mapping $T : C \to C$, where C is a nonempty closed convex subset of a real Hilbert space and $F(T) \neq \emptyset$. Furthermore, we prove that if $T : C \to C$ is a uniformly L-Lipschitzian asymptotically pseudocontractive mapping, then $(I-T)$ is demiclosed at 0. We then introduce an iterative algorithm which converges strongly to a fixed point of a uniformly L-Lipschitzian asymptotically pseudocontractive mapping $T : C \to C$ with $F(T) \neq \emptyset$. The technique of proof of our convergence theorems follows the one proposed by Maingé [14].

2 Preliminaries

In what follows, we shall need the following results.

Lemma 2.1 [16] *Let* $\{a_n\}_{n=1}^{\infty}$ *be a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\lambda_n)a_n + \lambda_n \gamma_n + \sigma_n, \quad n \geq 1,$$

where $\{\lambda_n\} \subseteq (0,1)$, $\{\gamma_n\} \subseteq \Re$, $\{\sigma_n\}$ *is a sequence of nonnegative real numbers and*

(i) $\sum_{n=0}^{\infty} \lambda_n = \infty$, *or equivalently*, $\prod_{n=0}^{\infty}(1-\lambda_n) = 0$,

(ii) $\limsup_{n\to\infty} \gamma_n \leq 0$, *and*

(iii) $\sum_{n=0}^{\infty} \sigma_n < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Let C be a closed convex subset of a real Hilbert space H. Let $P_C : H \to C$ denote the metric projection (the proximity map) which assigns to each point $x \in H$ the unique nearest point in C, denoted by $P_C(x)$. It is well known that

$$z = P_C(x) \quad \text{if and only if} \quad \langle x - z, z - y \rangle \geq 0, \quad \forall y \in C, \tag{2.1}$$

and that P_C is nonexpansive.

It is also well known that in real Hilbert spaces H, we have the following (see, for example, [17]):

(i) $$\|x+y\|^2 \leq \|y\|^2 + 2\langle x, x+y\rangle, \quad \forall x, y \in H; \tag{2.2}$$

(ii) $$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2,$$
$$\forall x, y \in H \text{ and } \alpha \in [0,1]; \tag{2.3}$$

(iii) if $\{x_n\}_{n=1}^{\infty}$ is a sequence in H which converges weakly to z, then
$$\limsup_{n\to\infty} \|x_n - y\|^2 = \limsup_{n\to\infty} \|x_n - z\|^2 + \|z-y\|^2, \quad \forall y \in H. \tag{2.4}$$

3 Main results

3.1 Strong convergence of an iterative algorithm for asymptotically *k*-strictly pseudocontractive maps

We now introduce the following iterative algorithm analogous to one studied in [13].

Modified averaging Mann algorithm Let C be a nonempty closed convex subset of a real Hilbert space H, and let $T : C \to C$ be a given mapping. For arbitrary $x_1 \in C$, our iteration sequence $\{x_n\}$ is given by

$$\begin{cases} v_n = P_C((1-t_n)x_n), \\ x_{n+1} = (1-\alpha_n)v_n + \alpha_n T^n v_n, \end{cases} \tag{3.1}$$

where $\{t_n\}$ and $\{\alpha_n\}$ are suitable real sequences in $(0,1)$ satisfying some appropriate conditions that will be made precise in our strong convergence theorem.

We now prove the following convergence theorem.

Theorem 3.1 *Let C be a nonempty closed convex subset of a real Hilbert space, and let $T : C \to C$ be an asymptotically k-strictly pseudocontractive mapping with a sequence $\{k_n\}_{n=1}^{\infty} \subseteq [1,\infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $F(T) \neq \emptyset$, and let $\{t_n\}_{n=1}^{\infty}$ and $\{\alpha_n\}_{n=1}^{\infty}$ be sequences in $(0,1)$ satisfying the conditions*:

(c1) $\lim_{n\to\infty} t_n = 0$;
(c2) $\sum_{n=1}^{\infty} t_n = \infty$;
(c3) $\lim_{n\to\infty} \frac{1}{t_n}(k_n - 1) = 0$;
(c4) $0 < \epsilon \leq \alpha_n < \frac{1}{2}(1-t_n)(1-k)$, $\forall n \geq 1$ *and for some* ϵ.

Then the modified averaging iteration sequence $\{x_n\}_{n=1}^{\infty}$ generated from $x_1 \in C$ by (3.1) *converges strongly to a fixed point of T.*

Proof Observe that (1.3) is equivalent to each of the following inequalities:

$$2\langle (I-T^n)x - (I-T^n)y, x-y\rangle \geq (1-k)\|(I-T^n)x - (I-T^n)y\|^2$$
$$- (k_n - 1)\|x-y\|^2, \tag{3.2}$$

$$2\langle T^n x - T^n y, x-y\rangle \leq (k_n+1)\|x-y\|^2 - (1-k)\|(I-T^n)x - (I-T^n)y\|^2. \tag{3.3}$$

Let $p \in F(T)$ be arbitrary. Then, using (1.3), (2.3), (3.1) and (3.3), we obtain

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|(1-\alpha_n)(v_n - p) + \alpha_n(T^n v_n - p)\|^2 \\ &= (1-\alpha_n)\|v_n - p\|^2 + \alpha_n\|T^n v_n - p\|^2 - \alpha_n(1-\alpha_n)\|v_n - T^n v_n\|^2 \\ &\le [1 + \alpha_n(k_n - 1)]\|v_n - p\|^2 - \alpha_n(1-\alpha_n - k)\|v_n - T^n v_n\|^2. \end{aligned} \tag{3.4}$$

Hence

$$\begin{aligned}\|x_{n+1} - p\| &\le [1 + \alpha_n(k_n - 1)]\|v_n - p\| \\ &= [1 + \alpha_n(k_n - 1)]\|P_C((1-t_n)x_n) - p\| \\ &\le [1 + \alpha_n(k_n - 1)]\|(1-t_n)x_n - p\| \\ &= [1 + \alpha_n(k_n - 1)]\|(1-t_n)(x_n - p) - t_n p\| \\ &\le [1 + \alpha_n(k_n - 1)][(1-t_n)\|x_n - p\| + t_n\|p\|] \\ &\le [1 + \alpha_n(k_n - 1)]\max\{\|x_n - p\|, \|p\|\} \\ &\vdots \\ &\le \prod_{j=1}^{n}[1 + \alpha_j(k_j - 1)]\max\{\|x_1 - p\|, \|p\|\}. \end{aligned}$$

Since $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, it follows from (3.4) that $\{x_n\}_{n=1}^{\infty}$ is bounded. Hence $\{v_n\}_{n=1}^{\infty}$ is also bounded. Furthermore, it follows from (2.2) that

$$\begin{aligned}\|x_n - x_{n+1}\|^2 &= \|x_n - v_n + v_n - x_{n+1}\|^2 \\ &\le \|v_n - x_{n+1}\|^2 + 2\langle x_n - v_n, x_n - x_{n+1}\rangle \\ &\le \|v_n - x_{n+1}\|^2 + 2\|x_n - v_n\|\|x_n - x_{n+1}\| \\ &\le \|v_n - x_{n+1}\|^2 + 2t_n\|x_n\|\|x_n - x_{n+1}\|. \end{aligned} \tag{3.5}$$

From (3.1) and (3.5) we obtain

$$\begin{aligned}\|v_n - T^n v_n\|^2 &= \frac{1}{\alpha_n^2}\|v_n - x_{n+1}\|^2 \\ &\ge \frac{1}{\alpha_n^2}[\|x_n - x_{n+1}\|^2 - 2t_n\|x_n\|\|x_n - x_{n+1}\|]. \end{aligned} \tag{3.6}$$

Since $\{v_n\}_{n=1}^{\infty}$ is bounded, then

$$\|v_n - p\|^2 \le D, \quad \forall n \ge 1 \text{ and for some } D > 0,$$

and hence using condition (c4) and (3.6) in (3.4), we obtain

$$\begin{aligned}\|x_{n+1} - p\|^2 \le{}& [1 + \alpha_n(k_n - 1)]\|v_n - p\|^2 \\ & - \alpha_n(1 - \alpha_n - k)\|v_n - T^n v_n\|^2 \end{aligned}$$

$$
\begin{aligned}
&\le \|v_n - p\|^2 - \frac{(1-\alpha_n - k)}{\alpha_n}\big[\|x_n - x_{n+1}\|^2 \\
&\quad - 2t_n\|x_n\|\|x_n - x_{n+1}\|\big] + \alpha_n(k_n - 1)D \\
&\le \big\|(1-t_n)x_n - p\big\|^2 - \frac{(1-\alpha_n - k)}{\alpha_n}\big[\|x_n - x_{n+1}\|^2 \\
&\quad - 2t_n\|x_n\|\|x_n - x_{n+1}\|\big] + \alpha_n(k_n - 1)D \\
&\le \|x_n - p\|^2 - 2t_n\langle x_n, x_n - p\rangle + t_n^2\|x_n\|^2 \\
&\quad - \sigma_1\|x_n - x_{n+1}\|^2 + 2\sigma_2 t_n\|x_n\|\|x_n - x_{n+1}\| \\
&\quad + \alpha_n(k_n - 1)D \quad \left(\text{where } \sigma_1 := \frac{1}{2}(1-k) > 0; \sigma_2 = \frac{1}{\epsilon}\right) \\
&= \|x_n - p\|^2 - \sigma_1\|x_n - x_{n+1}\|^2 + t_n\big[t_n\|x_n\|^2 \\
&\quad + 2\sigma_2\|x_n\|\|x_n - x_{n+1}\| - 2\langle x_n, x_n - p\rangle\big] \\
&\quad + \alpha_n(k_n - 1)D.
\end{aligned}
\tag{3.7}
$$

Since $\{x_n\}_{n=1}^{\infty}$ is bounded, we have that there exists $M > 0$ such that

$$
t_n\|x_n\|^2 + 2\sigma_2\|x_n\|\|x_n - x_{n+1}\| - 2\langle x_n, x_n - p\rangle \le M, \quad \forall n \ge 1. \tag{3.8}
$$

From (3.7) and (3.8) we obtain

$$
\|x_{n+1} - p\|^2 - \|x_n - p\|^2 + \sigma_1\|x_n - x_{n+1}\|^2 \le Mt_n + \alpha_n(k_n - 1)D. \tag{3.9}
$$

To complete the proof, we now consider the following two cases.

Case 1. Suppose that $\{\|x_n - p\|\}_{n=1}^{\infty}$ is a monotone sequence, then we may assume that $\{\|x_n - p\|\}$ is monotone decreasing. Then $\lim_{n\to\infty}\|x_n - p\|$ exists and it follows from (3.9), conditions (c1) and $\lim_{n\to\infty} k_n = 1$ that

$$
\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0. \tag{3.10}
$$

Furthermore,

$$
\begin{aligned}
&\|v_n - x_n\| \le t_n\|x_n\| \to 0 \quad \text{as } n \to \infty, \text{ and} \\
&\|v_n - x_{n+1}\| \le \|v_n - x_n\| + \|x_n - x_{n+1}\| \to 0 \quad \text{as } n \to \infty.
\end{aligned}
$$

Hence

$$
\big\|v_n - T^n v_n\big\| \le \frac{1}{\alpha_n}\|v_n - x_{n+1}\| \le \frac{1}{\epsilon}\|v_n - x_{n+1}\| \to 0 \quad \text{as } n \to \infty,
$$

and

$$
\begin{aligned}
\big\|x_n - T^n x_n\big\| &\le \|x_n - v_n\| + \big\|v_n - T^n v_n\big\| + \big\|T^n v_n - T^n x_n\big\| \\
&\le (1 + k_n)\|x_n - v_n\| + \big\|v_n - T^n v_n\big\| \to 0 \quad \text{as } n \to \infty.
\end{aligned}
$$

Observe also that since T is uniformly L-Lipschitzian, we obtain

$$\begin{aligned}
\|v_n - Tv_n\| &\le \|v_n - T^n v_n\| + \|T^n v_n - Tv_n\| \\
&\le \|v_n - T^n v_n\| + L\|T^{n-1}v_n - v_n\| \\
&\le \|v_n - T^n v_n\| + L\|T^{n-1}v_n - T^{n-1}v_{n-1}\| \\
&\quad + L\|T^{n-1}v_{n-1} - v_{n-1}\| + L\|v_{n-1} - v_n\| \\
&\le \|v_n - T^n v_n\| + L\|T^{n-1}v_{n-1} - v_{n-1}\| \\
&\quad + L(1+L)\|v_n - v_{n-1}\| \\
&\le \|v_n - T^n v_n\| + L\|T^{n-1}v_{n-1} - v_{n-1}\| \\
&\quad + L(1+L)\big[\|v_n - x_n\| + \|x_n - x_{n-1}\| \\
&\quad + \|x_{n-1} - v_{n-1}\|\big] \to 0 \quad \text{as } n \to \infty.
\end{aligned} \tag{3.11}$$

Furthermore,

$$\begin{aligned}
\|x_n - Tx_n\| &\le \|x_n - T^n x_n\| + \|T^n x_n - Tx_n\| \\
&\le \|x_n - T^n x_n\| + L\|T^{n-1}x_n - x_n\| \\
&\le \|x_n - T^n x_n\| + L\|T^{n-1}x_n - T^{n-1}x_{n-1}\| \\
&\quad + L\|T^{n-1}x_{n-1} - x_{n-1}\| + L\|x_{n-1} - x_n\| \\
&\le \|x_n - T^n x_n\| + L\|T^{n-1}x_{n-1} - x_{n-1}\| \\
&\quad + L(1+L)\|x_n - x_{n-1}\| \to 0 \quad \text{as } n \to \infty.
\end{aligned} \tag{3.12}$$

Since $\lim_{n\to\infty} \|x_n - Tx_n\| = \lim_{n\to\infty} \|v_n - Tv_n\| = \lim_{n\to\infty} \|v_n - x_n\| = 0$, then the demiclosedness property of $(I - T)$, (2.4) and the usual standard argument yield that $\{x_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ converge weakly to some $x^* \in F(T)$.

Since $\alpha_n(1-\alpha_n-k) \ge \frac{1}{\epsilon}(1-k) > 0$, and since $\|v_n - x^*\|^2 \le D_2$, $\forall n \ge 1$, and for some $D_2 > 0$, then using (3.4) we obtain

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le \|v_n - x^*\|^2 + \alpha_n(k_n - 1)D_2 \\
&\le \|(1-t_n)(x_n - x^*) - t_n x^*\|^2 + \alpha_n(k_n - 1)D_2 \\
&= (1-t_n)^2\|x_n - x^*\|^2 - 2t_n(1-t_n)\langle x_n - x^*, x^*\rangle \\
&\quad + t_n^2\|x^*\|^2 + \alpha_n(k_n - 1)D_2 \\
&\le (1-t_n)\|x_n - x^*\|^2 - 2t_n(1-t_n)\langle x_n - x^*, x^*\rangle \\
&\quad + t_n^2\|x^*\|^2 + \alpha_n(k_n - 1)D_2.
\end{aligned} \tag{3.13}$$

Thus

$$\|x_{n+1} - x^*\|^2 \le (1-t_n)\|x_n - x^*\|^2 + t_n\gamma_n + \sigma_n, \quad \forall n \ge 1,$$

where $\gamma_n := -2(1-t_n)\langle x_n - x^*, x^*\rangle + t_n\|x^*\|^2$, and $\sigma_n = \alpha_n(k_n - 1)D_2$, with $\sum_{n=1}^{\infty} \sigma_n < \infty$. Since $\{x_n\}_{n=1}^{\infty}$ converges weakly to x^*, then $\lim_{n\to\infty}\langle x_n - x^*, x^*\rangle = 0$, and this together with condition (c1) (*i.e.*, $\lim_{n\to\infty} t_n = 0$) implies that $\gamma_n := -2(1-t_n)\langle x_n - x^*, x^*\rangle + t_n\|x^*\|^2 \to 0$ as $n \to \infty$. It now follows from Lemma 2.1 that $\{x_n\}_{n=1}^{\infty}$ converges strongly to x^*. Consequently, $\{v_n\}_{n=1}^{\infty}$ converges strongly to x^*.

Case 2. Suppose that $\{\|x_n - p\|\}_{n=1}^{\infty}$ is not a monotone decreasing sequence, then set $\Gamma_n := \|x_n - p\|^2$, and let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping defined for all $n \geq N_0$ for some sufficiently large N_0 by

$$\tau(n) := \max\{k \in \mathbb{N} : k \leq n, \Gamma_k \leq \Gamma_{k+1}\}.$$

Then τ is a non-decreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ for $n \geq N_0$. Using (c1) and (c2) in (3.9), we obtain

$$\|x_{\tau(n)+1} - x_{\tau(n)}\|^2 \leq \frac{1}{\sigma_1}\bigl[Mt_{\tau(n)} + \alpha_{\tau(n)}(k_{\tau(n)} - 1)D\bigr] \to 0 \quad \text{as } n \to \infty. \tag{3.14}$$

Following the same argument as in Case 1, we obtain

$$\|v_{\tau(n)} - Tv_{\tau(n)}\| \to 0 \quad \text{as } n \to \infty \quad \text{and} \quad \|x_{\tau(n)} - Tx_{\tau(n)}\| \to 0 \quad \text{as } n \to \infty.$$

As in Case 1, we also obtain that $\{x_{\tau(n)}\}$ and $\{v_{\tau(n)}\}$ converge weakly to some x^* in $F(T)$. Furthermore, for all $n \geq N_0$, we obtain from (3.13) that

$$\begin{aligned} 0 &\leq \|x_{\tau(n)+1} - x^*\|^2 - \|x_{\tau(n)} - x^*\|^2 \\ &\leq t_{\tau(n)}\Bigl[-2(1-t_{\tau(n)})\langle x_{\tau(n)} - x^*, x^*\rangle + t_{\tau(n)}\|x^*\|^2 \\ &\quad + D_2\alpha_{\tau(n)}\frac{(k_{\tau(n)} - 1)}{t_{\tau(n)}} - \|x_{\tau(n)} - x^*\|^2\Bigr]. \end{aligned} \tag{3.15}$$

It follows from (3.15) that

$$\begin{aligned} \|x_{\tau(n)} - x^*\|^2 &\leq 2(1-t_{\tau(n)})\langle x^* - x_{\tau(n)}, x^*\rangle + t_{\tau(n)}\|x^*\|^2 \\ &\quad + D_2\alpha_{\tau(n)}\frac{(k_{\tau(n)} - 1)}{t_{\tau(n)}} \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Thus

$$\lim_{n\to\infty} \Gamma_{\tau(n)} = \lim_{n\to\infty} \Gamma_{\tau(n)+1}.$$

Furthermore, for $n \geq N_0$, we have $\Gamma_n \leq \Gamma_{\tau(n)+1}$ if $n \neq \tau(n)$ (*i.e.*, $\tau(n) < n$), because $\Gamma_j > \Gamma_{j+1}$ for $\tau(n) + 1 \leq j \leq n$. It then follows that for all $n \geq N_0$ we have

$$0 \leq \Gamma_n \leq \max\{\Gamma_{\tau(n)}, \Gamma_{\tau(n)+1}\} = \Gamma_{\tau(n)+1}.$$

This implies $\lim_{n\to\infty} \Gamma_n = 0$, and hence $\{x_n\}_{n=1}^{\infty}$ converges strongly to $x^* \in F(T)$. □

Corollary 3.1 *Let H be a real Hilbert space, and let C be a nonempty closed convex subset of H with $0 \in C$. Let $T : C \to C$ be an asymptotically k-strictly pseudocontractive mapping* *with a sequence* $\{k_n\}_{n=1}^{\infty} \subseteq [1,\infty)$ *such that* $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. *Let* $F(T) \neq \emptyset$, *and let* $\{t_n\}_{n=1}^{\infty}$ *and* $\{\alpha_n\}_{n=1}^{\infty}$ *be sequences in* $(0,1)$ *satisfying the conditions*:

(c1) $\lim_{n\to\infty} t_n = 0$;

(c2) $\sum_{n=1}^{\infty} t_n = \infty$;

(c3) $\lim_{n\to\infty} \frac{1}{t_n}(k_n - 1) = 0$;

(c4) $0 < \epsilon \leq \alpha_n < \frac{1}{2}(1-t_n)(1-k)$, $\forall n \geq 1$ *and for some* ϵ.

Then the modified averaging iteration sequence $\{x_n\}_{n=1}^{\infty}$, *generated from* $x_1 \in C$ *by*

$$\begin{cases} v_n := (1-t_n)x_n, \\ x_{n+1} := (1-\alpha_n)v_n + \alpha_n T^n v_n, \end{cases}$$

converges strongly to a fixed point of T.

Remark 3.1 Prototypes for our real sequences $\{t_n\}_{n=1}^{\infty}$ and $\{\alpha_n\}_{n=1}^{\infty}$ are:

$$t_n := \sqrt{k_n - 1 + \frac{1}{n+1}}, \quad n \geq 1; \qquad \alpha_n := \frac{n}{2(n+1)}(1-k)(1-t_n), \quad n \geq 1.$$

Corollary 3.2 *Let* C *be a nonempty closed convex subset of a real Hilbert space, and let* $T: C \to C$ *be an asymptotically nonexpansive mapping with a sequence* $\{k_n\}_{n=1}^{\infty} \subseteq [1,\infty)$. *Let* $F(T) \neq \emptyset$, *and let* $\{t_n\}_{n=1}^{\infty}$ *and* $\{\alpha_n\}_{n=1}^{\infty}$ *be sequences in* $(0,1)$ *satisfying the conditions*:

(c1) $\lim_{n\to\infty} t_n = 0$;

(c2) $\sum_{n=1}^{\infty} t_n = \infty$;

(c3) $0 < \epsilon \leq \alpha_n < \frac{1}{2}(1-t_n)$, $\forall n \geq 1$ *and for some* ϵ.

Then the modified averaging iteration sequence $\{x_n\}_{n=1}^{\infty}$, *generated from* $x_1 \in C$ *by* (3.1), *converges strongly to a fixed point of* T.

3.2 Demiclosedness principle and strong convergence results for uniformly Lipschitzian asymptotically pseudocontractive maps

Let H be a real Hilbert space, and let C be a nonempty closed convex subset of H. For uniformly L-Lipschitzian asymptotically pseudocontractive maps $T: C \to C$, we first prove that $(I - T)$ is demiclosed at 0 and then introduce a modified averaging Ishikawa iteration algorithm and prove that it converges strongly to a fixed point of $T: C \to C$ without any compactness assumption on T or C and without further requirement on $F(T)$. Our demiclosedness principle does not require the boundedness of C imposed in the result of [18].

Theorem 3.2 *Let* H *be a real Hilbert space, and let* C *be a nonempty closed convex subset of* H. *Let* $T: C \to C$ *be a uniformly* L*-Lipschitzian asymptotically pseudocontractive mapping. Then* $(I - T)$ *is demiclosed at* 0.

Proof Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in C which converges weakly to p and $\{x_n - Tx_n\}_{n=1}^{\infty}$ converges strongly to 0. We prove that $p \in F(T)$. Since $\{x_n\}_{n=1}^{\infty}$ converges weakly, it is bounded. For each $x \in H$, define $f: H \to [0,\infty)$ by

$$f(x) := \limsup_{n\to\infty} \|x_n - x\|^2. \tag{3.16}$$

Observe that for arbitrary but fixed integer $m \geq 1$, we have

$$\begin{aligned}\|x_n - T^m x_n\| &\leq \|x_n - Tx_n\| + \|Tx_n - T^2 x_n\| + \cdots + \|T^{m-1}x_n - T^m x_n\| \\ &\leq mL\|x_n - Tx_n\| \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Set

$$G_m x := T^m\big((1-\beta)x + \beta T^m x\big),$$

where $\beta \in (0, \frac{2}{(1+\lambda)+\sqrt{(1+\lambda)^2+4L^2}})$, and $\lambda := \sup_{n\geq 1} k_n$. Then

$$\|(1-\beta)x_n + \beta T^m x_n - T^m x_n\| = (1-\beta)\|x_n - T^m x_n\| \to 0 \quad \text{as } n \to \infty,$$

and

$$\begin{aligned}\|T^m x_n - G_m x_n\| &= \|T^m x_n - T^m\big((1-\beta)x_n + \beta T^m x_n\big)\| \\ &\leq L\beta\|x_n - T^m x_n\| \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Hence

$$\begin{aligned}\|(1-\beta)x_n + \beta T^m x_n - G_m x_n\| &\leq \|(1-\beta)x_n + \beta T^m x_n - T^m x_n\| \\ &\quad + \|T^m x_n - G_m x_n\| \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Also

$$\|x_n - G_m x_n\| \leq \|x_n - T^m x_n\| + \|T^m x_n - G_m x_n\| \to 0 \quad \text{as } n \to \infty.$$

From (2.4) we obtain

$$f(x) = \limsup_{n\to\infty} \|x_n - p\|^2 + \|p - x\|^2, \quad \forall x \in H.$$

Thus

$$f(x) = f(p) + \|p - x\|^2, \quad \forall x \in H,$$

and hence

$$f(G_m p) = f(p) + \|p - G_m p\|^2. \tag{3.17}$$

Observe that

$$\begin{aligned}f(G_m p) &= \limsup_{n\to\infty} \|x_n - G_m p\|^2 \\ &= \limsup_{n\to\infty} \|x_n - G_m x_n + G_m x_n - G_m p\|^2\end{aligned}$$

$$
\begin{aligned}
&= \limsup_{n\to\infty} \|G_m x_n - G_m p\|^2 \\
&= \limsup_{n\to\infty} \big\| T^m\big((1-\beta)x_n + \beta T^m x_n\big) - T^m\big((1-\beta)p + \beta T^m p\big)\big\|^2 \\
&\le \limsup_{n\to\infty} \Big[k_m \big\|(1-\beta)x_n + \beta T^m x_n - \big((1-\beta)p + \beta T^m p\big)\big\|^2 \\
&\quad + \big\|(1-\beta)x_n + \beta T^m x_n - G_m x_n - \big((1-\beta)p + \beta T^m p - G_m p\big)\big\|^2\Big] \\
&= \limsup_{n\to\infty} \Big[k_m \big\|(1-\beta)(x_n - p) + \beta\big(T^m x_n - T^m p\big)\big\|^2 \\
&\quad + \big\|(1-\beta)(p - G_m p) + \beta\big(T^m p - G_m p\big)\big\|^2\Big] \\
&= \limsup_{n\to\infty} \Big[k_m \big((1-\beta)\|x_n - p\|^2 + \beta\big\|T^m x_n - T^m p\big\|^2 \\
&\quad - \beta(1-\beta)\big\|x_n - T^m x_n - \big(p - T^m p\big)\big\|^2\big) + (1-\beta)\|p - G_m p\|^2 \\
&\quad + \beta\big\|T^m p - G_m p\big\|^2 - \beta(1-\beta)\big\|p - T^m p\big\|^2\Big] \\
&\le \limsup_{n\to\infty} \Big[k_m(1-\beta+k_m\beta)\|x_n - p\|^2 + k_m\beta\big\|x_n - T^m x_n - \big(p - T^m p\big)\big\|^2 \\
&\quad - k_m\beta(1-\beta)\big\|x_n - T^m x_n - \big(p - T^m p\big)\big\|^2 + (1-\beta)\|p - G_m p\|^2 \\
&\quad + \beta^3 L^2\big\|p - T^m p\big\|^2 - \beta(1-\beta)\big\|p - T^m p\big\|^2\Big] \\
&= \limsup_{n\to\infty} \Big[k_m\big(1 + \beta(k_m - 1)\big)\|x_n - p\|^2 + (1-\beta)\|p - G_m p\|^2 \\
&\quad - \beta\big[1 - \beta(1+k_m) - \beta^2 L^2\big]\big\|p - T^m p\big\|^2\Big] \\
&\le k_m\big(1 + \beta(k_m - 1)\big) f(p) + (1-\beta)\|p - G_m p\|^2.
\end{aligned}
\tag{3.18}
$$

Equations (3.17) and (3.18) imply that

$$
f(p) + \|p - G_m p\|^2 \le k_m\big(1 + \beta(k_m - 1)\big) f(p) + (1-\beta)\|p - G_m p\|^2,
$$

from which it follows that

$$
\beta\|p - G_m p\|^2 \le \big[k_m\big(1 + \beta(k_m - 1)\big) - 1\big] f(p) \to 0 \quad \text{as } m \to \infty.
$$

Thus

$$
\begin{aligned}
&\|p - G_m p\| \to 0 \quad \text{as } m \to \infty, \\
&\big\|p - T^m p\big\| \le \|p - G_m p\| + \big\|G_m p - T^m p\big\| \\
&\qquad \le \|p - G_m p\| + L\big\|(1-\beta)p + \beta T^m p - p\big\| \\
&\qquad = \|p - G_m p\| + L\beta\big\|p - T^m p\big\|.
\end{aligned}
$$

Hence

$$
(1 - L\beta)\big\|p - T^m p\big\| \le \|p - G_m p\| \to 0 \quad \text{as } m \to \infty.
$$

It now follows that $T^m p \to p$ as $m \to \infty$. Since T is continuous, we have that $T^{m+1} p \to Tp$ as $m \to \infty$, and hence $Tp = p$. □

We now introduce the following iterative algorithm for uniformly L-Lipschitzian asymptotically pseudocontractive maps.

Modified averaging Ishikawa algorithm For arbitrary $x_1 \in C$, the sequence $\{x_n\}_{n=1}^{\infty}$ is given by

$$\begin{cases} v_n = P_C((1-t_n)x_n), & n \geq 1, \\ y_n = (1-\beta_n)v_n + \beta_n T^n v_n, & n \geq 1, \\ x_{n+1} = (1-\alpha_n)v_n + \alpha_n T^n y_n, & n \geq 1. \end{cases} \tag{3.19}$$

Theorem 3.3 *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $T: C \to C$ be a uniformly L-Lipschitzian asymptotically pseudocontractive mapping with a sequence $\{k_n\}_{n=1}^{\infty} \subseteq [1,\infty)$, $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $F(T) \neq \emptyset$. Let $\{t_n\}_{n=1}^{\infty}$, $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be real sequences in $(0,1)$ satisfying the conditions*:

(c1) $\lim_{n\to\infty} t_n = 0$;
(c2) $\sum_{n=1}^{\infty} t_n = \infty$;
(c3) $0 < \epsilon \leq \alpha_n \leq (1-t_n)\beta_n \leq \beta_n \leq b < \frac{2}{(1+\lambda)+\sqrt{(1+\lambda)^2+4L^2}}$, *where* $\lambda = \sup_n k_n$;
(c4) $\lim_{n\to\infty} \frac{(k_n-1)}{t_n} = 0$.

Then the sequence $\{x_n\}_{n=1}^{\infty}$ generated from an arbitrary $x_1 \in C$ by (3.19) *converges strongly to a fixed point of T.*

Proof Since T is asymptotically pseudocontractive, it follows that

$$2\langle (I-T^n)x - (I-T^n)y, x-y\rangle \geq -(k_n-1)\|x-y\|^2 \tag{3.20}$$

and

$$2\langle T^n x - T^n y, x-y\rangle \leq (k_n+1)\|x-y\|^2. \tag{3.21}$$

Set

$$G_n v_n = T^n\big((1-\beta_n)v_n + \beta_n T^n v_n\big), \quad n \geq 1.$$

Then, for arbitrary $p \in F(T)$, we obtain

$$\begin{aligned} \|G_n v_n - p\|^2 &= \big\|T^n\big((1-\beta_n)v_n + \beta_n T^n v_n\big) - T^n p\big\|^2 \\ &\leq k_n\big\|(1-\beta_n)(v_n-p) + \beta_n\big(T^n v_n - p\big)\big\|^2 \\ &\quad + \big\|(1-\beta_n)v_n + \beta_n T^n v_n - G_n v_n\big\|^2 \\ &= k_n(1-\beta_n)\|v_n-p\|^2 + k_n\beta_n\big\|T^n v_n - p\big\|^2 \\ &\quad - k_n\beta_n(1-\beta_n)\big\|v_n - T^n v_n\big\|^2 \\ &\quad + \big\|(1-\beta_n)(v_n - G_n v_n) + \beta_n\big(T^n v_n - G_n v_n\big)\big\|^2 \\ &= \big[k_n(1-\beta_n) + k_n^2\beta_n\big]\|v_n-p\|^2 + k_n\beta_n\big\|v_n - T^n v_n\big\|^2 \\ &\quad - k_n(1-\beta_n)\beta_n\big\|v_n - T^n v_n\big\|^2 + (1-\beta_n)\|v_n - G_n v_n\|^2 \end{aligned}$$

$$
\begin{aligned}
&+\beta_n\|T^n v_n - G_n v_n\|^2 - \beta_n(1-\beta_n)\|v_n - T^n v_n\|^2\\
&\le \left[1+\left(k_n^2-1\right)\right]\|v_n-p\|^2 + k_n\beta_n\|v_n - T^n v_n\|^2\\
&\quad - k_n(1-\beta_n)\beta_n\|v_n - T^n v_n\|^2 + (1-\beta_n)\|v_n - G_n v_n\|^2\\
&\quad + \beta_n^3L^2\|v_n - T^n v_n\|^2 - \beta_n(1-\beta_n)\|v_n - T^n v_n\|^2\\
&= \left[1+\left(k_n^2-1\right)\right]\|v_n-p\|^2 + (1-\beta_n)\|v_n - G_n v_n\|^2\\
&\quad - \beta_n\left[1-(1+k_n)\beta_n - \beta_n^2L^2\right]\|v_n - T^n v_n\|^2.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\|G_n v_n - p\|^2 &\le \left[1+\left(k_n^2-1\right)\right]\|v_n-p\|^2 + (1-\beta_n)\|v_n - G_n v_n\|^2\\
&\quad - \beta_n\left[1-(1+k_n)\beta_n - \beta_n^2L^2\right]\|v_n - T^n v_n\|^2. \qquad (3.22)
\end{aligned}
$$

From (3.22) we obtain

$$
\begin{aligned}
2\langle v_n - G_n v_n, v_n - p\rangle &\ge \beta_n\|v_n - G_n v_n\|^2 + \beta_n\left[1-(1+k_n)\beta_n - \beta_n^2L^2\right]\|v_n - Tv_n\|^2\\
&\quad - \left(k_n^2-1\right)\|v_n-p\|^2 \qquad (3.23)
\end{aligned}
$$

and

$$
\begin{aligned}
2\langle G_n v_n - p, v_n - p\rangle &\le \left(1+k_n^2\right)\|v_n-p\|^2 - \beta_n\|v_n - G_n v_n\|^2\\
&\quad - \beta_n\left[1-(1+k_n)\beta_n - \beta_n^2L^2\right]\|v_n - T^n v_n\|^2. \qquad (3.24)
\end{aligned}
$$

Observe that

$$
\begin{aligned}
\|x_{n+1}-p\|^2 &= \|(1-\alpha_n)v_n + \alpha_n G_n v_n - p\|^2\\
&= (1-\alpha_n)\|v_n-p\|^2 + \alpha_n\|G_n v_n - p\|^2 - \alpha_n(1-\alpha_n)\|v_n - G_n v_n\|^2\\
&\le (1-\alpha_n)\|v_n-p\|^2 + \alpha_n\Big[\left[1+\left(k_n^2-1\right)\right]\|v_n-p\|^2 + (1-\beta_n)\|v_n - G_n v_n\|^2\\
&\quad - \beta_n\left[1-(1+k_n)\beta_n - \beta_n^2L^2\right]\|v_n - T^n v_n\|^2\Big]\\
&\quad - \alpha_n(1-\alpha_n)\|v_n - G_n v_n\|^2\\
&\le \left[1+\left(k_n^2-1\right)\right]\|v_n-p\|^2 - \alpha_n(\beta_n-\alpha_n)\|v_n - G_n v_n\|^2\\
&\quad - \alpha_n\beta_n\left[1-(1+k_n)\beta_n - \beta_n^2L^2\right]\|v_n - T^n v_n\|^2. \qquad (3.25)
\end{aligned}
$$

Hence

$$
\|x_{n+1}-p\|^2 \le \left[1+\left(k_n^2-1\right)\right]\|v_n-p\|^2
$$

and it follows, as in the proof of Theorem 3.1, that $\{x_n\}_{n=1}^{\infty}$ is bounded. Observe that

$$
\begin{aligned}
\|x_n - x_{n+1}\|^2 &= \|x_n - v_n + v_n - x_{n+1}\|^2\\
&\le \|v_n - x_{n+1}\|^2 + 2\langle x_n - v_n, x_n - x_{n+1}\rangle
\end{aligned}
$$

$$
\begin{aligned}
&\le \|v_n - x_{n+1}\|^2 + 2\|x_n - v_n\|\|x_n - x_{n+1}\| \\
&\le \|v_n - x_{n+1}\|^2 + 2\big\|P_C\big((1-t_n)x_n\big) - x_n\big\|\|x_n - x_{n+1}\| \\
&\le \|v_n - x_{n+1}\|^2 + 2t_n\|x_n\|\|x_n - x_{n+1}\|.
\end{aligned}
$$

Hence

$$
\|v_n - x_{n+1}\|^2 \ge \|x_n - x_{n+1}\|^2 - 2t_n\|x_n\|\|x_n - x_{n+1}\|. \tag{3.26}
$$

Furthermore,

$$
\begin{aligned}
\|v_n - G_n v_n\| &\le \big\|v_n - T^n v_n\big\| + \big\|T^n v_n - G_n v_n\big\| \\
&\le \big\|v_n - T^n v_n\big\| + L\beta_n\big\|v_n - T^n v_n\big\| \\
&= (1 + L\beta_n)\big\|v_n - T^n v_n\big\|.
\end{aligned}
$$

Thus

$$
\big\|v_n - T^n v_n\big\|^2 \ge \frac{1}{(1+L\beta_n)^2}\|v_n - G_n v_n\|^2. \tag{3.27}
$$

Observe also that

$$
\|x_{n+1} - v_n\| = \big\|(1-\alpha_n)v_n + \alpha_n G_n v_n - v_n\big\| = \alpha_n\|v_n - G_n v_n\|. \tag{3.28}
$$

Using (3.26) and (3.28), we obtain

$$
\begin{aligned}
\|v_n - G_n v_n\|^2 &= \frac{1}{\alpha_n^2}\|x_{n+1} - v_n\|^2 \\
&\ge \frac{1}{\alpha_n^2}\big[\|x_n - x_{n+1}\|^2 - 2t_n\|x_n\|\|x_n - x_{n+1}\|\big].
\end{aligned}
$$

It now follows from (3.27) that

$$
\begin{aligned}
\big\|v_n - T^n v_n\big\|^2 &\ge \frac{1}{(1+L\beta_n)^2}\|v_n - G_n v_n\|^2 \\
&\ge \frac{1}{\alpha_n^2(1+L\beta_n)^2}\big[\|x_n - x_{n+1}\|^2 - 2t_n\|x_n\|\|x_n - x_{n+1}\|\big].
\end{aligned} \tag{3.29}
$$

Using (3.29) in (3.25), we obtain

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &\le \big[1 + \big(k_n^2 - 1\big)\big]\|v_n - p\|^2 \\
&\quad - \alpha_n\beta_n\big[1 - (1+k_n)\beta_n - \beta_n^2 L^2\big]\bigg\{\frac{1}{\alpha_n^2(1+L\beta_n)^2}\big[\|x_n - x_{n+1}\|^2 \\
&\quad - 2t_n\|x_n\|\|x_n - x_{n+1}\|\big]\bigg\} \\
&\le \|v_n - p\|^2 + \big(k_n^2 - 1\big)D
\end{aligned}
$$

$$\begin{aligned}
&-\frac{\beta_n[1-(1+k_n)\beta_n-\beta_n^2L^2]}{\alpha_n(1+L\beta_n)^2}\big[\|x_n-x_{n+1}\|^2 \\
&-2t_n\|x_n\|\|x_n-x_{n+1}\|\big] \\
\le{}& \|x_n-p\|^2-2t_n\langle x_n,x_n-p\rangle+t_n^2\|x_n\|^2-\sigma_3\|x_n-x_{n+1}\|^2 \\
&+2\sigma_4t_n\|x_n\|\|x_n-x_{n+1}\| \\
&\left(\text{where } \sigma_3=\frac{\epsilon[1-(1+\lambda)b-b^2L^2]}{b(1+L\epsilon)^2},\sigma_4=\frac{1}{\epsilon(1+L\epsilon)^2}\right) \\
={}& \|x_n-p\|^2-\sigma_3\|x_n-x_{n+1}\|+t_n\big[-\langle x_n,x_n-p\rangle \\
&+t_n\|x_n\|^2+2\sigma_4\|x_n\|\|x_n-x_{n+1}\|\big]+\big(k_n^2-1\big)D. \qquad (3.30)
\end{aligned}$$

Since $\{x_n\}$ is bounded, we have that there exists $M>0$ such that

$$-\langle x_n,x_n-p\rangle+t_n\|x_n\|^2+2\sigma_4\|x_n\|\|x_n-x_{n+1}\|\le M,\quad \forall n\ge 1. \qquad (3.31)$$

From (3.30) and (3.31) we obtain

$$\|x_{n+1}-p\|^2-\|x_n-p\|^2+\sigma_3\|x_n-x_{n+1}\|\le Mt_n+\big(k_n^2-1\big)D. \qquad (3.32)$$

To complete the proof, we now consider the following two cases.

Case 1. Suppose that $\{\|x_n-p\|\}_{n=1}^\infty$ is a monotone sequence, then we may assume that $\{\|x_n-p\|\}$ is monotone decreasing. Then $\lim_{n\to\infty}\|x_n-p\|$ exists and it follows from (3.32), conditions (c1) and $\lim_{n\to\infty}k_n=1$ that

$$\lim_{n\to\infty}\|x_n-x_{n+1}\|=0. \qquad (3.33)$$

Furthermore,

$$\begin{aligned}
&\|v_n-x_n\|\le t_n\|x_n\|\to 0 \quad \text{as } n\to\infty, \text{ and} \\
&\|v_n-x_{n+1}\|\le\|v_n-x_n\|+\|x_n-x_{n+1}\|\to 0 \quad \text{as } n\to\infty.
\end{aligned}$$

Hence

$$\|v_n-G_nv_n\|=\frac{1}{\alpha_n}\|v_n-G_nv_n\|\le\frac{1}{\epsilon}\|v_n-x_{n+1}\|\to 0 \quad \text{as } n\to\infty.$$

Furthermore,

$$\begin{aligned}
\big\|v_n-T^nv_n\big\| &\le \|v_n-G_nv_n\|+\big\|G_nv_n-T^nv_n\big\| \\
&\le \|v_n-G_nv_n\|+L\beta_n\big\|v_n-T^nv_n\big\|.
\end{aligned}$$

Thus

$$\big\|v_n-T^nv_n\big\|^2\le\frac{1}{1-L\beta_n}\|v_n-G_nv_n\|\le\frac{1}{1-Lb}\|v_n-G_nv_n\|\to 0 \quad \text{as } n\to\infty,$$

and

$$\begin{aligned}\left\|x_n - T^n x_n\right\| &\le \|x_n - v_n\| + \left\|v_n - T^n v_n\right\| \\ &\quad + \left\|T^n v_n - T^n x_n\right\| \\ &\le (1 + k_n)\|x_n - v_n\| \\ &\quad + \left\|v_n - T^n v_n\right\| \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Observe also that since T is uniformly L-Lipschitzian, we obtain

$$\begin{aligned}\|v_n - Tv_n\| &\le \left\|v_n - T^n v_n\right\| + \left\|T^n v_n - Tv_n\right\| \\ &\le \left\|v_n - T^n v_n\right\| + L\left\|T^{n-1} v_n - v_n\right\| \\ &\le \left\|v_n - T^n v_n\right\| + L\left\|T^{n-1} v_n - T^{n-1} v_{n-1}\right\| \\ &\quad + L\left\|T^{n-1} v_{n-1} - v_{n-1}\right\| + L\|v_{n-1} - v_n\| \\ &\le \left\|v_n - T^n v_n\right\| + L\left\|T^{n-1} v_{n-1} - v_{n-1}\right\| + L(1 + L)\|v_n - v_{n-1}\| \\ &\le \left\|v_n - T^n v_n\right\| + L\left\|T^{n-1} v_{n-1} - v_{n-1}\right\| \\ &\quad + L(1 + L)\big[\|v_n - x_n\| + \|x_n - x_{n-1}\| \\ &\quad + \|x_{n-1} - v_{n-1}\|\big] \to 0 \quad \text{as } n \to \infty.\end{aligned} \tag{3.34}$$

Furthermore,

$$\begin{aligned}\|x_n - Tx_n\| &\le \left\|x_n - T^n x_n\right\| + \left\|T^n x_n - Tx_n\right\| \\ &\le \left\|x_n - T^n x_n\right\| + L\left\|T^{n-1} x_n - x_n\right\| \\ &\le \left\|x_n - T^n x_n\right\| + L\left\|T^{n-1} x_n - T^{n-1} x_{n-1}\right\| \\ &\quad + L\left\|T^{n-1} x_{n-1} - x_{n-1}\right\| + L\|x_{n-1} - x_n\| \\ &\le \left\|x_n - T^n x_n\right\| + L\left\|T^{n-1} x_{n-1} - x_{n-1}\right\| \\ &\quad + L(1 + L)\|x_n - x_{n-1}\| \to 0 \quad \text{as } n \to \infty.\end{aligned} \tag{3.35}$$

Since $\lim_{n\to\infty} \|x_n - Tx_n\| = \lim_{n\to\infty} \|v_n - Tv_n\| = \lim_{n\to\infty} \|v_n - x_n\| = 0$, then the demiclosedness property of $(I - T)$, (2.4) and the usual standard argument yield that $\{x_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ converge weakly to some $x^* \in F(T)$. Since $\|v_n - x^*\|^2 \le D_2$, $\forall n \ge 1$, and for some $D_2 > 0$, then using (3.25) we obtain

$$\begin{aligned}\left\|x_{n+1} - x^*\right\|^2 &\le \left\|v_n - x^*\right\|^2 + \alpha_n\left(k_n^2 - 1\right)D_2 \\ &\le \left\|(1 - t_n)\left(x_n - x^*\right) - t_n x^*\right\|^2 + \alpha_n\left(k_n^2 - 1\right)D_2 \\ &= (1 - t_n)^2\left\|x_n - x^*\right\|^2 - 2t_n(1 - t_n)\left\langle x_n - x^*, x^*\right\rangle \\ &\quad + t_n^2\left\|x^*\right\|^2 + \alpha_n\left(k_n^2 - 1\right)D_2 \\ &\le (1 - t_n)\left\|x_n - x^*\right\|^2 - 2t_n(1 - t_n)\left\langle x_n - x^*, x^*\right\rangle \\ &\quad + t_n^2\left\|x^*\right\|^2 + \alpha_n\left(k_n^2 - 1\right)D_2.\end{aligned} \tag{3.36}$$

Thus

$$\|x_{n+1} - x^*\|^2 \le (1 - t_n)\|x_n - x^*\|^2 + t_n\gamma_n + \sigma_n, \quad \forall n \ge 1,$$

where $\gamma_n := -2(1 - t_n)\langle x_n - x^*, x^*\rangle + t_n\|x^*\|^2 \to 0$ as $n \to \infty$, and $\sigma_n = \alpha_n(k_n^2 - 1)D_2$ with $\sum_{n=1}^{\infty} \sigma_n < \infty$. It now follows from Lemma 2.1 that $\{x_n\}_{n=1}^{\infty}$ converges strongly to x^*. Consequently, $\{v_n\}_{n=1}^{\infty}$ converges strongly to x^*.

Case 2. Suppose that $\{\|x_n - p\|\}_{n=1}^{\infty}$ is not a monotone decreasing sequence, then set $\Gamma_n := \|x_n - p\|^2$, and let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping defined for all $n \ge N_0$ for some sufficiently large N_0 by

$$\tau(n) := \max\{k \in \mathbb{N} : k \le n, \Gamma_k \le \Gamma_{k+1}\}.$$

Then τ is a non-decreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and $\Gamma_{\tau(n)} \le \Gamma_{\tau(n)+1}$ for $n \ge N_0$. Using (c1) and (c2) in (3.32), we obtain

$$\|x_{\tau(n)+1} - x_{\tau(n)}\|^2 \le \frac{1}{\sigma_4}\left[Mt_{\tau(n)} + \left(k_{\tau(n)}^2 - 1\right)D\right] \to 0 \quad \text{as } n \to \infty. \tag{3.37}$$

Following the same argument as in Case 1, we obtain

$$\|v_{\tau(n)} - Tv_{\tau(n)}\| \to 0 \quad \text{as } n \to \infty \quad \text{and} \quad \|x_{\tau(n)} - Tx_{\tau(n)}\| \to 0 \quad \text{as } n \to \infty.$$

As in Case 1 we also obtain that $\{x_{\tau(n)}\}$ and $\{v_{\tau(n)}\}$ converge weakly to some x^* in $F(T)$.

Furthermore, for all $n \ge N_0$, we obtain from (3.36) that

$$\begin{aligned} 0 &\le \|x_{\tau(n)+1} - x^*\|^2 - \|x_{\tau(n)} - x^*\|^2 \\ &\le t_{\tau(n)}\Bigg[-2(1 - t_{\tau(n)})\langle x_{\tau(n)} - x^*, x^*\rangle + t_{\tau(n)}\|x^*\|^2 \\ &\quad + D_2\alpha_{\tau(n)}\frac{(k_{\tau(n)}^2 - 1)}{t_{\tau(n)}} - \|x_{\tau(n)} - x^*\|^2\Bigg]. \end{aligned} \tag{3.38}$$

It follows from (3.38) that

$$\begin{aligned} \|x_{\tau(n)} - x^*\|^2 &\le 2(1 - t_{\tau(n)})\langle x^* - x_{\tau(n)}, x^*\rangle + t_{\tau(n)}\|x^*\|^2 \\ &\quad + D_2\alpha_{\tau(n)}\frac{(k_{\tau(n)}^2 - 1)}{t_{\tau(n)}} \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Thus

$$\lim_{n\to\infty} \Gamma_{\tau(n)} = \lim_{n\to\infty} \Gamma_{\tau(n)+1}.$$

Furthermore, for $n \ge N_0$, we have $\Gamma_n \le \Gamma_{\tau(n)+1}$ if $n \ne \tau(n)$ (*i.e.*, $\tau(n) < n$), because $\Gamma_j > \Gamma_{j+1}$ for $\tau(n) + 1 \le j \le n$. It then follows that for all $n \ge N_0$ we have

$$0 \le \Gamma_n \le \max\{\Gamma_{\tau(n)}, \Gamma_{\tau(n)+1}\} = \Gamma_{\tau(n)+1}.$$

This implies $\lim_{n\to\infty} \Gamma_n = 0$, and hence $\{x_n\}_{n=1}^{\infty}$ converges strongly to $x^* \in F(T)$. □

Corollary 3.3 *Let C be a nonempty closed convex subset of a real Hilbert space H with $0 \in C$, and let $T : C \to C$ be a uniformly L-Lipschitzian asymptotically pseudocontractive mapping with a sequence $\{k_n\}_{n=1}^{\infty} \subseteq [1,\infty)$, $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $F(T) \neq \emptyset$. Let $\{t_n\}_{n=1}^{\infty}$, $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be real sequences in $(0,1)$ satisfying the conditions*:

(c1) $\lim_{n\to\infty} t_n = 0$;
(c2) $\sum_{n=1}^{\infty} t_n = \infty$;
(c3) $0 < \epsilon \leq \alpha_n \leq (1 - t_n)\beta_n \leq \beta_n \leq b < \frac{2}{(1+\lambda)+\sqrt{(1+\lambda)^2+4L^2}}$, *where* $\lambda = \sup_n k_n$;
(c4) $\lim_{n\to\infty} \frac{(k_n-1)}{t_n} = 0$.

Then the sequence $\{x_n\}_{n=1}^{\infty}$ generated from an arbitrary $x_1 \in C$ by

$$\begin{cases} v_n = (1 - t_n)x_n, \quad n \geq 1, \\ y_n = (1 - \beta_n)v_n + \beta_n T^n v_n, \quad n \geq 1, \\ x_{n+1} = (1 - \alpha_n)v_n + \alpha_n T^n y_n, \quad n \geq 1, \end{cases}$$

converges strongly to a fixed point of T.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
MO conceived of the study. All authors carried out the research, read and approved the final manuscript.

Acknowledgements
The work was completed when the first author was visiting the Abdus Salam International Centre for Theoretical Physics (ICTP), Trieste, Italy as an Associate. He is grateful to the Centre for the invaluable facilities at the Centre and for hospitality. The authors are grateful to the referees for their valuable comments.

Received: 6 August 2013 Accepted: 5 November 2013 Published: 10 Dec 2013

References

1. Goebel, K, Kirk, WA: A fixed point theorem for asymptotically nonexpansive mappings. Proc. Am. Math. Soc. **35**(1), 171-174 (1972)
2. Qihou, L: Convergence theorems of the sequence of iterates for asymptotically demicontractive and hemicontractive mappings. Nonlinear Anal. **26**(11), 1835-1842 (1996)
3. Osilike, MO, Aniagbosor, SC, Akuchu, BG: Fixed point of asymptotically demicontractive mappings in arbitrary Banach spaces. Panam. Math. J. **12**(2), 77-88 (2002)
4. Osilike, MO: Iterative approximation of fixed points of asymptotically demicontractive mappings. Indian J. Pure Appl. Math. **29**(12), 1291-1300 (1998)
5. Osilike, MO, Udomene, A, Igbokwe, DI, Akuchu, BG: Demiclosedness principle and convergence theorems for *k*-strictly asymptotically pseudocontractive maps. J. Math. Anal. Appl. **326**, 1334-1345 (2007)
6. Schu, J: Iterative construction of fixed points of asymptotically nonexpansive mappings. J. Math. Anal. Appl. **158**, 407-413 (1991)
7. Schu, J: Weak and strong convergence to fixed points of asymptotically nonexpansive mappings. Bull. Aust. Math. Soc. **43**(1), 153-159 (1991)
8. Zhang, Y, Xie, Z: Convergence of the modified Mann's iterative method for asymptotically *k*-strictly. Fixed Point Theory Appl. **2011**, Article ID 100 (2011)
9. Long, QX, Qun, WC, Juan, SM: Strong convergence theorems for asymptotically strictly pseudocontractive maps in Hilbert spaces. J. Math. Res. Expo. **29**(1), 43-51 (2009)
10. Zhang, SS: Strong convergence theorem for strictly asymptotically pseudocontractive mappings in Hilbert spaces. Acta Math. Sin. Engl. Ser. **27**(7), 1367-1378 (2011)
11. Mann, WR: Mean value methods in iteration. Proc. Am. Math. Soc. **4**, 506-610 (1953)
12. Genel, A, Lindenstraus, J: An example concerning fixed points. Isr. J. Math. **22**(1), 81-86 (1975)
13. Yao, Y, Zhou, H, Liou, Y-C: Strong convergence of a modified Krasnoselski-Mann iterative algorithm for non-expansive mappings. J. Appl. Math. Comput. **29**, 383-389 (2009)
14. Maingé, PE: Regularized and inertial algorithms for common fixed points of nonlinear operators. J. Math. Anal. Appl. **344**, 876-887 (2008)
15. Maingé, PE, Măruşter, Ş: Convergence in norm of modified Krasnoselski-Mann iterations for fixed points of demicontractive mappings. Appl. Math. Comput. **217**(24), 9864-9874 (2011)
16. Xu, H-K: Iterative algorithm for nonlinear operators. J. Lond. Math. Soc. (2) **66**, 240-256 (2002)
17. Marino, G, Xu, H-K: Weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces. J. Math. Anal. Appl. **329**, 336-346 (2007)
18. Zhou, H: Demiclosedness principle with applications for asymptotically pseudo-contractions in Hilbert spaces. Nonlinear Anal. **70**, 3140-3145 (2009)

10.1186/1687-1812-2013-334
Cite this article as: Osilike et al.: Strong convergence of new iterative algorithms for certain classes of asymptotically pseudocontractions. *Fixed Point Theory and Applications* 2013, 2013:334