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Convergence of perturbed composite implicit iteration process for a finite family of asymptotically nonexpansive mappings

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Abstract

In this paper, we introduce a perturbed composite implicit iterative process with errors for a finite family of asymptotically nonexpansive mappings. Under Opial's condition, semicompact and $\liminf_{n\to\infty} d(x_n, F(T)) = 0$ conditions, respectively, we prove that this iterative scheme converges weakly or strongly to a common fixed point of a finite family of asymptotically nonexpansive mappings in uniformly convex Banach spaces. The results presented in this paper generalize and improve the corresponding results of Sun (J. Math. Anal. Appl. 286:351-358, 2003), Chang (J. Math. Anal. Appl. 313:273-283, 2006), Gu (J. Math. Anal. Appl. 329:766-776, 2007), Thakur (Appl. Math. Comput. 190:965-973, 2007), Rafiq (Rostock. Math. Kolloqu. 62:21-39, 2007) and some others.

MSC: 47H9; 47H10

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1 Introduction

Let E be a real Banach space, K be a nonempty convex subset of E. Let $\{T_1, T_2, \ldots, T_N\}$ be a finite family of mappings from K into itself, and $F(T_i)$ be the set of fixed points of T_i ($i \in I = \{1, 2, \ldots, N\}$). $F(T)$ denotes the set of common fixed points of $\{T_1, T_2, \ldots, T_N\}$.

Recently, Xu and Ori [1] have introduced an implicit iteration process for a finite family of nonexpansive mappings as follows:

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_n x_n, \quad \forall n \geq 1, \tag{1}$$

where $T_n = T_{n(\operatorname{mod} N)}$ (here the $\operatorname{mod} N$ function takes values in I), $\{\alpha_n\}$ be a real sequence in $[0, 1]$, x_0 be an initial point in K.

Sun [2] have extended this iterative process defined by Xu and Ori to a new iterative process for a finite family of asymptotically nonexpansive mappings, which is defined as follows:

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_i^k x_n, \quad n \geq 1, \tag{2}$$

where $n = (k - 1)N + i$, $i \in I$.

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Chang [3] have discussed the convergence of the implicit iteration process with errors for a finite family of asymptotically nonexpansive mappings as follows:

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_{i(n)}^{k(n)} x_n + u_n, \quad n \geq 1, \tag{3}$$

where $n = (k(n)-1)N + i(n)$, $i(n) \in I$, and $k(n) \geq 1$ with $k(n) \to \infty$ as $n \to \infty$. Under the hypotheses $\sum_{n=1}^{\infty} \|u_n\| < \infty$ and some appropriate conditions, they proved some results of weak and strong convergence for $\{x_n\}$ defined by (3). However, the condition $\sum_{n=1}^{\infty} \|u_n\| < \infty$ is not too reasonable, because this implies that $\{u_n\}$ are very small for n sufficiently big.

Gu [4] has extended the above implicit iteration processes. A composite implicit iteration process with random errors was introduced as follows:

$$\begin{cases} x_n = (1-\alpha_n-\gamma_n)x_{n-1} + \alpha_n T_n y_n + \gamma_n u_n, & n \geq 1; \\ y_n = (1-\beta_n-\delta_n)x_n + \beta_n T_n x_n + \delta_n v_n, & n \geq 1, \end{cases} \tag{4}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ are four real sequences in $[0,1]$ satisfying $\alpha_n + \gamma_n \leq 1$ and $\beta_n + \delta_n \leq 1$ for all $n \geq 1$, $\{u_n\}$, $\{v_n\}$ are two sequences in K and x_0 is an initial point. Some theorems were established on the strong convergence of the composite implicit iteration process defined by (4) for a finite family of mappings in real Banach spaces.

Thakur [5] has improved the composite implicit iteration process defined by (4) as follows:

$$\begin{cases} x_n = (1-\alpha_n)x_{n-1} + \alpha_n T_{i(n)}^{k(n)} y_n, & n \geq 1; \\ y_n = (1-\beta_n)x_n + \beta_n T_{i(n)}^{k(n)} x_n, & n \geq 1. \end{cases} \tag{5}$$

Some theorems were proved on the weak and strong convergence of the composite implicit iteration process defined by (5) for a finite family of mappings in real uniformly convex Banach spaces.

Rafiq [6] have improved the implicit iterative process. The Mann type implicit iteration process was introduced in Hilbert spaces as follows:

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)T v_n, \quad n \geq 1, \tag{6}$$

where v_n is a perturbation of x_n, and satisfy $\sum_{n\geq 1} \|x_n - v_n\| < \infty$. Moreover, Ciric [7] also did some work in this respect.

Inspired and motivated by the above works, in this paper we will extend and improve the above iterative process to a perturbed composite implicit iterative process for a finite family of asymptotically nonexpansive mappings as follows:

$$\begin{cases} x_n = (1-\alpha_n-\gamma_n)x_{n-1} + \alpha_n T_{i(n)}^{k(n)} y_n + \gamma_n u_n, & n \geq 1; \\ y_n = (1-\beta_n-\delta_n)x_{n-1} + \beta_n T_{i(n)}^{k(n)} \tilde{x}_n + \delta_n v_n, & n \geq 1, \end{cases} \tag{7}$$

where $n = (k(n)-1)N + i(n)$, $i(n) \in I$, $T_n = T_{n(\mathrm{mod}\, N)}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ are four real sequences in $[0,1]$ satisfying $\alpha_n + \gamma_n \leq 1$ and $\beta_n + \delta_n \leq 1$ for all $n \geq 1$, $\{u_n\}$, $\{v_n\}$ are two sequences in K and x_0 is an initial point. $\{\tilde{x}_n\}$ be a sequence in K satisfying $\sum_{n\geq 1} \|x_n - \tilde{x}_n\| <$

∞, which implies that $\|x_n - \tilde{x}_n\| \to 0$ $(n \to \infty)$. Therefore, $\tilde{x}_n$ is known as the perturbation of x_n, and $\{\tilde{x}_n\}$ is known as the perturbed sequence of $\{x_n\}$. This sequence $\{x_n\}$ defined by (7) is said to be the perturbed composite implicit iterative sequence with random errors.

Especially, (I) in the iterative process defined by (7), when $\beta_n = 0$, $\delta_n = 0$ for all $n \geq 1$, we have

$$x_n = (1 - \alpha_n - \gamma_n)x_{n-1} + \alpha_n T_{i(n)}^{k(n)} x_{n-1} + \gamma_n u_n, \quad n \geq 1. \tag{8}$$

At this time, the perturbed composite implicit iterative sequence generated by (7) becomes a Mann-type iterative sequence with random errors.

(II) In the iterative process defined by (7), when $\beta_n = 1$, $\delta_n = 0$ for all $n \geq 1$, we have

$$x_n = (1 - \alpha_n - \gamma_n)x_{n-1} + \alpha_n T_{i(n)}^{2k(n)} \tilde{x}_n + \gamma_n u_n, \quad n \geq 1. \tag{9}$$

At this time, the perturbed composite implicit iterative sequence generated by (7) becomes a perturbed implicit iterative sequence with random errors.

(III) In the iterative process defined by (7), when $x_{n-1} = \tilde{x}_n$ for all $n \geq 1$, we have

$$\begin{cases} x_n = (1 - \alpha_n - \gamma_n)x_{n-1} + \alpha_n T_{i(n)}^{k(n)} y_n + \gamma_n u_n, \quad n \geq 1; \\ y_n = (1 - \beta_n - \delta_n)x_{n-1} + \beta_n T_{i(n)}^{k(n)} x_{n-1} + \delta_n v_n, \quad n \geq 1. \end{cases} \tag{10}$$

At this time, the perturbed composite implicit iterative sequence generated by (7) becomes an Ishikawa-type iterative sequence with random errors for a finite family of asymptotically nonexpansive mappings $\{T_i, i \in I\}$.

From the above iterative processes defined by (1)-(6) and (8)-(10), we know that the iterative process (7) improves and extends some iterative process introduced by the recent literature. Moreover, we point out that the iterative process, defined by (7), in which it is not necessary to compute the value of the given operator at x_n, but compute an approximate point of x_n, are particularly useful in the numerical analysis. Therefore, the iterative sequence generated by (7) is better than some implicit iterative sequences at the existent aspect.

The main purpose of this paper is to study the convergence of the perturbed composite implicit iterative sequence $\{x_n\}$ defined by (7) for a finite family of asymptotically nonexpansive mappings under Opial's condition, semicompact and $\liminf_{n\to\infty} d(x_n, F(T)) = 0$ conditions, respectively. The results presented in this paper generalized and improve the corresponding results of Sun [2], Chang [3], Gu [4], Thakur [5], Rafiq [6], and some others [1, 7–15].

2 Preliminaries

For the sake of convenience, we first recall some definitions and conclusions.

Definition 2.1 Let K be a closed subset of the real Banach space E and $T : K \to K$ be a mapping.

1. T is said to be semicompact, if for any bounded sequence $\{x_n\}$ in K such that $\|Tx_n - x_n\| \to 0$ $(n \to \infty)$, then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to x^* \in E$;

2. T is said to be demiclosed at the origin, if for each sequence $\{x_n\}$ in K, the conditions $x_n \rightharpoonup x_0$ weakly and $Tx_n \to 0$ strongly imply $Tx_0 = 0$;
3. T is said to be asymptotically nonexpansive, if there exists a sequence $h_n \in [1, +\infty)$ with $\lim_{n\to\infty} h_n = 1$ such that

$$\|T^n x - T^n y\| \le h_n \|x - y\|, \quad \forall x, y \in K, n \ge 1. \tag{11}$$

4. Let T is said to be uniformly L-Lipschitizian if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \le L\|x - y\|, \quad \forall x, y \in E, n \ge 1.$$

Definition 2.2 [16] A Banach space X is said to satisfy Opial's condition if $x_n \rightharpoonup x$ weakly as $n \to \infty$ and $x \neq y$ imply that $\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$.

Lemma 2.1 *Let K be a nonempty subset of E, $T_1, T_2, \ldots, T_N : K \to K$ be N asymptotically nonexpansive mappings. Then*

(i) *there exists a sequence $\{h_n\} \subset [1, +\infty)$ with $\lim_{n\to\infty} h_n = 1$ such that*

$$\|T_i^n x - T_i^n y\| \le h_n \|x - y\|, \quad \forall x, y \in K, i \in I, n \ge 1; \tag{12}$$

(ii) *$\{T_1, T_2, \ldots, T_N\}$ is uniformly Lipschitzian, i.e., there exists a constant L such that*

$$\|T_i^n x - T_i^n y\| \le L\|x - y\|, \quad \forall x, y \in K, i \in I, n \ge 1. \tag{13}$$

Proof Since $T_1, T_2, \ldots, T_N : K \to K$ are N asymptotically nonexpansive mappings, then for every $i \in I$ and $n \in N$, there exists $h_n^{(i)} \in [1, +\infty)$ with $\lim_{n\to\infty} h_n^{(i)} = 1$ such that

$$\|T_i^n x - T_i^n y\| \le h_n^{(i)} \|x - y\|, \quad \forall x, y \in E.$$

Taking $h_n = \max\{h_n^{(1)}, h_n^{(2)}, \ldots, h_n^{(N)}\}$, then $h_n \subset [1, +\infty)$, $\lim_{n\to\infty} h_n = 1$ and (12) holds.

An asymptotically nonexpansive mapping must is a uniformly Lipschitzian mapping. Hence, for every $i \in I$ and $n \in N$, there exists L_i such that

$$\|T_i^n x - T_i^n y\| \le L_i \|x - y\|, \quad \forall x, y \in E.$$

Taking $L = \max\{L_1, L_2, \ldots, L_N\}$, it is obvious that (13) holds. □

Lemma 2.2 [17] *Let E be a uniformly convex Banach space, K be a nonempty, closed and convex subset of E and $T : K \to K$ be an asymptotically nonexpansive mapping. Then $I - T$ is demi-closed at zero, i.e., for each sequence $\{x_n\}$ in K, if $\{x_n\}$ convergence weakly to $q \in E$ and $\{(I - T)x_n\}$ converges strongly to 0, then $(I - T)q = 0$.*

Lemma 2.3 [18] *Let E be a Banach space satisfying Opial's condition, $\{x_n\}$ be a sequence in E. Let $u, v \in E$ be such that $\lim_{n\to\infty} \|x_n - u\|$ and $\lim_{n\to\infty} \|x_n - v\|$ exist. If $\{x_{n_k}\}$ and $\{x_{n_l}\}$ are two subsequences of $\{x_n\}$ which converge weakly to u and v, respectively, then $u = v$.*

Lemma 2.4 [19] *Let E be a uniformly convex Banach space, b, c be two constants with $0 < b < c < 1$. Suppose that $\{t_n\}$ is a sequence in $[b, c]$ and $\{x_n\}$, $\{y_n\}$ are two sequences in E. Then the conditions $\lim_{n\to\infty} \|t_n x_n + (1 - t_n)y_n\| = d$, $\limsup_{n\to\infty} \|x_n\| \le d$, $\limsup_{n\to\infty} \|y_n\| \le d$ imply that $\lim_{n\to\infty} \|x_n - y_n\| = 0$, where d is a nonnegative constant.*

Lemma 2.5 [20] *Let $\{a_n\}$, $\{b_n\}$, $\{\delta_n\}$ are three sequences of nonnegative real numbers, if there exists n_0 such that*

$$a_{n+1} \le (1 + \delta_n)a_n + b_n, \quad \forall n > n_0,$$

where $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$. Then
(i) $\lim_{n\to\infty} a_n$ *exists*;
(ii) $\lim_{n\to\infty} a_n = 0$ *whenever* $\liminf_{n\to\infty} a_n = 0$.

Lemma 2.6 *Let E be a real Banach space and K be a nonempty closed convex subset of E. Let $T_1, T_2, \ldots, T_N : K \to K$ be N asymptotically nonexpansive mappings with $F(T) = \bigcap_{i=1}^{N} F(T_i) \ne \varnothing$. Let $\{u_n\}$ and $\{v_n\}$ are two bounded sequences in K. If $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ be four real sequences in $[0, 1]$ satisfying the following conditions*:
(i) $\alpha_n + \gamma_n \le 1$ *and* $\beta_n + \delta_n \le 1$ *for all* $n \ge 1$;
(ii) $\limsup_{n\to\infty} \alpha_n = \alpha < 1$ *or* $\limsup_{n\to\infty} \beta_n = \beta < 1$;
(iii) $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \delta_n < \infty$, $\sum_{n=1}^{\infty}(h_n - 1) < \infty$;
(iv) $\sum_{n=1}^{\infty} \|\tilde{x}_n - x_n\| < \infty$.
Let $\{x_n\}$ be the perturbed composite implicit iterative sequence defined by (7), *then $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F(T)$.*

Proof Take $p \in F(T)$, it follows from (7) and Lemma 2.1 that

$$\begin{aligned} \|x_n - p\| &\le \bigl\|(1 - \alpha_n - \gamma_n)x_{n-1} + \alpha_n T_{i(n)}^{k(n)} y_n + \gamma_n u_n - p\bigr\| \\ &\le (1 - \alpha_n - \gamma_n)\|x_{n-1} - p\| + \alpha_n h_n \|y_n - p\| + \gamma_n \|u_n - p\| \end{aligned} \tag{14}$$

and

$$\begin{aligned} \|y_n - p\| &\le \bigl\|(1 - \beta_n - \delta_n)x_{n-1} + \beta_n T_{i(n)}^{k(n)} \tilde{x}_n + \delta_n v_n - p\bigr\| \\ &\le (1 - \beta_n - \delta_n)\|x_{n-1} - p\| + \beta_n h_n \|\tilde{x}_n - p\| + \delta_n \|v_n - p\| \\ &\le (1 - \beta_n - \delta_n)\|x_{n-1} - p\| + \beta_n h_n \|\tilde{x}_n - x_n\| + \beta_n h_n \|x_n - p\| + \delta_n \|v_n - p\|. \end{aligned} \tag{15}$$

Substituting (15) into (14) and simplifying, we obtain

$$\begin{aligned} \bigl(1 - \alpha_n \beta_n h_n^2\bigr)\|x_n - p\| &\le \bigl[1 - \alpha_n - \gamma_n + \alpha_n h_n (1 - \beta_n - \delta_n)\bigr]\|x_{n-1} - p\| \\ &\quad + \alpha_n \beta_n h_n^2 \|\tilde{x}_n - x_n\| + \alpha_n \delta_n h_n \|v_n - p\| + \gamma_n \|u_n - p\|. \end{aligned} \tag{16}$$

We notice the hypotheses on $\{\alpha_n\}$, $\{\beta_n\}$ and $\{h_n\}$, by $\limsup_{n\to\infty} \alpha_n = \alpha < 1$, there exists $n_0 \in N$ such that

$$1 - \alpha_n \beta_n h_n^2 \ge 1 - \alpha_n h_n^2 \ge \frac{1}{2}(1 - \alpha) > 0, \quad n \ge n_0.$$

It follows from (16) that for $n \geq n_0$

$$\begin{aligned}
\|x_n - p\| &\leq \frac{1-\alpha_n-\gamma_n+\alpha_n h_n(1-\beta_n-\delta_n)}{1-\alpha_n\beta_n h_n^2}\|x_{n-1}-p\| \\
&\quad + \frac{1}{1-\alpha_n\beta_n h_n^2}\big(\alpha_n\beta_n h_n^2\|\tilde{x}_n - x_n\| + \alpha_n\delta_n h_n\|v_n - p\| + \gamma_n\|u_n - p\|\big) \\
&\leq \left[1 + \frac{\alpha_n\beta_n h_n^2 - \alpha_n + \alpha_n h_n(1-\beta_n)}{1-\alpha_n\beta_n h_n^2}\right]\|x_{n-1}-p\| \\
&\quad + \frac{2}{1-\alpha}\big(\alpha_n\beta_n h_n^2\|\tilde{x}_n - x_n\| + \alpha_n\delta_n h_n\|v_n - p\| + \gamma_n\|u_n - p\|\big) \\
&\leq \left\{1 + \frac{2}{1-\alpha}\big[\alpha_n\beta_n h_n(h_n-1) + \alpha_n(h_n-1)\big]\right\}\|x_{n-1}-p\| \\
&\quad + \frac{2}{1-\alpha}\big(\alpha_n\beta_n h_n^2\|\tilde{x}_n - x_n\| + \alpha_n\delta_n h_n\|v_n - p\| + \gamma_n\|u_n - p\|\big).
\end{aligned}$$

Hence, we have

$$\|x_n - p\| \leq (1+\theta_n)\|x_{n-1}-p\| + \eta_n, \quad n \geq n_0, \tag{17}$$

where

$$\theta_n = \frac{2}{1-\alpha}\big[\alpha_n\beta_n h_n(h_n-1) + \alpha_n(h_n-1)\big], \quad n \geq n_0$$

and

$$\eta_n = \frac{2}{1-\alpha}\big(\alpha_n\beta_n h_n^2\|\tilde{x}_n - x_n\| + \alpha_n\delta_n h_n\|v_n - p\| + \gamma_n\|u_n - p\|\big), \quad n \geq n_0.$$

From condition (iii), it is obvious that $\sum_{n=1}^{\infty}\theta_n < \infty$. In addition, since $\{\|u_n\|\}$, $\{\|v_n\|\}$ are all bounded, we deduce that $\sum_{n=1}^{\infty}\eta_n < \infty$ form (iii)-(iv). By virtue of (17) and Lemma 2.5, we obtain that $\lim_{n\to\infty}\|x_n - p\|$ exists. This completes the proof of Lemma 2.6. □

3 Main results and proofs

Theorem 3.1 *Let E be a real Banach space and K be a nonempty, closed and convex subset of E. Let $T_1, T_2, \ldots, T_N : K \to K$ be N asymptotically nonexpansive mappings with $F(T) = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $\{u_n\}$ and $\{v_n\}$ are two bounded sequences in K. If $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ be four real sequences in $[0,1]$ satisfying the following conditions*:

(i) $\alpha_n + \gamma_n \leq 1$ *and* $\beta_n + \delta_n \leq 1$ *for all* $n \geq 1$;
(ii) $\limsup_{n\to\infty}\alpha_n < 1$ *or* $\limsup_{n\to\infty}\beta_n < 1$;
(iii) $\sum_{n=1}^{\infty}\gamma_n < \infty$, $\sum_{n=1}^{\infty}\delta_n < \infty$, $\sum_{n=1}^{\infty}(h_n - 1) < \infty$;
(iv) $\sum_{n=1}^{\infty}\|\tilde{x}_n - x_n\| < \infty$.

Then the perturbed composite implicit iterative sequence $\{x_n\}$ defined by (7) *converges strongly to a common fixed point of $\{T_1, T_2, \ldots, T_N\}$ if and only if $\liminf_{n\to\infty} d(x_n, F(T)) = 0$.*

Proof The necessity of Theorem 3.1 is obvious. Now we prove the sufficiency of Theorem 3.1.

For arbitrary $p \in F(T)$, it follows from (17) in Lemma 2.6 that

$$\|x_n - p\| \le (1+\theta_n)\|x_{n-1} - p\| + \eta_n, \quad \forall n \ge n_0,$$

where $\sum_{n=1}^{\infty} \theta_n < \infty$ and $\sum_{n=1}^{\infty} \eta_n < \infty$. Hence, we have

$$d\bigl(x_n, F(T)\bigr) \le (1+\theta_n) d\bigl(x_{n-1}, F(T)\bigr) + \eta_n, \quad \forall n \ge n_0. \tag{18}$$

It follows from (18) and Lemma 2.5 that limit $\lim_{n\to\infty} d(x_n, F(T))$ exists. By the assumption, we have $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Consequently, for any given $\varepsilon > 0$, there exists a positive integer N_1 ($N_1 > n_0$) such that

$$d\bigl(x_n, F(T)\bigr) < \frac{\varepsilon}{8}, \qquad \sum_{k=n}^{\infty} \eta_k < \frac{\varepsilon}{8}, \qquad \sum_{k=n}^{\infty} \theta_k < 1, \quad \forall n \ge N_1,$$

and there exists $p_1 \in F(T)$ such that $\|x_n - p_1\| < \varepsilon/8$, $\forall n \ge N_1$. By (18) and the inequality $1 + x \le \mathrm{e}^x$ ($x \ge 0$), for any $n \ge N_1$ and all $m \ge 1$, we have

$$\begin{aligned}
\|x_{n+m} - x_n\| &\le \exp\{\theta_{n+m-1}\}\|x_{n+m-1} - p_1\| + \eta_{n+m-1} + \|x_n - p_1\| \\
&\le \exp\{\theta_{n+m-1} + \theta_{n+m-2}\}\|x_{n+m-2} - p_1\| + \exp\{\theta_{n+m-1}\}\eta_{n+m-2} \\
&\quad + \eta_{n+m-1} + \|x_n - p_1\| \le \cdots \\
&\le \left[\exp\left\{\sum_{k=n}^{n+m-1} \theta_k\right\} + 1\right]\|x_n - p_1\| + \exp\left\{\sum_{k=n}^{n+m-1} \theta_k\right\} \sum_{k=n}^{n+m-1} \eta_k < \varepsilon.
\end{aligned}$$

Hence, $\{x_n\}$ is a Cauchy sequence in E. By the completeness of E, we can assume that $x_n \to x^* \in K$. Next we prove that $F(T)$ is a close subset of K. Let $\{p_n\}$ is a sequence in $F(T)$ which converges strongly to some p, then we have for any $i \in I$

$$\|p - T_i p\| \le \|p - p_n\| + \|p_n - T_i p\| \le (1+L)\|p - p_n\| \to 0 \quad (n \to \infty).$$

Thus, $p \in F(T)$, and $F(T)$ is closed. Since $\lim_{n\to\infty} d(x_n, F(T)) = 0$, then $x^* \in F(T)$. Consequently, $\{x_n\}$ defined by (7) converges strongly to a common fixed point of $\{T_1, T_2, \ldots, T_N\}$ in K. This completes the proof of Theorem 3.1. □

Theorem 3.2 *Let E be a real uniformly convex Banach space satisfying Opial's condition and K be a nonempty closed convex subset of E. Let $T_1, T_2, \ldots, T_N : K \to K$ be N asymptotically nonexpansive mappings with $F(T) = \bigcap_{i=1}^{N} F(T_i) \neq \varnothing$. Let $\{u_n\}$ and $\{v_n\}$ are two bounded sequences in K. If $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ be four real sequences in* $[0,1]$ *satisfying the following conditions*:

(i) $\alpha_n + \gamma_n \le 1$ *and* $\beta_n + \delta_n \le 1$ *for all* $n \ge 1$;
(ii) $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$, $\limsup_{n\to\infty} \beta_n < 1$;
(iii) $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \delta_n < \infty$, $\sum_{n=1}^{\infty} (h_n - 1) < \infty$;
(iv) $\sum_{n=1}^{\infty} \|\tilde{x}_n - x_n\| < \infty$.

Then the perturbed composite implicit iterative sequence $\{x_n\}$ defined by (7) *converges weakly to a common fixed point of* $\{T_1, T_2, \ldots, T_N\}$ *in K.*

Proof First, we prove that $\lim_{n\to\infty}\|x_n - T_j x_n\| = 0$ for all $j \in I$.

For any $p \in F(T)$, it follows from Lemma 2.6 that $\lim_{n\to\infty}\|x_n - p\|$ exists. Suppose that $\lim_{n\to\infty}\|x_n - p\| = d$, we have from (7)

$$\begin{aligned}\lim_{n\to\infty}\|x_n - p\| = \lim_{n\to\infty}\big\|(1-\alpha_n)\big[x_{n-1} - p + \gamma_n(u_n - x_{n-1})\big] \\ + \alpha_n\big[T_{i(n)}^{k(n)}y_n - p + \gamma_n(u_n - x_{n-1})\big]\big\| = d.\end{aligned}\tag{19}$$

Since $\lim_{n\to\infty}\|x_n - p\| = d$, then $\{x_n\}$ be a bounded sequence. By virtue of the condition (iii) and the boundedness of sequences $\{x_n\}$ and $\{u_n\}$, we have

$$\begin{aligned}&\limsup_{n\to\infty}\big\|x_{n-1} - p + \gamma_n(u_n - x_{n-1})\big\| \\ &\quad\le \limsup_{n\to\infty}\|x_{n-1} - p\| + \limsup_{n\to\infty}\gamma_n\|u_n - x_{n-1}\| = d.\end{aligned}\tag{20}$$

It follows from $\sum_{n=1}^{\infty}\|\tilde{x}_n - x_n\| < \infty$ that $\lim_{n\to\infty}\|\tilde{x}_n - p\| = \lim_{n\to\infty}\|x_n - p\| = d$. We have

$$\begin{aligned}&\limsup_{n\to\infty}\big\|T_{i(n)}^{k(n)}y_n - p + \gamma_n(u_n - x_{n-1})\big\| \\ &\quad\le \limsup_{n\to\infty}h_n\|y_n - p\| + \limsup_{n\to\infty}\gamma_n\|u_n - x_{n-1}\| \\ &\quad\le \limsup_{n\to\infty}\big[(1-\beta_n-\delta_n)\|x_{n-1} - p\| + \beta_n h_n\|\tilde{x}_n - p\| + \delta_n\|v_n - p\|\big] = d.\end{aligned}\tag{21}$$

Therefore, by (19), (20), (21), (ii) and Lemma 2.4, we obtain that

$$\lim_{n\to\infty}\big\|T_{i(n)}^{k(n)}y_n - x_{n-1}\big\| = 0.$$

Hence,

$$\lim_{n\to\infty}\|x_n - x_{n-1}\| \le \lim_{n\to\infty}\big[\alpha_n\big\|T_{i(n)}^{k(n)}y_n - x_{n-1}\big\| + \gamma_n\|u_n - x_{n-1}\|\big] = 0,\tag{22}$$

which implies that $\lim_{n\to\infty}\|x_n - x_{n+j}\| = 0$ for all $j \in I$. On the other hand, we also have

$$\begin{aligned}\lim_{n\to\infty}\big\|T_{i(n)}^{k(n)}x_n - x_n\big\| &\le \lim_{n\to\infty}\big[\|x_n - x_{n-1}\| + \big\|x_{n-1} - T_{i(n)}^{k(n)}y_n\big\| + \big\|T_{i(n)}^{k(n)}y_n - T_{i(n)}^{k(n)}x_n\big\|\big] \\ &\le \lim_{n\to\infty}h_n\|y_n - x_n\| \le \lim_{n\to\infty}\|y_n - x_{n-1}\| + \lim_{n\to\infty}\|x_n - x_{n-1}\| \\ &\le \lim_{n\to\infty}\big[\beta_n\big\|T_{i(n)}^{k(n)}\tilde{x}_n - x_{n-1}\big\| + \delta_n\|v_n - x_{n-1}\|\big] \\ &\le \lim_{n\to\infty}\big[\beta_n\big\|T_{i(n)}^{k(n)}\tilde{x}_n - T_{i(n)}^{k(n)}x_n\big\| + \beta_n\big\|T_{i(n)}^{k(n)}x_n - x_{n-1}\big\|\big] \\ &\le \lim_{n\to\infty}\big[\beta_n h_n\|\tilde{x}_n - x_n\| + \beta_n\big\|T_{i(n)}^{k(n)}x_n - x_n\big\| + \beta_n\|x_{n-1} - x_n\|\big].\end{aligned}\tag{23}$$

It follows from (22), (23), conditions (ii) and (iv) that

$$\lim_{n\to\infty}\big\|T_{i(n)}^{k(n)}x_n - x_n\big\| = 0.\tag{24}$$

Since for each $n > N$, $n = (n - N)(\operatorname{mod} N)$, $n = (k(n) - 1)N + i(n)$, hence $n - N = [(k(n) - 1) - 1]N + i(n - N)$, *i.e.* $k(n - N) = k(n) - 1$ and $i(n - N) = i(n)$. Therefore, we have

$$\left\| T_n^{k(n)-1} x_n - T_{n-N}^{k(n)-1} x_{n-N} \right\| = \left\| T_n^{k(n)-1} x_n - T_n^{k(n)-1} x_{n-N} \right\| \le L \|x_n - x_{n-N}\| \tag{25}$$

and

$$\left\| T_{n-N}^{k(n)-1} x_{n-N} - x_{n-N} \right\| = \left\| T_{n-N}^{k(n-N)} x_{n-N} - x_{n-N} \right\|. \tag{26}$$

In view of (25) and (26), we have

$$\begin{aligned} \|x_{n-1} - T_n x_n\| &\le \left\| x_{n-1} - T_n^{k(n)} x_n \right\| + \left\| T_n x_n - T_n^{k(n)} x_n \right\| \\ &\le \|x_n - x_{n-1}\| + \left\| x_n - T_n^{k(n)} x_n \right\| + L \left\| x_n - T_n^{k(n)-1} x_n \right\| \\ &\le \|x_n - x_{n-1}\| + \left\| x_n - T_n^{k(n)} x_n \right\| \\ &\quad + L \left(\left\| T_n^{k(n)-1} x_n - T_{n-N}^{k(n)-1} x_{n-N} \right\| + \left\| T_{n-N}^{k(n)-1} x_{n-N} - x_n \right\| \right) \\ &\le \|x_n - x_{n-1}\| + \left\| x_n - T_n^{k(n)} x_n \right\| \\ &\quad + \left(L^2 + L\right) \|x_n - x_{n-N}\| + L \left\| T_{n-N}^{k(n-N)} x_{n-N} - x_{n-N} \right\|. \end{aligned} \tag{27}$$

From (24) and (27), it is obviously that $\lim_{n\to\infty} \|x_{n-1} - T_n x_n\| = 0$, which implies that

$$\lim_{n\to\infty} \|x_n - T_n x_n\| \le \lim_{n\to\infty} \left(\|x_{n-1} - T_n x_n\| + \|x_n - x_{n-1}\| \right) = 0.$$

Consequently, we obtain that for all $i \in I$

$$\begin{aligned} \|x_n - T_{n+i} x_n\| &\le \|x_n - x_{n+i}\| + \|x_{n+i} - T_{n+i} x_{n+i}\| + \|T_{n+i} x_{n+i} - T_{n+i} x_n\| \\ &\le (1 + L)\|x_n - x_{n+i}\| + \|x_{n+i} - T_{n+i} x_{n+i}\| \to 0 \quad (n \to \infty). \end{aligned} \tag{28}$$

By virtue of (28), we have $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ for all $i \in I$.

Since E is uniformly convex, every bounded subset of E is weakly compact. Again since $\{x_n\}$ is a bounded subset in K, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to q in K, and $\lim_{n_k\to\infty} \|x_{n_k} - T_i x_{n_k}\| = 0$ for all $i \in I$. By Lemma 2.2, we have that $(I - T_i)q = 0$. Hence, $q \in F(T_i)$ for all $i \in I$. Therefore, $q \in F(T)$.

Next, we prove that $\{x_n\}$ converges weakly to q. Suppose that contrary, then there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to $q_1 \in K$ and $q \neq q_1$. Using the same method, we can prove that $q_1 \in F(T)$ and limit $\lim_{n\to\infty} \|x_n - q_1\|$ exists. Without loss generality, we assume that $\lim_{n\to\infty} \|x_n - q\| = d_1$, $\lim_{n\to\infty} \|x_n - q_1\| = d_2$, where d_1, d_2 are two nonnegative constants. By virtue of the Opial's condition of E, we have

$$\begin{aligned} d_1 &= \limsup_{n_k\to\infty} \|x_{n_k} - q\| < \limsup_{n_k\to\infty} \|x_{n_k} - q_1\| = \limsup_{n\to\infty} \|x_n - q_1\| \\ &= \limsup_{n_j\to\infty} \|x_{n_j} - q_1\| < \limsup_{n_j\to\infty} \|x_{n_j} - q\| = d_1. \end{aligned}$$

This is contradictory. Hence, $q = q_1$, which implies that $\{x_n\}$ converges weakly to q. The proof of Theorem 3.2 is completed. □

Theorem 3.3 *Let E be a real uniformly convex Banach space and K be a nonempty, closed and convex subset of E. Let $T_1, T_2, \dots, T_N : K \to K$ be N asymptotically nonexpansive mappings with $F(T) = \bigcap_{i=1}^{n} F(T_i) \neq \varnothing$ and at least there exists T_i ($i \in I$), it is semicompact. Let $\{u_n\}$ and $\{v_n\}$ are two bounded sequences in K. If $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ be four real sequences in* $[0,1]$ *satisfying the following conditions*:

(i) $\alpha_n + \gamma_n \leq 1$ *and* $\beta_n + \delta_n \leq 1$ *for all* $n \geq 1$;
(ii) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$, $\limsup_{n\to\infty} \beta_n < 1$;
(iii) $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \delta_n < \infty$, $\sum_{n=1}^{\infty} (h_n - 1) < \infty$;
(iv) $\sum_{n=1}^{\infty} \|\tilde{x}_n - x_n\| < \infty$.

Then the perturbed composite implicit iterative sequence $\{x_n\}$ defined by (7) *converges strongly to a common fixed point of $\{T_1, T_2, \dots, T_N\}$ in K.*

Proof Without loss of generality, we assume that T_1 is semicompact. By Theorem 3.2, we have $\lim_{n\to\infty} \|x_n - T_1 x_n\| = 0$. Hence, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\} \to x^*$ as $j \to \infty$. Therefore, we have for all $i \in I$

$$\left\| T_i x^* - x^* \right\| \leq \left\| T_i x^* - T_i x_{n_j} \right\| + \| T_i x_{n_j} - x_{n_j} \| + \left\| x_{n_j} - x^* \right\|. \tag{29}$$

It follows from (29) that $\|T_i x^* - x^*\| = 0$ for all $i \in I$. This implies that $x^* \in F(T)$. Therefore, x^* be a common fixed point of $\{T_i, i \in I\}$. By virtue of Lemma 2.6, $\lim_{n\to\infty} \|x_n - x^*\|$ exists. It follows from $x_{n_j} \to x^* \in E$ that $\lim_{n\to\infty} \|x_n - x^*\| = 0$. Hence, the perturbed composites implicit iterative sequence $\{x_n\}$ generated by (7) strongly converges to a common fixed point of $\{T_i, i \in I\}$. This completes the proof of Theorem 3.3. □

Corollary 3.4 *Let E be a real Banach space and K be a nonempty closed convex subset of E. Let $T_1, T_2, \dots, T_N : K \to K$ be N asymptotically nonexpansive mappings with $F(T) = \bigcap_{i=1}^{N} F(T_i) \neq \varnothing$ and let $\{u_n\}$ is a bounded sequence in K. If $\{\alpha_n\}$, $\{\gamma_n\}$ be two real sequences in* $[0,1]$ *satisfying the following conditions*:

(i) $\alpha_n + \gamma_n \leq 1$ *for all* $n \geq 1$;
(ii) $\limsup_{n\to\infty} \alpha_n < 1$;
(iii) $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} (h_n - 1) < \infty$;
(iv) $\sum_{n=1}^{\infty} \|\tilde{x}_n - x_n\| < \infty$.

Then the perturbed implicit iterative sequence $\{x_n\}$ defined by (9) *converges strongly to a common fixed point of $\{T_1, T_2, \dots, T_N\}$ if and only if* $\liminf_{n\to\infty} d(x_n, F(T)) = 0$.

Proof It is enough to take $\beta_n = 1$, $\delta_n = 0$ for all $n \in \mathbb{N}$ in Theorem 3.1. □

Corollary 3.5 *Let E be a real uniformly convex Banach space satisfying Opial's condition and K be a nonempty closed convex subset of E. Let $T_1, T_2, \dots, T_N : K \to K$ be N asymptotically nonexpansive mappings with $F(T) = \bigcap_{i=1}^{N} F(T_i) \neq \varnothing$ and let $\{u_n\}$ is a bounded sequence in K. If $\{\alpha_n\}$, $\{\gamma_n\}$ be two real sequences in* $[0,1]$ *satisfying the following conditions*:

(i) $\alpha_n + \gamma_n \leq 1$ *for all* $n \geq 1$;
(ii) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;
(iii) $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} (h_n - 1) < \infty$.

Then the Mann type iterative sequence $\{x_n\}$ defined by (8) *converges weakly to a common fixed point of $\{T_1, T_2, \dots, T_N\}$ in K.*

Proof It is sufficient to take $\beta_n = \delta_n = 0$ for all $n \in \mathbb{N}$ in Theorem 3.2. □

Corollary 3.6 *Let E be a real uniformly convex Banach space and K be a nonempty closed convex subset of E. Let $T_1, T_2, \ldots, T_N : K \to K$ be N asymptotically nonexpansive mappings with $F(T) = \bigcap_{i=1}^{n} F(T_i) \neq \emptyset$ and at least there exists T_i $(i \in I)$, it is semicompact. Let $\{u_n\}$ is a bounded sequence in K. If $\{\alpha_n\}$, $\{\gamma_n\}$ be two real sequences in $[0,1]$ satisfying the following conditions*:

(i) $\alpha_n + \gamma_n \leq 1$ *for all* $n \geq 1$;

(ii) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;

(iii) $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} (h_n - 1) < \infty$.

Then the Mann type iterative sequence $\{x_n\}$ defined by (8) *converges strongly to a common fixed point of $\{T_1, T_2, \ldots, T_N\}$ in K.*

Proof It is enough to take $\beta_n = \delta_n = 0$ for all $n \in \mathbb{N}$ in Theorem 3.3. □

Competing interests

The author did not provide this information.

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