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Oscillation results for certain forced fractional difference equations with damping term

Wei Nian Li*

*Correspondence: wnli@263.net
Department of Mathematics, Binzhou University, Binzhou, Shandong 256603, P.R. China

Abstract
In this paper, we establish two sufficient conditions for the oscillation of forced fractional difference equations with damping term of the form

$$(1+p(t))\Delta(\Delta^\alpha x(t)) + p(t)\Delta^\alpha x(t) + f(t, x(t)) = g(t), \quad t \in \mathbb{N}_0,$$

with initial condition $\Delta^{\alpha-1}x(t)|_{t=0} = x_0$, where $0 < \alpha < 1$ is a constant, $\Delta^\alpha x$ is the Riemann-Liouville fractional difference operator of order α of x, and $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$.

MSC: 26A33; 39A12; 39A21

Keywords: oscillation; forced fractional difference equation; damping term

1 Introduction

In the past few years, the theory of fractional differential equations and their applications have been investigated extensively. For example, see monographs [1–4]. In recent years, fractional difference equations, which are the discrete counterpart of the corresponding fractional differential equations, have comparably gained attention by some researchers. Many interesting results were established. For instance, see papers [5–20] and the references therein.

The oscillation theory as a part of the qualitative theory of differential equations and difference equations has been developed rapidly in the last decades, and there have been many results on the oscillatory behavior of integer-order differential equations and integer-order difference equations. In particular, we notice that the oscillation of fractional differential equations has been developed significantly in recent years. We refer the reader to [21–33] and the references therein. However, to the best of author's knowledge, up to now, very little is known regarding the oscillatory behavior of fractional difference equations [18–20]. Unfortunately, the main results of paper [18] are incorrect. The main reason for the mistakes in [18] is an incorrect relation of $t^{(\alpha-1)}$ and $t^{(1-\alpha)}$. In fact, noting the definition of $t^{(\alpha)} = \frac{\Gamma(t+1)}{\Gamma(t+1-\alpha)}$, it is easy to observe that $t^{(\alpha-1)}t^{(1-\alpha)} \neq 1$.

In this paper we investigate the oscillation of forced fractional difference equations with damping term of the form

$$(1+p(t))\Delta\left(\Delta^\alpha x(t)\right) + p(t)\Delta^\alpha x(t) + f\left(t, x(t)\right) = g(t), \quad t \in \mathbb{N}_0, \tag{1}$$

with initial condition $\Delta^{\alpha-1}x(t)|_{t=0} = x_0$, where $0 < \alpha < 1$ is a constant, $\Delta^\alpha x$ is the Riemann-Liouville difference operator of order α of x, and $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$.

Throughout this paper, we assume that

(A) $p(t)$ and $g(t)$ are real sequences, $p(t) > -1$, $f : \mathbb{N}_0 \times \mathbb{R} \to \mathbb{R}$, and $xf(t,x) > 0$ for $x \neq 0$, $t \in \mathbb{N}_0$.

A solution $x(t)$ of the Eq. (1) is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is nonoscillatory.

2 Preliminaries

In this section, we present some preliminary results of discrete fractional calculus.

Definition 2.1 ([7]) Let $\nu > 0$. The νth fractional sum f is defined by

$$\Delta^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{(\nu-1)} f(s), \tag{2}$$

where f is defined for $s = a \mod(1)$, $\Delta^{-\nu} f$ is defined for $t = (a+\nu) \mod(1)$, and $t^{(\nu)} = \frac{\Gamma(t+1)}{\Gamma(t+1-\nu)}$. The fractional sum $\Delta^{-\nu} f$ maps functions defined on $\mathbb{N}_a = \{a, a+1, a+2, \ldots\}$ to functions defined on $\mathbb{N}_{a+\nu} = \{a+\nu, a+\nu+1, a+\nu+2, \ldots\}$, where Γ is the gamma function.

Definition 2.2 ([7]) Let $\mu > 0$ and $m-1 < \mu < m$, where m is a positive integer, $m = \lceil \mu \rceil$. Set $\nu = m - \mu$. The μth fractional difference is defined as

$$\Delta^\mu f(t) = \Delta^{m-\nu} f(t) = \Delta^m \Delta^{-\nu} f(t), \tag{3}$$

where $\lceil \mu \rceil$ is the ceiling function of μ.

Lemma 2.3 ([7]) *Let f be a real-valued function defined on $\mathbb{N}_a$, and let $\mu, \nu > 0$. Then the following equalities hold*:

$$\Delta^{-\nu}\left[\Delta^{-\mu} f(t)\right] = \Delta^{-(\mu+\nu)} f(t) = \Delta^{-\mu}\left[\Delta^{-\nu} f(t)\right]; \tag{4}$$

$$\Delta^{-\nu} \Delta f(t) = \Delta \Delta^{-\nu} f(t) - \frac{(t-a)^{(\nu-1)}}{\Gamma(\nu)} f(a). \tag{5}$$

Lemma 2.4 *Let $x(t)$ be a solution of Eq.* (1), *and let*

$$E(t) = \sum_{s=t_0}^{t-1+\alpha} (t-s-1)^{(-\alpha)} x(s), \quad t \in \mathbb{N}_0. \tag{6}$$

Then

$$\Delta E(t) = \Gamma(1-\alpha) \Delta^\alpha x(t). \tag{7}$$

Proof Using Definition 2.1, it follows from (6) that

$$\begin{aligned} E(t) &= \sum_{s=t_0}^{t-1+\alpha} (t-s-1)^{(-\alpha)} x(s) = \sum_{s=t_0}^{t-(1-\alpha)} (t-s-1)^{((1-\alpha)-1)} x(s) \\ &= \Gamma(1-\alpha) \Delta^{-(1-\alpha)} x(t). \end{aligned}$$

Therefore,

$$\Delta E(t) = \Gamma(1-\alpha)\Delta\Delta^{-(1-\alpha)}x(t) = \Gamma(1-\alpha)\Delta^{\alpha}x(t).$$

The proof of Lemma 2.4 is complete. □

Lemma 2.5 ([6]) *Let $\mu \in \mathbb{R} \setminus \{\dots,-2,-1\}$. Then*

$$\Delta^{-\nu}t^{(\mu)} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)}t^{(\mu+\nu)}. \tag{8}$$

3 Main results

In this section, we establish the oscillation results for Eq. (1).

Theorem 3.1 *Suppose that, for $t_0 \in \mathbb{N}_0$,*

$$\liminf_{t\to\infty}\left\{\sum_{s=0}^{t-\alpha}\frac{(t-s-1)^{(\alpha-1)}}{V(s)}\left[M+\sum_{\xi=t_0}^{s-1}g(\xi)V(\xi)\right]\right\} < 0 \tag{9}$$

and

$$\limsup_{t\to\infty}\left\{\sum_{s=0}^{t-\alpha}\frac{(t-s-1)^{(\alpha-1)}}{V(s)}\left[M+\sum_{\xi=t_0}^{s-1}g(\xi)V(\xi)\right]\right\} > 0, \tag{10}$$

where M is a constant, and

$$V(t) = \prod_{s=t_0}^{t-1}\bigl(1+p(s)\bigr). \tag{11}$$

Then every solution $x(t)$ of Eq. (1) is oscillatory.

Proof Suppose to the contrary that there is a nonoscillatory solution $x(t)$ of Eq. (1) which has no zero in $\mathbb{N}_{t_0} = \{t_0, t_0+1, t_0+2, \dots\}$. Then $x(t) > 0$ or $x(t) < 0$ for $t \in \mathbb{N}_{t_0}$.

Case 1. $x(t) > 0$, $t \in \mathbb{N}_{t_0}$. Noting assumption (A), from Eq. (1) we have

$$\bigl(1+p(t)\bigr)\Delta\bigl(\Delta^{\alpha}x(t)\bigr) + p(t)\Delta^{\alpha}x(t) = -f\bigl(t,x(t)\bigr) + g(t) < g(t). \tag{12}$$

Therefore, using the fundamental property of Δ and noting the definition of $V(t)$, we get

$$\begin{aligned}\Delta\bigl(\bigl(\Delta^{\alpha}x(t)\bigr)V(t)\bigr) &= \Delta\bigl(\Delta^{\alpha}x(t)\bigr)V(t+1) + \Delta^{\alpha}x(t)\Delta V(t)\\ &= \Delta\bigl(\Delta^{\alpha}x(t)\bigr)\bigl(1+p(t)\bigr)V(t) + \Delta^{\alpha}x(t)p(t)V(t)\\ &< g(t)V(t).\end{aligned} \tag{13}$$

Summing both sides of (13) from t_0 to $t-1$, we obtain

$$\bigl(\Delta^{\alpha}x(t)\bigr)V(t) < \bigl(\Delta^{\alpha}x(t_0)\bigr)V(t_0) + \sum_{s=t_0}^{t-1}g(s)V(s) = M + \sum_{s=t_0}^{t-1}g(s)V(s),$$

where $M = (\Delta^\alpha x(t_0))V(t_0)$, that is,

$$\Delta^\alpha x(t) < \frac{M}{V(t)} + \frac{1}{V(t)}\sum_{s=t_0}^{t-1} g(s)V(s). \tag{14}$$

Applying the $\Delta^{-\alpha}$ operator to inequality (14), we have

$$\Delta^{-\alpha}\Delta^\alpha x(t) < \Delta^{-\alpha}\left[\frac{M}{V(t)} + \frac{1}{V(t)}\sum_{s=t_0}^{t-1} g(s)V(s)\right]. \tag{15}$$

On the one hand, applying Lemma 2.3 to the left-hand side of (15), we obtain

$$\begin{aligned}\Delta^{-\alpha}\Delta^\alpha x(t) &= \Delta^{-\alpha}\Delta\Delta^{-(1-\alpha)}x(t)\\ &= \Delta\Delta^{-\alpha}\Delta^{-(1-\alpha)}x(t) - \frac{t^{(\alpha-1)}}{\Gamma(\alpha)}x_0\\ &= x(t) - \frac{x_0}{\Gamma(\alpha)}t^{(\alpha-1)}.\end{aligned} \tag{16}$$

On the other hand, using Definition 2.1, it follows from the right-hand side of (15) that

$$\begin{aligned}&\Delta^{-\alpha}\left[\frac{M}{V(t)} + \frac{1}{V(t)}\sum_{s=t_0}^{t-1} g(s)V(s)\right]\\ &\quad= \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{t-\alpha}(t-s-1)^{(\alpha-1)}\left[\frac{M}{V(s)} + \frac{1}{V(s)}\sum_{\xi=t_0}^{s-1} g(\xi)V(\xi)\right].\end{aligned} \tag{17}$$

Combining (15)-(17), we get

$$x(t) < \frac{x_0}{\Gamma(\alpha)}t^{(\alpha-1)} + \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{t-\alpha}(t-s-1)^{(\alpha-1)}\left[\frac{M}{V(s)} + \frac{1}{V(s)}\sum_{\xi=t_0}^{s-1} g(\xi)V(\xi)\right]. \tag{18}$$

It follows from (18) that

$$\begin{aligned}\Gamma(\alpha)t^{1-\alpha}x(t) &< x_0 t^{(\alpha-1)}t^{1-\alpha}\\ &\quad+ t^{1-\alpha}\sum_{s=0}^{t-\alpha}(t-s-1)^{(\alpha-1)}\left[\frac{M}{V(s)} + \frac{1}{V(s)}\sum_{\xi=t_0}^{s-1} g(\xi)V(\xi)\right].\end{aligned} \tag{19}$$

By using the Stirling formula [20]

$$\lim_{t\to\infty}\frac{\Gamma(t)t^\varepsilon}{\Gamma(t+\varepsilon)} = 1, \quad \varepsilon > 0,$$

we obtain

$$\begin{aligned}\lim_{t\to\infty} t^{1-\alpha}t^{(\alpha-1)} &= \lim_{t\to\infty} t^{1-\alpha}\frac{\Gamma(t+1)}{\Gamma(t+1-\alpha+1)}\\ &= \lim_{t\to\infty} t^{1-\alpha}\frac{t\Gamma(t)}{(t+1-\alpha)\Gamma(t+(1-\alpha))}\end{aligned}$$

$$
\begin{aligned}
&= \lim_{t\to\infty} \frac{t}{t+1-\alpha} \frac{\Gamma(t)t^{1-\alpha}}{\Gamma(t+(1-\alpha))} \\
&= 1.
\end{aligned}
\tag{20}
$$

From (20), taking then limit as $t \to \infty$ in (19), we have

$$
\liminf_{t\to\infty} \{t^{1-\alpha} x(t)\} \le -\infty,
$$

which contradicts with $x(t) > 0$.

Case 2. $x(t) < 0$, $t \in \mathbb{N}_{t_0}$. By assumption (A), from Eq. (1) we have

$$
(1 + p(t))\Delta(\Delta^\alpha x(t)) + p(t)\Delta^\alpha x(t) = -f(t, x(t)) + g(t) > g(t). \tag{21}
$$

Therefore,

$$
\Delta((\Delta^\alpha x(t))V(t)) > g(t)V(t). \tag{22}
$$

Summing both sides of (22) from t_0 to $t-1$, we obtain

$$
(\Delta^\alpha x(t))V(t) > (\Delta^\alpha x(t_0))V(t_0) + \sum_{s=t_0}^{t-1} g(s)V(s) = M + \sum_{s=t_0}^{t-1} g(s)V(s),
$$

where $M = (\Delta^\alpha x(t_0))V(t_0)$, that is,

$$
\Delta^\alpha x(t) > \frac{M}{V(t)} + \frac{1}{V(t)} \sum_{s=t_0}^{t-1} g(s)V(s). \tag{23}
$$

Using the procedure of the proof of Case 1, we conclude that

$$
\begin{aligned}
\Gamma(\alpha)t^{1-\alpha}x(t) &> x_0 t^{(\alpha-1)} t^{1-\alpha} \\
&\quad + t^{1-\alpha} \sum_{s=0}^{t-\alpha} (t-s-1)^{(\alpha-1)} \left[\frac{M}{V(s)} + \frac{1}{V(s)} \sum_{\xi=t_0}^{s-1} g(\xi)V(\xi) \right].
\end{aligned}
\tag{24}
$$

By (20), taking the limit as $t \to \infty$ in (24), we have

$$
\limsup_{t\to\infty} \{t^{1-\alpha} x(t)\} \ge \infty,
$$

which contradicts with $x(t) < 0$. The proof of Theorem 3.1 is complete. □

Theorem 3.2 *Suppose that, for* $t_0 \in \mathbb{N}_0$,

$$
\liminf_{t\to\infty} \sum_{s=t_0}^{t-1} \frac{1}{V(s)} \left\{ M + \sum_{\xi=t_0}^{s-1} g(\xi)V(\xi) \right\} = -\infty \tag{25}
$$

and

$$
\limsup_{t\to\infty} \sum_{s=t_0}^{t-1} \frac{1}{V(s)} \left\{ M + \sum_{\xi=t_0}^{s-1} g(\xi)V(\xi) \right\} = \infty, \tag{26}
$$

where M is a constant, and $V(t)$ is defined by (11). *Then every solution $x(t)$ of Eq.* (1) *is oscillatory.*

Proof Suppose to the contrary that there is a nonoscillatory solution $x(t)$ of Eq. (1) that has no zero in $\mathbb{N}_{t_0}$. Then $x(t) > 0$ or $x(t) < 0$ for $t \in \mathbb{N}_{t_0}$.

Case 1. $x(t) > 0$, $t \in \mathbb{N}_{t_0}$. As in the proof of Case 1 in Theorem 3.1, we obtain (14). By Lemma 2.4 it follows from (14) that

$$\Delta E(t) < \frac{\Gamma(1-\alpha)}{V(t)}\left\{M + \sum_{s=t_0}^{t-1} g(s)V(s)\right\}. \tag{27}$$

Summing both sides of (27) from t_0 to $t-1$, we have

$$E(t) < E(t_0) + \Gamma(1-\alpha)\sum_{s=t_0}^{t-1}\frac{1}{V(s)}\left\{M + \sum_{\xi=t_0}^{s-1} g(\xi)V(\xi)\right\}. \tag{28}$$

Letting $t \to \infty$ in (28), we obtain a contradiction with $E(t) > 0$.

Case 2. $x(t) < 0$, $t \in \mathbb{N}_{t_0}$. As in the proof of Case 2 in Theorem 3.1, we obtain (23). By Lemma 2.4 it follows from (23) that

$$\Delta E(t) > \frac{\Gamma(1-\alpha)}{V(t)}\left\{M + \sum_{s=t_0}^{t-1} g(s)V(s)\right\}. \tag{29}$$

Summing both sides of (29) from t_0 to $t-1$, we have

$$E(t) > E(t_0) + \Gamma(1-\alpha)\sum_{s=t_0}^{t-1}\frac{1}{V(s)}\left\{M + \sum_{\xi=t_0}^{s-1} g(\xi)V(\xi)\right\}. \tag{30}$$

Letting $t \to \infty$ in (30), we obtain a contradiction with $E(t) < 0$. This completes the proof of Theorem 3.2. □

4 Examples

In this section, we conclude from the following two examples that the assumptions of Theorem 3.1 and Theorem 3.2 cannot be dropped.

Example 4.1 Consider the following fractional difference equation:

$$\frac{2}{3}\Delta\left(\Delta^{\frac{2}{3}}x(t)\right) + \left(-\frac{1}{3}\right)\Delta^{\frac{2}{3}}x(t) + \frac{\Gamma(t+\frac{1}{3})}{3t\Gamma(t)}x(t) = \frac{3-2\Gamma(\frac{2}{3})}{9}, \quad t \in \mathbb{N}_0, \tag{31}$$

with the initial condition $\Delta^{-\frac{1}{3}}x(t)|_{t=0} = 0$.

Here $\alpha = \frac{2}{3}$, $p(t) = -\frac{1}{3}$, $f(t,x(t)) = \frac{\Gamma(t+\frac{1}{3})}{3t\Gamma(t)}x(t)$, $g(t) = \frac{3-2\Gamma(\frac{2}{3})}{9}$. It is easy to see that

$$V(t) = \prod_{s=1}^{t-1}\left(1+p(t)\right) = \prod_{s=1}^{t-1}\frac{2}{3} = \left(\frac{2}{3}\right)^{t-1}, \qquad g(t) = \frac{3-2\Gamma(\frac{2}{3})}{9} > 0.$$

Therefore, we have

$$\begin{aligned}
&\sum_{s=0}^{t-\frac{2}{3}} \frac{(t-s-1)^{(-\frac{1}{3})}}{V(s)} \left[M + \sum_{\xi=1}^{s-1} g(\xi) V(\xi) \right] \\
&\quad = \sum_{s=0}^{t-\frac{2}{3}} (t-s-1)^{(-\frac{1}{3})} \left(\frac{3}{2} \right)^{s-1} \left[M + \sum_{\xi=1}^{s-1} \frac{3 - 2\Gamma(\frac{2}{3})}{9} \left(\frac{2}{3} \right)^{\xi-1} \right] \\
&\quad > 0,
\end{aligned} \tag{32}$$

which shows that condition (9) of Theorem 3.1 does not hold. It is not difficult to see that $x(t) = t^{(\frac{2}{3})}$ is a nonoscillatory solution of Eq. (31).

Indeed, on the one hand, using Lemma 2.5, we obtain

$$\begin{aligned}
\Delta^{\frac{2}{3}} x(t) = \Delta^{\frac{2}{3}} t^{(\frac{2}{3})} &= \Delta \bigl(\Delta^{-\frac{1}{3}} t^{(\frac{2}{3})} \bigr) \\
&= \Delta \left(\frac{\Gamma(\frac{2}{3}+1)}{\Gamma(\frac{2}{3}+\frac{1}{3}+1)} t^{(\frac{1}{3}+\frac{2}{3})} \right) \\
&= \Delta \left(\frac{2}{3} \Gamma\left(\frac{2}{3} \right) t^{(1)} \right) \\
&= \Delta \left(\frac{2}{3} \Gamma\left(\frac{2}{3} \right) t \right) \\
&= \frac{2}{3} \Gamma\left(\frac{2}{3} \right)
\end{aligned} \tag{33}$$

and

$$\Delta \bigl(\Delta^{\frac{2}{3}} x(t) \bigr) = \Delta \bigl(\Delta^{\frac{2}{3}} t^{(\frac{2}{3})} \bigr) = 0. \tag{34}$$

On the other hand, we have

$$x(t) = t^{(\frac{2}{3})} = \frac{\Gamma(t+1)}{\Gamma(t+1-\frac{2}{3})} = \frac{t\Gamma(t)}{\Gamma(t+\frac{1}{3})}. \tag{35}$$

Combining (33)-(35), we conclude that $x(t) = t^{(\frac{2}{3})}$ is a solution of Eq. (31).

Example 4.2 Consider the following fractional difference equation:

$$\frac{1}{2} \Delta \bigl(\Delta^{\frac{1}{3}} x(t) \bigr) + \left(-\frac{1}{2} \right) \Delta^{\frac{1}{3}} x(t) + \frac{\Gamma(t+\frac{2}{3})}{2t\Gamma(t)} x(t) = \frac{3-\Gamma(\frac{1}{3})}{6}, \quad t \in \mathbb{N}_0, \tag{36}$$

with the initial condition $\Delta^{-\frac{2}{3}} x(t)|_{t=0} = 0$.

Here $\alpha = \frac{1}{3}$, $p(t) = -\frac{1}{2}$, $f(t, x(t)) = \frac{\Gamma(t+\frac{2}{3})}{2t\Gamma(t)} x(t)$, $g(t) = \frac{3-\Gamma(\frac{1}{3})}{6}$. Obviously,

$$V(t) = \prod_{s=1}^{t-1} \bigl(1 + p(t) \bigr) = \prod_{s=1}^{t-1} \frac{1}{2} = \left(\frac{1}{2} \right)^{t-1}, \qquad g(t) = \frac{3-\Gamma(\frac{1}{3})}{6} > 0.$$

Therefore, we have

$$\begin{aligned}
&\sum_{s=1}^{t-1} \frac{1}{V(s)} \left[M + \sum_{\xi=1}^{s-1} g(\xi) V(\xi) \right] \\
&\quad = \sum_{s=1}^{t-1} 2^{s-1} \left[M + \sum_{\xi=1}^{s-1} \frac{3 - \Gamma(\frac{1}{3})}{6} \left(\frac{1}{2} \right)^{\xi-1} \right] \\
&\quad > 0. \qquad (37)
\end{aligned}$$

Thus, condition (25) of Theorem 3.2 does not hold. In fact, we can easily verify that $x(t) = t^{(\frac{1}{3})}$ is a nonoscillatory solution of Eq. (36).

Competing interests
The author declares that there are no competing interests.

Author's contributions
The author read and approved the final manuscript.

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