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Research article

Multiple finite-energy positive weak solutions to singular elliptic problems with a parameter

Tomas Godoy[*] **and Alfredo Guerin**

Facultad de Matematica, Astronomia y Fisica, Universidad Nacional de Cordoba, Ciudad Universitaria, 5000 Cordoba, Argentina

* **Correspondence:** Email: godoy@famaf.unc.edu.ar.

Abstract: Consider the problem $-\Delta u = a(x)\,u^{-\alpha} + f(\lambda, x, u)$ in Ω, $u = 0$ on $\partial\Omega$, $u > 0$ in Ω, where Ω is a bounded domain in $\mathbb{R}^n$ with C^2 boundary, $0 \le a \in L^\infty(\Omega)$, $0 < \alpha < 3$, and $f(\lambda, x, .)$ is nonnegative, and superlinear with subcritical growth at ∞. We prove that, if f satisfies some additional conditions, then, for some $\Lambda > 0$, there are at least two weak solutions in $H_0^1(\Omega) \cap C\left(\overline{\Omega}\right)$ if $\lambda \in (0, \Lambda)$, and there is no weak solution in $H_0^1(\Omega) \cap L^\infty(\Omega)$ if $\lambda > \Lambda$. We also prove that, for each $\lambda \in [0, \Lambda]$, there exists a unique minimal weak solution u_λ in $H_0^1(\Omega) \cap L^\infty(\Omega)$, which is strictly increasing in λ.

Keywords: singular elliptic problems; positive solutions; bifurcation problems; sub and supersolutions; fixed points; multiplicity theorems
Mathematics Subject Classification: 35J75, 35D30, 35J20

1. Introduction and statement of the main results

Consider the singular semilinear elliptic problem with a parameter λ:

$$\begin{cases} -\Delta u = au^{-\alpha} + f(\lambda, ., u) \ \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \\ u > 0 \text{ in } \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $0 \le \lambda < \infty$, $\alpha > 0$, and a, f are functions defined on Ω and $[0, \infty) \times \overline{\Omega} \times [0, \infty)$ respectively.

Singular elliptic problems like (1.1) appear in many fields, for instance in models of the temperature in electrical conductors, and also in models of chemical catalysts process and of non Newtonian flows (see e.g., [6], [10], [17], [20] and the references therein). Existence of solutions to problem (1.1) was studied, when $f \equiv 0$, by Fulks and Maybee [20], Crandall, Rabinowitz and Tartar [11], Lazer and McKenna [33], Diaz, Morel and Oswald [17], Del Pino [15], Bougherara, Giacomoni and Hernández

[3], and, when $f \equiv 0$ and a is a suitable measure, by Oliva and Petitta [36]. The existence of classical solutions to problem (1.1) was proved by Shi and Yao in [40], for the case when Ω and a are regular enough, and $f(\lambda, x, s) = \lambda s^p$, with $0 < \alpha < 1$, and $0 < p < 1$. Related free boundary singular elliptic problems of the form $-\Delta u = \chi_{\{u>0\}}(-u^{-\alpha} + \lambda g(., u))$ in Ω, $u = 0$ on $\partial\Omega$, $u \geq 0$ in Ω, $u \not\equiv 0$ in Ω (that is: $|\{x \in \Omega : u(x) > 0\}| > 0$) were studied by Dávila and Montenegro in [13].

Singular problems of the form

$$\begin{cases} -\Delta u = g(x, u) + h(x, \lambda u) \text{ in } \Omega, \\ \quad u = 0 \text{ on } \partial\Omega, \; u > 0 \text{ in } \Omega, \end{cases} \tag{1.2}$$

were addressed by Coclite and Palmieri in [9]. We would like to note that, as a particular case of their results, if $g(x, u) = au^{-\alpha}$, $a \in C^1\left(\overline{\Omega}\right)$, $a > 0$ in $\overline{\Omega}$, $h \in C^1\left(\overline{\Omega} \times [0, \infty)\right)$, and $\inf_{\overline{\Omega}\times[0,\infty)} \frac{h(x,s)}{1+s} > 0$, then there exists $\lambda^* > 0$ such that, for any $\lambda \in [0, \lambda^*)$, (1.2) has a positive classical solution $u \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$ and, for $\lambda > \lambda^*$, (1.2) has no positive classical solution.

The existence and nonexistence of positive solutions to problems of the form

$$\begin{cases} -\Delta u = -u^{-\gamma} + +\lambda f(x, u) \text{ in } \Omega, \\ \quad u = 0 \text{ on } \partial\Omega, \; u > 0 \text{ in } \Omega, \end{cases} \tag{1.3}$$

was studied by Papageorgiou and Rădulescu [37], in the case where Ω is a bounded domain in $\mathbb{R}^n$ with C^2 boundary, $\gamma \geq 0$, $\lambda \geq 0$, and f is a Carathéodory function. Under some additional assumptions on f, they proved that, if $0 < \gamma < 1$, then there exists $\lambda^* > 0$ such that (1.3) has a solution $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ when $\lambda > \lambda^*$, and has no solution in $H_0^1(\Omega) \cap L^\infty(\Omega)$ for $\lambda > \lambda^*$. Moreover, they proved also that, if $\gamma > 1$, then (1.3) has no solutions in $H_0^1(\Omega) \cap L^\infty(\Omega)$.

Godoy and Guerin ([28], [29] and [30]) considered singular elliptic problems of the form

$$\begin{cases} -\Delta u = \chi_{\{u>0\}} g(., u) + f(., u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \\ u \geq 0 \text{ in } \Omega, \; u \not\equiv 0 \text{ in } \Omega, \end{cases} \tag{1.4}$$

with $s \to g(x, s)$ singular at the origin, and $f : \Omega \times [0, \infty) \to \mathbb{R}$ sublinear at ∞. In [28] and [29] the singular part g was of the form $au^{-\alpha}$. In [30] a more general singular term was allowed; there conditions were established on g in order to limit the strength of the singularity to a level that guarantee the existence of finite Dirichlet energy weak solutions to problem (1.4).

Ghergu and Rădulescu [25] proved existence and nonexistence results for positive classical solutions of singular biparametric bifurcation problems of the form $-\Delta u = g(u) + \lambda |\nabla u|^p + \mu h(., u)$ in Ω, $u = 0$ on $\partial\Omega$, $u > 0$ in Ω, where Ω is a smooth bounded domain in $\mathbb{R}^n$, $0 < p \leq 2$, $\lambda, \mu \geq 0$, $h(x, s)$ is nondecreasing with respect to s, and g is unbounded around the origin. The asymptotic behaviour of the solution around the bifurcation point was also established, provided $g(s)$ behaves like $s^{-\alpha}$ around the origin, for some α in $(0, 1)$.

Dupaigne, Ghergu and Rădulescu [19] addressed Lane-Emden-Fowler equations with convection term and singular potential.

Rădulescu in [38] investigated the existence of blow-up boundary solutions for logistic equations; and for Lane-Emden-Fowler equations, with a singular nonlinearity, and a subquadratic convection term.

The problem $-\Delta u = ag(u)+\lambda h(u)$ in Ω, $u = 0$ on $\partial\Omega$, $u > 0$ in Ω was considered by Cîrstea, Ghergu and Rădulescu [12], in the case when Ω is a regular enough bounded domain in $\mathbb{R}^n$, $0 \leq a \in C^{\beta}\left(\overline{\Omega}\right)$, $0 < h \in C^{0,\beta}[0,\infty)$ for some $\beta \in (0,1)$, h is nondecreasing on $[0,\infty)$, $h(s)/s$ is nonincreasing for $s > 0$, g is nonincreasing on $(0,\infty)$, $\lim_{s\to 0^+} g(s) = +\infty$; and $\sup_{s\in(0,\sigma_0)} s^{\alpha} g(s) < \infty$ for some $\alpha \in (0,1)$ and $\sigma_0 > 0$.

Ghergu and Rădulescu [22], addressed the Lane-Emden-Fowler singular equation $-\Delta u = \lambda f(u) + a(x) g(u)$ in Ω, $u = 0$ on $\partial\Omega$, where Ω is a bounded and regular enough domain in $\mathbb{R}^n$, λ is a positive parameter, f is a nondecreasing function such that $s^{-1} f(s)$ is nondecreasing, $a \in C^{\alpha}\left(\overline{\Omega}\right)$ for some $\alpha \in (0,1)$, and g is unbounded around the origin. Under suitable additional assumptions on a, f, and g, they proved that, for some $\lambda^* > 0$,

(i) There exists a unique solution u_λ in $\mathcal{E} := \left\{u \in C^2(\Omega) \cap C^{1,1-\alpha}\left(\overline{\Omega}\right) \text{ such that } \Delta u \in L^1(\Omega)\right\}$, whenever $0 \leq \lambda < \lambda^*$.

(ii) For $\lambda \geq \lambda^*$ the problem has no solution in $\mathcal{E}$.

Moreover, they obtained an explicit characterization of λ^*, and, in the case $0 \leq \lambda < \lambda^*$, a precise description of the behavior of the solution u_λ near $\partial\Omega$ was also given.

Ghergu and Rădulescu [24], proved the existence of a ground state solution to the singular Lane-Emden-Fowler equation with sublinear convection term $-\Delta u = p(x)(g(u) + f(u) + |\nabla u|^{\alpha})$ in $\mathbb{R}^n$, $u > 0$ in $\mathbb{R}^n$, $\lim_{|x|\to\infty} u(x) = 0$, in the case where $n \geq 3$, $0 < \alpha < 1$, p is a positive function, f is positive, nondecreasing, with sublinear growth, and g is positive, decreasing and unbounded around the origin.

Ghergu and Rădulescu [23], obtained existence and nonexistence results for the two parameter singular problem $-\Delta u + K(x)g(u) = \lambda f(x,u) + \mu h(x)$ in Ω, $u = 0$ on $\partial\Omega$, where Ω is a smooth bounded domain in $\mathbb{R}^n$, λ and μ are positive parameters, h is a positive function, f has sublinear growth, K may change sign, and g is nonnegative and unbounded around the origin.

Aranda and Godoy [2] obtained a multiplicity result for positive solutions in $W^{1,p}_{loc}(\Omega) \cap C\left(\overline{\Omega}\right)$ to problems of the form $-\Delta_p u = g(u) + \lambda h(u)$ in Ω, $u = 0$ on $\partial\Omega$, in the case when Ω is a C^2 bounded and strictly convex domain in $\mathbb{R}^n$, $1 < p \leq 2$; and g, h are locally Lipschitz functions on $(0,\infty)$ and $[0,\infty)$ respectively, with g nonincreasing, and allowed to be singular at the origin; and h nondecreasing, with subcritical growth, and satisfying $\inf_{s>0} s^{-p+1} h(s) > 0$.

Kaufmann and Medri [32] obtained existence and nonexistence results for positive solutions of one dimensional singular problems of the form $-\left((u')^{p-2} u'\right)' = m(x) u^{-\gamma}$ in Ω, $u = 0$ on $\partial\Omega$, where $\Omega \subset \mathbb{R}$ is a bounded open interval, $p > 1$, $\gamma > 0$, and $m : \Omega \to \mathbb{R}$ is a function that may change sign in Ω.

Chhetri, Drábek and Shivaji [8] considered the problem $-\Delta_p u = K(x) f(u) u^{-\delta}$ in $\mathbb{R}^n \setminus \Omega$, $u = 0$ on $\partial\Omega$, $\lim_{|x|\to\infty} u(x) = 0$, in the case where Ω is a simply connected bounded domain in $\mathbb{R}^n$ containing the origin, $n \geq 2$, $1 < p < n$, and $0 \leq \delta < 1$. Under a suitable decay assumption on K at infinity and a growth restriction on f, they proved the existence of a weak solution $u \in C^1\left(\overline{\mathbb{R}^n \setminus \Omega}\right)$ such that $u = 0$ on $\partial\Omega$ pointwise. Moreover, under an additional condition on K, they also proved the uniquennes of such a solution. The existence of radial solutions in the case when Ω is a ball centered at the origin was also addressed.

Recently, Saoudi, Agarwal and Mursaleenin [39], proved that, for λ positive and small enough, at least two positive weak solutions in $H^1_0(\Omega)$ exist for singular elliptic problems of the form $-\operatorname{div}(A(x)\nabla u) = u^{-\alpha} + \lambda u^p$ in Ω, $u = 0$ on $\partial\Omega$, with $0 < \alpha < 1 < p < \frac{n+2}{n-2}$.

Giacomoni, Schindler and Takac [26] considered the problem $-\Delta_p u = \lambda u^{-\alpha} + u^q$ in Ω, $u = 0$ on $\partial\Omega$, $u > 0$ in Ω, in the case $0 < \alpha < 1$, $1 < p < \infty$, $q < \infty$ and $p - 1 < q \leq p^* - 1$, with p^* defined by $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$ if $p < n$ and $p^* = \infty$ otherwise. There it was proved that there exists $\Lambda \in (0, \infty)$ such that this problem has a weak solution if $\lambda \in (0, \Lambda]$, has no weak solution if $\lambda > \Lambda$, and has at least two weak solutions if $\lambda \in (0, \Lambda)$.

Finally, let us mention that in [31], existence and multiplicity results were obtained for positive solutions of problem (1.1) for $0 < \alpha < 3$, $0 \leq a \in L^\infty(\Omega)$, $a \not\equiv 0$ in Ω, and for some nonlinearities f satisfying that $f(\lambda, x, .)$ is superlinear with subcritical growth at ∞ (a precise statement of these results is given in Remark 1.1 below).

Additional references, and a comprehensive treatment of the subject, can be found in [21] and [38], see also [16].

Unless otherwise stated, the notion of weak solution that we use is the usual one: If $h : \Omega \to \mathbb{R}$ is a measurable function we say that $u : \Omega \to \mathbb{R}$ is a weak solution of the problem

$$-\Delta u = h \text{ in } \Omega, \; u = 0 \text{ on } \partial\Omega \tag{1.5}$$

if $u \in H_0^1(\Omega)$ and, for any $\varphi \in H_0^1(\Omega)$, $h\varphi \in L^1(\Omega)$ and $\int_\Omega \langle \nabla u, \nabla\varphi \rangle = \int_\Omega h\varphi$.

Since our results heavily rely on those in [31]; in the next remark we summarize some of the main results included in that work:

Remark 1.1. *(See [31], Theorems 1.1 and 1.2, and Lemmas 2.9 and 4.3). Assume that Ω is a bounded domain in $\mathbb{R}^n$ with C^2 boundary, and that the following conditions H1)-H5) hold:*
H1) $0 < \alpha < 3$.
H2) $a \in L^\infty(\Omega)$, *and there exists* $\delta > 0$ *such that* $\inf_{A_\delta} a > 0$,
where, for $\rho > 0$,

$$A_\rho := \{x \in \Omega : d_\Omega(x) \leq \rho\},$$

where $d_\Omega := dist(., \partial\Omega)$; *and where, for a measurable subset E of Ω,* $\inf_E$ *means the essential infimum on E.*
H3) $0 \leq f \in C\left([0, \infty) \times \overline{\Omega} \times [0, \infty)\right)$, *and* $f(0, ., .) \equiv 0$ *on* $\overline{\Omega} \times [0, \infty)$.
H4) There exist numbers $\eta_0 > 0$, $q \geq 1$, *and a nonnegative function* $b \in L^\infty(\Omega)$, *such that* $b \not\equiv 0$, *and* $f(\lambda, ., s) \geq \lambda b s^q$ *a.e. in* Ω, *whenever* $\lambda \geq \eta_0$ *and* $s \geq 0$.
H5) There exist $p \in \left(1, \frac{n+2}{n-2}\right)$, *and* $h \in C\left((0, \infty) \times \overline{\Omega}\right)$ *that satisfy* $\inf_{[\eta, \infty) \times \overline{\Omega}} h > 0$ *for any* $\eta > 0$, *and such that, for every* $\sigma > 0$,

$$\lim_{(\lambda, s) \to (\sigma, \infty)} s^{-p} f(\lambda, ., s) = h(\sigma, .) \text{ uniformly on } \overline{\Omega}.$$

Then there exist positive numbers Λ, *and* $\Lambda^* \leq \Lambda$, *such that:*
i) Problem (1.1) has at least one weak solution $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ *if and only if* $0 \leq \lambda \leq \Lambda$. *Moreover, for* $\lambda = 0$ *there is only one such solution.*
ii) For each $\lambda \in [0, \Lambda]$, *if* $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ *is a weak solution of problem (1.1), then* $u \in C\left(\overline{\Omega}\right)$, *and satisfies* $u \geq c d_\Omega^\kappa$ *in* Ω, *where* $d_\Omega := dist(., \partial\Omega)$, $\kappa := 1$ *if* $0 < \alpha \leq 1$ *and* $\kappa := \frac{2}{1+\alpha}$ *if* $1 < \alpha < 3$, *and in both cases c is a positive constant independent of λ and u.*
iii) If $\lambda \in (0, \Lambda^*)$, *then problem (1.1) has at least two positive weak solutions in* $H_0^1(\Omega) \cap C\left(\overline{\Omega}\right)$.

Our aim in this work is to prove the following two Theorems, which complement the results quoted in Remark 1.1.

Theorem 1.2. *Let Ω be a bounded domain in $\mathbb{R}^n$ with C^2 boundary. Assume the following conditions H1)-H6) :*
H1) $0 < \alpha < 3$.
H2) $a \in L^\infty(\Omega)$, *and there exists* $\delta > 0$ *such that* $\inf_{A_\delta} a > 0$,
where, for $\rho > 0$,

$$A_\rho := \{x \in \Omega : d_\Omega(x) \leq \rho\},$$

where $d_\Omega := dist(., \partial\Omega)$; *and where, for a measurable subset E of* Ω, inf_E *means the essential infimum on E.*
H3) $0 \leq f \in C\left([0,\infty) \times \overline{\Omega} \times [0,\infty)\right)$, *and* $f(0,.,.) \equiv 0$ *on* $\overline{\Omega} \times [0,\infty)$.
H4) There exist numbers $\eta_0 > 0$, $q \geq 1$, *and a nonnegative function* $b \in L^\infty(\Omega)$, *such that* $b \not\equiv 0$, *and* $f(\lambda, ., s) \geq \lambda b s^q$ *a.e. in* Ω, *whenever* $\lambda \geq \eta_0$ *and* $s \geq 0$.
H5) There exist $p \in \left(1, \frac{n+2}{n-2}\right)$, *and* $h \in C\left((0,\infty) \times \overline{\Omega}\right)$ *that satisfy* $\inf_{[\eta,\infty)\times\overline{\Omega}} h > 0$ *for any* $\eta > 0$, *and such that, for every* $\sigma > 0$,

$$\lim_{(\lambda,s)\to(\sigma,\infty)} s^{-p} f(\lambda, ., s) = h(\sigma, .) \text{ uniformly on } \overline{\Omega}.$$

H6) For any $(\lambda, x) \in (0,\infty) \times \Omega$, *the function* $f(\lambda, x, .)$ *is nondecreasing on* $(0,\infty)$ *and, for any* $(x, s) \in \Omega \times (0,\infty)$, *the function* $f(., x, s)$ *is strictly increasing on* $(0,\infty)$.
Let Λ *be as given in Remark 1.1. Then, for any* $\lambda \in [0,\Lambda]$, *problem (1.1) has a minimal weak solution* $u_\lambda \in H_0^1(\Omega) \cap L^\infty(\Omega)$ *that satisfies* $u_\lambda \leq v$ *for any weak solution* $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$ *of (1.1). Moreover,* $u_\lambda \in C\left(\overline{\Omega}\right)$ *and, if* $0 \leq \lambda_1 < \lambda_2 \leq \Lambda$, *then there exists a positive constant c such that* $u_{\lambda_1} + c d_\Omega \leq u_{\lambda_2}$ *in* Ω; *in particular,* $\lambda \to u_\lambda$ *is strictly increasing from* $[0,\Lambda]$ *into* $C\left(\overline{\Omega}\right)$.

Theorem 1.3. *Assume the hypothesis of Theorem 1.2 and let* Λ *be as in Remark 1.1. Then, for each* $\lambda \in (0,\Lambda)$, *problem (1.1) has at least two positive weak solutions in* $H_0^1(\Omega) \cap C\left(\overline{\Omega}\right)$.

The following two corollaries are direct consequences of Theorems 1.2 and 1.3, and of Remark 1.1:

Corollary 1. *Let* Ω *be a bounded domain in* $\mathbb{R}^n$ *with* C^2 *boundary. Consider the problem:*

$$\begin{cases} -\Delta u = a u^{-\alpha} + \lambda g(., u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \\ u > 0 \text{ in } \Omega. \end{cases} \tag{1.6}$$

Assume that the conditions H1) and H2) of Theorem 1.2 hold, and that $g : \overline{\Omega} \times [0,\infty) \to \mathbb{R}$ *satisfies the following conditions H3')-H5'):*
H3') $0 \leq g \in C\left(\overline{\Omega} \times [0,\infty)\right)$ *and, for any* $x \in \Omega$, $g(x, .)$ *is strictly increasing on* $(0,\infty)$.
H4') There exist $q \in [1,\infty)$ *and a nonnegative* $b \in L^\infty(\Omega)$, *with* $b \not\equiv 0$, *such that, for any* $s \geq 0$, $g(., s) \geq b s^q$ *a.e. in* Ω.
H5') $\lim_{s\to\infty} \frac{g(.,s)}{s^p} = h$ *uniformly on* $\overline{\Omega}$ *for some* $p \in \left(1, \frac{n+2}{n-2}\right)$ *and some* $h \in C\left(\overline{\Omega}\right)$ *such that* $\min_{\overline{\Omega}} h > 0$.
Then there exists $\Lambda \in (0,\infty)$ *such that problem (1.6):*
i) Has at least two positive weak solutions in $H_0^1(\Omega) \cap C\left(\overline{\Omega}\right)$ *if* $\lambda \in (0,\Lambda)$,

ii) Has no positive weak solution in $H_0^1(\Omega) \cap L^\infty(\Omega)$ *if* $\lambda > \Lambda$,
iii) Has at least one positive weak solution in $H_0^1(\Omega) \cap C\left(\overline{\Omega}\right)$ *if* $\lambda = \Lambda$,
iv) Has a unique positive weak solution in $H_0^1(\Omega) \cap L^\infty(\Omega)$ *if* $\lambda = 0$, *and it belongs to* $C\left(\overline{\Omega}\right)$.
Moreover, for such a Λ, *the conclusions of Theorems 1.2 and 1.3 hold for problem (1.6).*

Corollary 2. *Let* Ω *be a bounded domain in* $\mathbb{R}^n$ *with* C^2 *boundary. Consider the problem*

$$\begin{cases} -\Delta u = au^{-\alpha} + g(., \lambda u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \\ u > 0 \text{ in } \Omega. \end{cases} \tag{1.7}$$

Assume that the conditions H1) and H2) of Theorem 1.2 hold; and that $g : \overline{\Omega} \times [0, \infty) \to \mathbb{R}$ *satisfies the conditions H3')-H5') of Corollary 1, and the following additional condition:*
H6') $g(., 0) = 0$.
Then there exists $\Lambda \in (0, \infty)$ *such that the conclusions of Corollary 1 hold for problem (1.7).*

The paper is organized as follows: At the beginning of Section 2 we recall some results from [31] that we need in order to prove Theorems 1.2 and 1.3. Lemma 2.5 provides a sub-supersolution result adapted to our singular problem and, in Lemma 2.9, we use results from [17] to prove a version, suitable for our purposes, of the strong maximum principle in the presence of a singular potential.

In Section 3 we prove Theorems 1.2 and 1.3. Concerning Theorem 1.2, the minimal solution u_λ is found by adapting, to our singular setting, ideas from [35], and using the sub and supersolutions method (applied to suitable nonsingular approximations to problem (1.1)). The sub and supersolutions method also gives that $\lambda \to u_\lambda$ is nondecreasing. Next, Lemma 2.9 is used to prove the stronger monotonicity assertion of Theorem 1.2.

In Remark 3.1 we recall a sub-supersolution theorem from [34], which allows singular nonlinearities, and provides solutions, in the sense of distributions, to problems like (1.1). Lemma 3.2 states that, under suitable assumptions, a solution, in the sense of distributions, to problem (1.1), is also a weak solution in $H_0^1(\Omega)$.

Theorem 1.3 is proved by using a classical fixed point theorem from [1], combined with an a priori bound (obtained in [31]) for the L^∞ norm of the solutions of problem (1.1), as well as the results of Theorem 1.2, and the sub-supersolutions method developed in [34].

2. Preliminaries

We assume from now on that Ω is a bounded domain in $\mathbb{R}^n$ with C^2 boundary; and that the conditions *H1)-H6)* of Theorem 1.2 hold. Let us summarize in the next lemmas some facts proved in [31].

Lemma 2.1. *(See [31], Lemmas 2.6 and 2.12) For any nonnegative* $\zeta \in L^\infty(\Omega)$ *and* $\varepsilon \geq 0$, *the problem*

$$\begin{cases} -\Delta u = a(u+\varepsilon)^{-\alpha} + \zeta \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \\ u > 0 \text{ in } \Omega, \end{cases} \tag{2.1}$$

has a unique weak solution $u \in H_0^1(\Omega)$, *and it belongs to* $L^\infty(\Omega)$.

Let $P_\infty := \{\zeta \in L^\infty(\Omega) : \zeta \geq 0 \ a.e. \text{ in } \Omega\}$ and, for any $\varepsilon \geq 0$, let $S_\varepsilon : P_\infty \to H_0^1(\Omega) \cap L^\infty(\Omega)$ be defined by $S_\varepsilon(\zeta) := u$, where u is the unique weak solution to problem (2.1) given by Lemma 2.1. Define also $S : P_\infty \times [0,\infty) \to H_0^1(\Omega) \cap L^\infty(\Omega)$ by $S(\zeta,\varepsilon) := S_\varepsilon(\zeta)$.
Unless explicit mention to the contrary, we will consider P_∞ endowed with the topology of the L^∞ norm.

Lemma 2.2. *(See [31], Lemmas 2.14, 2.7, 2.12 and 2.9):*
i) $\zeta \to S_\varepsilon(\zeta)$ *is nondecreasing on* P_∞ *for any* $\varepsilon \geq 0$.
ii) $\varepsilon \to S_\varepsilon(\zeta)$ *is nonincreasing on* $[0,\infty)$ *for any* $\zeta \in P_\infty$.
iii) $S(P_\infty \times [0,\infty)) \subset C(\overline{\Omega})$, *and* $S : P_\infty \times [0,\infty) \to C(\overline{\Omega})$ *is continuous.*
iv) $S : P_\infty \times [0,\infty) \to C(\overline{\Omega})$ *is a compact map.*
v) There exists a positive constant c such that $S_\varepsilon(\zeta) \geq c d_\Omega$ *in* Ω *for any* $\varepsilon \in [0,1]$ *and* $\zeta \in P_\infty$.
vi) If $1 < \alpha < 3$, *then there exists a positive constant c such that* $S_0(\zeta) \geq c d_\Omega^{\frac{2}{1+\alpha}}$ *in* Ω *for any* $\zeta \in P_\infty$.
vii) For any $\zeta \in P_\infty$, $\varepsilon \geq 0$, *and* $\gamma \in (0,1)$, *there exists a positive constant c such that* $S_\varepsilon(\zeta) \leq c d_\Omega^\gamma$ *in* Ω.

Lemma 2.3. *(See [31], Lemma 4.8) Let* $\lambda_0 > 0$, *let* $\{\lambda_j\}_{j\in\mathbb{N}}$ *be a sequence in* $[\lambda_0,\infty)$, *let* $\{\varepsilon_j\}_{j\in\mathbb{N}}$ *be a sequence in* $[0,1]$, *and for each* $j \in \mathbb{N}$, *let* $w_j \in H_0^1(\Omega) \cap L^\infty(\Omega)$ *be a weak solution of the following problem*

$$\begin{cases} -\Delta w_j = a\left(w_j + \varepsilon_j\right)^{-\alpha} + f\left(\lambda_j, ., w_j\right) \ \text{in } \Omega, \\ w_j = 0 \ \text{on } \partial\Omega, \\ w_j > 0 \ \text{in } \Omega. \end{cases}$$

Then i) $\{w_j\}_{j\in\mathbb{N}}$ *is bounded in* $H_0^1(\Omega)$.
ii) If $\{w_{j_k}\}_{k\in\mathbb{N}}$ *is a subsequence of* $\{w_j\}_{j\in\mathbb{N}}$ *that converges weakly in* $H_0^1(\Omega)$ *to some* $w \in H_0^1(\Omega) \cap L^\infty(\Omega)$, *and if* $\lim_{k\to\infty}\left(\lambda_{j_k}, \varepsilon_{j_k}\right) = (\lambda, \varepsilon)$, *then w is a weak solution of the problem*

$$\begin{cases} -\Delta w = a(w + \varepsilon)^{-\alpha} + f(\lambda, ., w) \ \text{in } \Omega, \\ w = 0 \ \text{on } \partial\Omega, \\ w > 0 \ \text{in } \Omega; \end{cases}$$

and, moreover, there exists a positive constant c such that $w \geq c d_\Omega$ *in* Ω.

For $u \in H^1(\Omega)$, we write $u \geq 0$ on $\partial\Omega$ (respectively $u \leq 0$ on $\partial\Omega$), to mean that $u^- \in H_0^1(\Omega)$ (resp. $u^+ \in H_0^1(\Omega)$). The notions of weak subsolutions and supersolutions, to be used from now on in this work, are the usual ones: If $h : \Omega \to \mathbb{R}$ is a measurable function such that $h\varphi \in L^1(\Omega)$ for any $\varphi \in H_0^1(\Omega)$, we say that $u : \Omega \to \mathbb{R}$ is a weak subsolution (respectively a weak supersolution) of (1.5) if $u \in H_0^1(\Omega)$, $u \leq 0$ on $\partial\Omega$, and $\int_\Omega \langle \nabla u, \nabla\varphi\rangle \leq \int_\Omega h\varphi$ (resp. $u \geq 0$ on $\partial\Omega$ and $\int_\Omega \langle \nabla u, \nabla\varphi\rangle \geq \int_\Omega h\varphi$) for any nonnegative $\varphi \in H_0^1(\Omega)$.

Remark 2.4. *If U is an open set in* $\mathbb{R}^n$, $u \in H^1(U)$ *and* $h \in L^1_{loc}(U)$, *we will write* $-\Delta u \geq h$ *in U (respectively* $-\Delta u \leq h$ *in U) to mean that*

$$\int_U \langle \nabla u, \nabla\varphi\rangle \geq \int_U h\varphi \ \ (\textit{resp.} \ \int_U \langle \nabla u, \nabla\varphi\rangle \leq \int_U h\varphi) \ \textit{for any nonnegative } \varphi \in C_c^\infty(U). \tag{2.2}$$

Note that if, in addition, $h \in H^{-1}(U) := \left(H_0^1(U)\right)'$ (i.e., if the map $\varphi \to \int_U h\varphi$ is continuous on $H_0^1(U)$), then, by a standard density argument, from (2.2) it follows that $\int_U \langle \nabla u, \nabla \varphi \rangle \geq \int_U h\varphi$ (resp. $\int_U \langle \nabla u, \nabla \varphi \rangle \leq \int_U h\varphi$) also holds for any nonnegative $\varphi \in H_0^1(U)$.

We will also need the following auxiliary results.

Lemma 2.5. *Let $\lambda > 0$, and suppose that u and v are weak nonnegative supersolutions in $H_0^1(\Omega) \cap L^\infty(\Omega)$ of problem (1.1). Then there exists a weak solution $z \in H_0^1(\Omega) \cap C\left(\overline{\Omega}\right)$ of problem (1.1) such that $z \leq \min\{u, v\}$ in Ω.*

Proof. Let $\{\varepsilon_j\}_{j \in N}$ be a sequence in $(0, 1]$ such that $\lim_{j\to\infty} \varepsilon_j = 0$. Then, for any j, u and v are weak supersolutions of the (nonsingular) problem

$$\begin{cases} -\Delta w = a\left(w + \varepsilon_j\right)^{-\alpha} + f(\lambda, ., w) \text{ in } \Omega, \\ w = 0 \text{ on } \partial\Omega, \\ w > 0 \text{ in } \Omega, \end{cases} \tag{2.3}$$

and therefore (see, e.g., [18], Lemma 4.10), $\min\{u, v\}$ is a weak supersolution of (2.3). Note that $S_{\varepsilon_j}(0)$ is a weak subsolution of the same problem, and that, by Lemma 2.2, $S_{\varepsilon_j}(0) \leq S_{\varepsilon_j}(f(\lambda, ., u)) \leq S_0(f(\lambda, ., u)) = u$. Similarly, $S_{\varepsilon_j}(0) \leq S_{\varepsilon_j}(f(\lambda, ., v)) \leq S_0(f(\lambda, ., v)) = v$, therefore $S_{\varepsilon_j}(0) \leq \min\{u, v\}$. Thus (see e.g., [18], Theorem 4.9), there exists a weak solution z_j of problem (2.3) such that $z_j \leq \min\{u, v\}$. As, by Lemma 2.3, $\{z_j\}_{j \in N}$ is bounded in $H_0^1(\Omega)$, there exist $z \in H_0^1(\Omega)$, and a subsequence $\{z_{j_k}\}_{k \in N}$, such that $\{z_{j_k}\}_{k \in N}$ converges to z in $L^2(\Omega)$ and $\{\nabla z_{j_k}\}_{k \in N}$ converges weakly in $L^2(\Omega, \mathbb{R}^n)$ to ∇z. Taking a subsequence if necessary, we can assume that $\{z_{j_k}\}_{k \in N}$ converges to z *a.e.* in Ω. Then $z \leq \min\{u, v\}$ *a.e.* in Ω and, by Lemma 2.3, z is a weak solution of (1.1); now Remark 1.1 says $z \in C\left(\overline{\Omega}\right)$. □

Remark 2.6. *Following [5], for $\mu \in L^1(\Omega)$ we say that $u : \Omega \to \mathbb{R}$ is a solution of the problem*

$$\begin{cases} -\Delta u = \mu \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{2.4}$$

if $u \in L^1(\Omega)$ and $\int_\Omega u(-\Delta\varphi) = \int_\Omega \mu\varphi$, for any $\varphi \in C_0^2\left(\overline{\Omega}\right)$, where $C_0^2\left(\overline{\Omega}\right) := \left\{\varphi \in C^2\left(\overline{\Omega}\right) : \varphi = 0 \text{ on } \partial\Omega\right\}$.
From [5], Theorem B.1, for any $\mu \in L^1(\Omega)$, problem (2.4) has a unique solution u (in the above sense). Moreover, $u \in W_0^{1,1}(\Omega)$ and, for any $\varphi \in C_c^\infty(\Omega)$,

$$\int_\Omega \langle \nabla u, \nabla \varphi \rangle = \int_\Omega \mu\varphi.$$

Remark 2.7. *Let us recall the Hardy inequality (see e.g., [4], p. 313): There exists a positive constant c such that $\left\|\frac{\varphi}{d_\Omega}\right\|_{L^2(\Omega)} \leq c \|\nabla\varphi\|_{L^2(\Omega)}$ for all $\varphi \in H_0^1(\Omega)$.*

Let us introduce some notation: φ_1 will denote the positive principal eigenfunction of $-\Delta$ in Ω with homogeneous Dirichlet boundary condition, normalized by $\|\varphi_1\|_\infty = 1$. We recall that, for some positive constant c, $\frac{1}{c}d_\Omega \le \varphi_1 \le cd_\Omega$ in Ω (for the definitions and properties of principal eigenvalues and principal eigenfunctions see, e.g., Chapter 1 in [14]).
For $h \in L^1(\Omega)$, $N(h)$ will denote the unique solution $u \in W_0^{1,1}(\Omega)$, in the sense of Remark 2.6, of the problem $-\Delta u = h$ in Ω, $u = 0$ on $\partial\Omega$.

Remark 2.8. *Let us recall the following result from [17] (see [17], Theorem 1 and Corollary 1): If $\gamma \in (0,1)$, $0 \le h \in L^1(\Omega)$, and $|\{x \in \Omega : h(x) > 0\}| > 0$, then there exists $\tau_0 > 0$ such that, for any $t \ge \tau_0$, the problem*

$$\begin{cases} -\Delta v = -v^{-\gamma} + th \text{ in } \Omega, \\ v = 0 \text{ on } \partial\Omega, \\ v > 0 \text{ in } \Omega \end{cases} \tag{2.5}$$

has a maximal solution v_t, in the sense of Remark 2.6, and as such, $v_t \in W_0^{1,1}(\Omega)$ and $-v_t^{-\gamma} + h \in L^1(\Omega)$. If, in addition, $h \in L^\infty(\Omega)$, then, by ([17], Lemma 2), $v_t \in H_0^1(\Omega)$. Moreover, as observed in the proof of ([17], Theorem 1), $v_t \le N(h)$ in Ω, and so there exists a positive constant r' such that $v_t \le r'd_\Omega$ in Ω. Also, within the proof of ([17], Theorem 3) it is proved that if $\tau_0 \le t' < t$, then, for some $\varepsilon > 0$, $v_t \ge v_{t'} + \varepsilon\varphi_1$ in Ω, and so, for $t > \tau_0$, there exists a positive constant r such that $v_t \ge rd_\Omega$ in Ω. Thus, for $t > \tau_0$,

$$rd_\Omega \le v_t \le r'd_\Omega \text{ in } \Omega. \tag{2.6}$$

Since v_t is a solution in the sense of distributions of (2.5), and since, from (2.6), $v_t \in L^\infty(\Omega)$ and $-v_t^{-\gamma} + th \in L^\infty_{loc}(\Omega)$, the inner elliptic estimates (see e.g., in [27], Theorem 8.24) give that $v_t \in C(\Omega)$. From (2.6), v_t is continuous at $\partial\Omega$, and so $v_t \in C\left(\overline{\Omega}\right)$. Also, from (2.6), there exists a positive constant c such that $\left|-v_t^{-\gamma} + th\right| \le cd_\Omega^{-\gamma}$ in Ω. Then, for any $\varphi \in H_0^1(\Omega)$,

$$\int_\Omega \left|\left(-v_t^{-\gamma} + th\right)\varphi\right| \le c\int_\Omega d_\Omega^{1-\gamma}\left|\frac{\varphi}{d_\Omega}\right| \le c''\left\|\frac{\varphi}{d_\Omega}\right\|_2,$$

where c'' is a constant independent of φ. Thus, by the Hardy inequality, the functional $\varphi \to \int_\Omega \left(-v_t^{-\gamma} + th\right)\varphi$ is continuous on $H_0^1(\Omega)$. Therefore, taking into account that $v_t \in H_0^1(\Omega)$, and that

$$\int_\Omega \langle \nabla v_t, \nabla\varphi\rangle = \int_\Omega \left(-v_t^{-\gamma} + th\right)\varphi \qquad \text{for any } \varphi \in C_c^\infty(\Omega) \tag{2.7}$$

it follows that (2.7) remains valid for any $\varphi \in H_0^1(\Omega)$; therefore v_t is a weak solution of (2.5).

Lemma 2.9. *Let $k > 0$, $\eta \in (0,2)$, and let $g \in C(\Omega) \cap L^\infty(\Omega)$ be a function such that $g(x) > 0$ for all $x \in \Omega$. If $w \in H_0^1(\Omega)$ satisfies $-\Delta w + kd_\Omega^{-\eta}w \ge g$ in Ω, then there exists a positive constant c such that $w \ge cd_\Omega$ a.e. in Ω.*

Proof. Note that if $w \in H_0^1(\Omega)$ satisfies $-\Delta w + kd_\Omega^{-\eta}w \ge g$ in Ω, then, for $\tau > 0$, $-\Delta(\tau w) + kd_\Omega^{-\eta}\tau w \ge \tau g$ in Ω. Thus the lemma will follow if we show that, if τ is large enough and if $w \in H_0^1(\Omega)$ satisfies $-\Delta w + kd_\Omega^{-\eta}w \ge \tau g$ in Ω, then there exists a positive constant c such that $w \ge cd_\Omega$ in Ω.
We consider first the case $1 < \eta < 2$. Let $\theta := \frac{1}{2}(2-\eta)$ and let $\gamma := \frac{\eta}{2}$. Notice that $\eta + \theta < 2$ and $0 < \gamma < 1$. According to Remark 2.8 there exists $t_0 = t_0(\eta, g) > 0$ such that, for $t = t_0$ and $h = g$, (2.5)

has a positive maximal weak solution $v_{t_0} \in H_0^1(\Omega)$, which satisfies, for some positive constants c_1 and c_2, $c_1 d_\Omega \le v_{t_0} \le c_2 d_\Omega$ in Ω. Assume temporarily $k \ge c_2^{-\eta-\theta} \|d_\Omega\|_\infty^{-\theta}$. Fix $\delta \in \left(0, \frac{1}{2}\left(k c_2^{\eta+\theta}\right)^{-\frac{1}{\theta}}\right)$; and for $\rho > 0$ let $A_\rho := \{x \in \Omega : d_\Omega(x) < \rho\}$ and $\Omega_\rho := \{x \in \Omega : d_\Omega(x) > \rho\}$. We have, in $A_{2\delta}$,

$$\begin{aligned}
-\Delta v_{t_0} &= -v_{t_0}^{-\gamma} + t_0 g \\
&= -v_{t_0}^{-(\eta+\theta)} v_{t_0} + t_0 g \le -\left(c_2 d_\Omega\right)^{-(\eta+\theta)} v_{t_0} + t_0 g \\
&= -c_2^{-(\eta+\theta)} d_\Omega^{-\theta} d_\Omega^{-\eta} v_{t_0} + t_0 g \\
&\le -c_2^{-(\eta+\theta)} (2\delta)^{-\theta} d_\Omega^{-\eta} v_{t_0} + t_0 g \le -k d_\Omega^{-\eta} v_{t_0} + t_0 g,
\end{aligned}$$

therefore,

$$-\Delta v_{t_0} + k d_\Omega^{-\eta} v_{t_0} \le t_0 g \quad \text{in } A_{2\delta}. \tag{2.8}$$

We have also, for any $x \in \Omega_\delta$,

$$\begin{aligned}
\left(k d_\Omega^{-\eta}(x) - \left(c_2 d_\Omega(x)\right)^{-\eta-\theta}\right) v_{t_0}(x) &= \left(k - c_2^{-\eta-\theta} d_\Omega^{-\theta}(x)\right) d_\Omega^{-\eta}(x)\, v_{t_0}(x) \\
&\le \left(k - c_2^{-\eta-\theta} \|d_\Omega\|_\infty^{-\theta}\right) d_\Omega^{-\eta}(x)\, v_{t_0}(x) \\
&\le c_2 \left(k - c_2^{-\eta-\theta} \|d_\Omega\|_\infty^{-\theta}\right) \delta^{-\eta} d_\Omega(x);
\end{aligned}$$

that is,

$$\left(k d_\Omega^{-\eta} - \left(c_2 d_\Omega\right)^{-\eta-\theta}\right) v_{t_0} \le c_2 \left(k - c_2^{-\eta-\theta} \|d_\Omega\|_\infty^{-\theta}\right) \delta^{-\eta} d_\Omega \quad \text{in } \Omega_\delta. \tag{2.9}$$

Define $\tau_0 := t_0 + c_2 \left(k - c_2^{-\eta-\theta} \|d_\Omega\|_\infty^{-\theta}\right) \left(\min_{\Omega_\delta} g\right)^{-1} \delta^{-\eta} \|d_\Omega\|_\infty$. For $t > \tau_0$, from (2.9), we have, in Ω_δ,

$$\begin{aligned}
(t - t_0)\, g \ge (t - t_0) \min_{\Omega_\delta} g &\ge c_2 \left(k - c_2^{-\eta-\theta} \|d_\Omega\|_\infty^{-\theta}\right) \delta^{-\eta} \|d_\Omega\|_\infty \qquad (2.10) \\
&\ge c_2 \left(k - c_2^{-\eta-\theta} \|d_\Omega\|_\infty^{-\theta}\right) \delta^{-\eta} d_\Omega \\
&\ge \left(k - c_2^{-\eta-\theta} \|d_\Omega\|_\infty^{-\theta}\right) \delta^{-\eta} v_{t_0} \\
&\ge \left(k - c_2^{-\eta-\theta} \|d_\Omega\|_\infty^{-\theta}\right) d_\Omega^{-\eta} v_{t_0} \\
&\ge \left(k d_\Omega^{-\eta} - c_2^{-\eta-\theta} d_\Omega^{-\eta-\theta}\right) v_{t_0},
\end{aligned}$$

therefore, for $t > \tau_0$,

$$\begin{aligned}
-\Delta v_{t_0} + k d_\Omega^{-\eta} v_{t_0} &= -v_{t_0}^{-\gamma} + t_0 g + k d_\Omega^{-\eta} v_{t_0} = -v_{t_0}^{-\eta-\theta} v_{t_0} + t_0 g + k d_\Omega^{-\eta} v_{t_0} \qquad (2.11) \\
&\le -\left(c_2 d_\Omega\right)^{-\eta-\theta} v_{t_0} + t_0 g + k d_\Omega^{-\eta} v_{t_0} \le t g \quad \text{in } \Omega_\delta,
\end{aligned}$$

the last inequality by (2.10). Then, from (2.8) and (2.11), we have, for $t > \tau_0$,

$$-\Delta v_{t_0} + k d_\Omega^{-\eta} v_{t_0} \le t g \quad \text{in } \Omega. \tag{2.12}$$

Let $w \in H_0^1(\Omega)$ be such that, for some $t \ge \tau_0$, $-\Delta w + k d_\Omega^{-\eta} w \ge t g$ in Ω, then, from (2.12), we have $-\Delta\left(w - v_{t_0}\right) + k d_\Omega^{-\eta}\left(w - v_{t_0}\right) \ge 0$ in Ω; i.e.,

$$\int_\Omega \left\langle \nabla\left(w - v_{t_0}\right), \nabla\varphi \right\rangle + \int_\Omega k d_\Omega^{-\eta}\left(w - v_{t_0}\right)\varphi \ge 0 \tag{2.13}$$

for any nonnegative $\varphi \in C_c^\infty(\Omega)$. Also, since $\eta < 2$, from the Hölder and Hardy inequalities there exists a positive constant c such that, for any $\varphi \in H_0^1(\Omega)$, $\left|\int_\Omega kd_\Omega^{-\eta}(w - v_{t_0})\varphi\right| \le \int_\Omega kd_\Omega^{2-\eta}\left|\frac{w-v_{t_0}}{d_\Omega}\right|\left|\frac{\varphi}{d_\Omega}\right| \le c\left\|w - v_{t_0}\right\|_{H_0^1(\Omega)}\|\varphi\|_{H_0^1(\Omega)}$. Thus $kd_\Omega^{-\eta}(w - v_{t_0}) \in H^{-1}(\Omega)$, and then, as observed in Remark 2.4, (2.13) holds for any $\varphi \in H_0^1(\Omega)$. Now, taking $\varphi = (w - v_{t_0})^-$ in (2.13), we get

$$-\int_\Omega \left|\nabla(w - v_{t_0})^-\right|^2 - \int_\Omega kd_\Omega^{-\eta}\left((w - v_{t_0})^-\right)^2 \ge 0$$

which gives $(w - v_{t_0})^- = 0$ in Ω. Thus $w \ge v_{t_0}$ in Ω, and, since $v_{t_0} \ge c_1 d_\Omega$ in Ω, the lemma is proved when $1 < \eta < 2$ and $k \ge c_2^{-\eta-\theta}\|d_\Omega\|_\infty^{-\theta}$.
If $1 < \eta < 2$ and $k \le c_2^{-\eta-\theta}\|d_\Omega\|_\infty^{-\theta}$, define $\overline{k} := k + c_2^{-\eta-\theta}\|d_\Omega\|_\infty^{-\theta}$. Note that, if $w \in H_0^1(\Omega)$ satisfies $-\Delta w + kd_\Omega^{-\eta}w \ge tg$ in Ω, then $-\Delta w + \overline{k}d_\Omega^{-\eta}w \ge tg$ in Ω, and and thus the lemma follows, in this case, from the previous case $1 < \eta < 2$.
Finally, note that the case $0 < \eta \le 1$ reduces to the case $1 < \eta < 2$. Indeed, since $0 < \eta \le 1$ and d_Ω is bounded on Ω, there exists a positive constant q such that $d_\Omega^{-\eta} \le qd_\Omega^{-\frac{3}{2}}$ in Ω, and so, if $w \in H_0^1(\Omega)$ satisfies $-\Delta w + kd_\Omega^{-\eta}w \ge tg$ in Ω, then $-\Delta w + qkd_\Omega^{-\frac{3}{2}}w \ge tg$ in Ω, therefore the case $1 < \eta < 2$ gives a positive constant c such that $w \ge cd_\Omega$ in Ω. □

Remark 2.10. *Let Λ be as in Remark 1.1; and for $\lambda \in [0, \Lambda]$, let $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ be a weak solution of (1.1). Then $u \in C^1(\Omega)$. Indeed, from Remark 1.1, $u \ge cd_\Omega$ in Ω for some positive constant c. Thus $au^{-\alpha} + f(\lambda, ., u) \in L_{loc}^\infty(\Omega)$. Also $u \in L^\infty(\Omega)$, and so, by the inner elliptic estimates (as stated e.g., in [7], Proposition 1.4.2), $u \in W_{loc}^{2,p}(\Omega)$ for any $p \in (1, \infty)$ and then $u \in C^1(\Omega)$.*

3. Proof of the main results

Proof of Theorem 1.2. Let Λ be as in Remark 1.1. We first prove that, for any $\lambda \in [0, \Lambda]$, problem (1.1) has a weak solution $u_\lambda \in H_0^1(\Omega) \cap L^\infty(\Omega)$, minimal in the sense stated in the theorem, i.e., such that $u_\lambda \le v$ for any weak solution $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$ of (1.1). Let

$$\beta_\lambda := \inf\left\{\int_\Omega w : w \in H_0^1(\Omega) \cap L^\infty(\Omega) \text{ and } w \text{ is a weak solution of (1.1)}\right\}$$

For each $\lambda \in [0, \Lambda]$, if $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ is a weak solution of (1.1), then, by Remark 1.1, $u \ge cd_\Omega$ in Ω, for some positive c independent of λ and u; therefore $\beta_\lambda > 0$. Let $\left\{w_j\right\}_{j\in N}$ be a minimizing sequence for the above infimum. By Lemma 2.3, $\left\{w_j\right\}_{j\in N}$ is bounded in $H_0^1(\Omega)$; then there exists $u_\lambda \in H_0^1(\Omega)$, and a subsequence $\left\{w_{j_k}\right\}_{k\in N}$, such that $\left\{w_{j_k}\right\}_{k\in N}$ converges to u_λ in $L^2(\Omega)$ and $\left\{\nabla w_{j_k}\right\}_{k\in N}$ converges weakly in $L^2(\Omega, \mathbb{R}^n)$ to ∇u_λ. Taking a further subsequence we can assume that $\left\{w_{j_k}\right\}_{k\in N}$ converges to u_λ *a.e.* in Ω. Again by Lemma 2.3, u_λ is a weak solution of (1.1) and, by Lemma 2.2, $u_\lambda \in C\left(\overline{\Omega}\right)$. Moreover, since $\left\{w_{j_k}\right\}_{k\in N}$ converges to u_λ in $L^2(\Omega)$, we have $\beta_\lambda = \lim_{k\to\infty}\int_\Omega w_{j_k} = \int_\Omega u_\lambda$. Let $\left\{\varepsilon_j\right\}_{j\in N}$ be a sequence in $(0, 1]$ such that $\lim_{j\to\infty}\varepsilon_j = 0$. Let $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$ be a weak solution of (1.1). From Lemma 2.5, there exists a weak solution $z \in H_0^1(\Omega) \cap C\left(\overline{\Omega}\right)$ to problem (1.1) such that $z \le \min\{u_\lambda, v\}$

in Ω. Thus $\int_\Omega z \leq \beta_\lambda$. Also, from the definition of β_λ, $\beta_\lambda \leq \int_\Omega z$, and so $\int_\Omega z = \int_\Omega u_\lambda$. Thus $u_\lambda = z \leq v$; therefore u_λ is a minimal solution of (1.1), and clearly such a minimal solution is unique.

To see that $\lambda \to u_\lambda$ is nondecreasing, suppose $0 \leq \lambda_1 < \lambda_2 \leq \Lambda$; from *H6)* we have $f(\lambda_2, x, s) \geq f(\lambda_1, x, s)$ for any $(x, s) \in \Omega \times [0, \infty)$, and so u_{λ_2} is a weak supersolution of the problem

$$\begin{cases} -\Delta w = aw^{-\alpha} + f(\lambda_1, ., w) \text{ in } \Omega, \\ w = 0 \text{ on } \partial\Omega, \\ w > 0 \text{ in } \Omega. \end{cases} \tag{3.1}$$

Since u_{λ_1} is a weak supersolution of the same problem, Lemma 2.5 says that there exists a weak solution $\widetilde{z} \in H_0^1(\Omega) \cap C\left(\overline{\Omega}\right)$ to problem (3.1) such that $\widetilde{z} \leq \min\{u_{\lambda_1}, u_{\lambda_2}\}$; which implies $\widetilde{z} = u_{\lambda_1}$, since u_{λ_1} is minimal; then $u_{\lambda_1} \leq u_{\lambda_2}$.

To complete the proof of the theorem it remains to prove that if $0 \leq \lambda_1 < \lambda_2 \leq \Lambda$, then

$$u_{\lambda_1} + cd_\Omega \leq u_{\lambda_2} \text{ in } \Omega \text{ for some constant } c > 0. \tag{3.2}$$

Suppose $0 \leq \lambda_1 < \lambda_2 \leq \Lambda$. From the first part of the proof we have $u_{\lambda_1} \leq u_{\lambda_2}$ in Ω. If $u_{\lambda_1} \equiv u_{\lambda_2}$ in Ω, then $f(\lambda_2, ., u_{\lambda_2}) = f(\lambda_1, ., u_{\lambda_1}) = f(\lambda_1, ., u_{\lambda_2})$ in Ω (the first of these equalities from the equations satisfied by u_{λ_1} and u_{λ_2} and the second one because $u_{\lambda_1} \equiv u_{\lambda_2}$ in Ω), and therefore $f(\lambda_2, x, u_{\lambda_2}(x)) = f(\lambda_1, x, u_{\lambda_2}(x))$ for any $x \in \Omega$, which contradicts *H6)*. Thus $u_{\lambda_1} \not\equiv u_{\lambda_2}$ in Ω. To prove (3.2) we consider first the case $1 \leq \alpha < 3$. Let $\varepsilon > 0$ be such that $\alpha + \varepsilon < 3$. We have, for $i = 1, 2$,

$$\begin{cases} -\Delta u_{\lambda_i} = au_{\lambda_i}^{-\alpha} + f(\lambda_i, ., u_{\lambda_i}) = au_{\lambda_i}^{\varepsilon} u_{\lambda_i}^{-\alpha-\varepsilon} + f(\lambda_i, ., u_{\lambda_i}) \quad \text{in } \Omega, \\ u_{\lambda_i} = 0 \quad \text{on } \partial\Omega, \\ u_{\lambda_i} > 0 \quad \text{in } \Omega. \end{cases}$$

Notice that, since $u_{\lambda_1} \leq u_{\lambda_2}$, the mean value theorem gives

$$\begin{aligned} au_{\lambda_2}^{\varepsilon} u_{\lambda_2}^{-\alpha-\varepsilon} - au_{\lambda_1}^{\varepsilon} u_{\lambda_1}^{-\alpha-\varepsilon} &\geq au_{\lambda_1}^{\varepsilon}\left(u_{\lambda_2}^{-\alpha-\varepsilon} - u_{\lambda_1}^{-\alpha-\varepsilon}\right) \\ &= -(\alpha + \varepsilon)\, au_{\lambda_1}^{\varepsilon} \theta^{-\alpha-\varepsilon-1}(u_{\lambda_2} - u_{\lambda_1}) \end{aligned}$$

for some measurable $\theta : \Omega \to \mathbb{R}$ such that $u_{\lambda_1} \leq \theta \leq u_{\lambda_2}$. Thus

$$\begin{cases} -\Delta(u_{\lambda_2} - u_{\lambda_i}) + (\alpha + \varepsilon)\, au_{\lambda_1}^{\varepsilon} \theta^{-\alpha-\varepsilon-1}(u_{\lambda_2} - u_{\lambda_i}) \\ = f(\lambda_2, ., u_{\lambda_2}) - f(\lambda_i, ., u_{\lambda_i}) \text{ in } \Omega, \\ u_{\lambda_2} - u_{\lambda_1} = 0 \text{ on } \partial\Omega, \\ u_{\lambda_2} - u_{\lambda_1} \geq 0 \text{ in } \Omega. \end{cases} \tag{3.3}$$

By Lemma 2.2, for any $\gamma \in (0, 1)$, there exists a positive constant c_1 such that $\max\{u_{\lambda_1}, u_{\lambda_2}\} \leq c_1 d_\Omega^\gamma$ in Ω. Lemma 2.2 also gives a positive constant c_2 such that $u_{\lambda_1} \geq c_2 d_\Omega^{\frac{2}{1+\alpha}}$ in Ω. Let $\eta_{\gamma,\varepsilon} := \gamma\varepsilon + \gamma - \frac{2(\alpha+1+\varepsilon)}{1+\alpha}$. A computation shows that if we take $\gamma = 1 - \varepsilon$, with ε positive and small enough, then $2\left(\eta_{\gamma,\varepsilon} + 1\right) > -1$; for such values of ε and γ, and for any $\varphi \in H_0^1(\Omega)$, Hölder's and Hardy's inequalities give

$$\left\| ad_\Omega^{\gamma\varepsilon} u_{\lambda_1}^{-\alpha-\varepsilon-1}(u_{\lambda_2} - u_{\lambda_i})\varphi \right\|_1 \leq \|a\|_\infty c_1 c_2^{-\alpha-\varepsilon-1} \left\| d_\Omega^{\gamma\varepsilon+\gamma} d_\Omega^{-\frac{2(\alpha+\gamma+\varepsilon)}{1+\alpha}+1} \frac{\varphi}{d_\Omega} \right\| \tag{3.4}$$

$$\leq \|a\|_{\infty} c_1 c_2^{-\alpha-\varepsilon-1} \left\| d_{\Omega}^{\eta_{\gamma,\varepsilon}+1} \right\|_2 \left\| \frac{\varphi}{d_{\Omega}} \right\|_2 < \infty.$$

As $\theta \geq u_{\lambda_1}$, we also have $\left\| a d_{\Omega}^{\gamma\varepsilon} \theta^{-\alpha-\varepsilon-1} \left(u_{\lambda_2} - u_{\lambda_i}\right) \varphi \right\|_1 < \infty$.
From (3.3) and (3.4) we conclude that, in weak sense,

$$\begin{aligned} & -\Delta\left(u_{\lambda_2} - u_{\lambda_i}\right) + (\alpha+\varepsilon) a c_1^{\varepsilon} d_{\Omega}^{\gamma\varepsilon} u_{\lambda_1}^{-\alpha-\varepsilon-1}\left(u_{\lambda_2} - u_{\lambda_i}\right) \\ & \geq -\Delta\left(u_{\lambda_2} - u_{\lambda_i}\right) + (\alpha+\varepsilon) a c_1^{\varepsilon} d_{\Omega}^{\gamma\varepsilon} \theta^{-\alpha-\varepsilon-1}\left(u_{\lambda_2} - u_{\lambda_i}\right) \\ & \geq f\left(\lambda_2, ., u_{\lambda_2}\right) - f\left(\lambda_i, ., u_{\lambda_i}\right) \text{ in } \Omega. \end{aligned} \tag{3.5}$$

Notice that u_{λ_1} satisfies

$$\begin{cases} -\Delta u_{\lambda_1} = a u_{\lambda_1}^{\varepsilon} u_{\lambda_1}^{-\alpha-\varepsilon} + f\left(\lambda_1, ., u_{\lambda_1}\right) \text{ in } \Omega, \\ u_{\lambda_1} = 0 \text{ on } \partial\Omega, \\ u_{\lambda_1} > 0 \text{ in } \Omega, \end{cases}$$

and that $0 \leq a u_{\lambda_1}^{\varepsilon} \in L^{\infty}(\Omega)$, $a u_{\lambda_1}^{\varepsilon} \not\equiv 0$ in Ω, and $1 < \alpha + \varepsilon < 3$; therefore Remark 1.1 says (with a replaced by $a u_{\lambda_1}^{\varepsilon}$) that there exists a constant $c_2 > 0$ such that $u_{\lambda_1} \geq c_2 d_{\Omega}^{\frac{2}{1+\alpha+\varepsilon}}$ in Ω. Thus, for some constant $c_3 > 0$, $u_{\lambda_1}^{-\alpha-\varepsilon-1} \leq c_3 d_{\Omega}^{-2}$ in Ω. Therefore, for some constant $c_4 > 0$,

$$0 \leq (\alpha+\varepsilon) a c_1^{\varepsilon} u_{\lambda_1}^{-\alpha-\varepsilon-1} d_{\Omega}^{\gamma\varepsilon} \leq c_4 d_{\Omega}^{-2+\gamma\varepsilon} \text{ in } \Omega. \tag{3.6}$$

Since $u_{\lambda_2} \geq u_{\lambda_1}$ in Ω, from *H6)* we get

$$\begin{aligned} & f\left(\lambda_2, ., u_{\lambda_2}\right) - f\left(\lambda_1, ., u_{\lambda_1}\right) \\ & \geq f\left(\lambda_2, ., u_{\lambda_1}\right) - f\left(\lambda_1, ., u_{\lambda_1}\right) > 0 \text{ in } \Omega. \end{aligned} \tag{3.7}$$

Then, taking into account (3.5), (3.6) and (3.7), Lemma 2.9 gives a positive constant c such that $u_{\lambda_2} - u_{\lambda_i} \geq c d_{\Omega}$ in Ω.
Consider now the case $0 < \alpha < 1$. Let $m : \Omega \to \mathbb{R}$ be defined by

$$m := -\chi_{\{u_{\lambda_2} > u_{\lambda_1}\}} a \left(u_{\lambda_2}^{-\alpha} - u_{\lambda_1}^{-\alpha}\right)\left(u_{\lambda_2} - u_{\lambda_1}\right)^{-1},$$

and let $w := u_{\lambda_2} - u_{\lambda_1}$. Thus w satisfies, in weak sense,

$$\begin{cases} -\Delta w + m w = f\left(\lambda_2, ., u_{\lambda_2}\right) - f\left(\lambda_1, ., u_{\lambda_1}\right) \text{ in } \Omega, \\ w = 0 \text{ on } \partial\Omega, \\ w > 0 \text{ in } \Omega, \end{cases} \tag{3.8}$$

and, by Remark 2.10 and Remark 1.1, $w \in C^1(\Omega) \cap C\left(\overline{\Omega}\right)$. The mean value theorem gives $m = -\alpha a \theta^{-\alpha-1}$ in $\{x \in \Omega : u_{\lambda_2}(x) > u_{\lambda_1}(x)\}$, for some measurable function θ such that $u_{\lambda_1} \leq \theta \leq u_{\lambda_2}$. Also, by Remark 1.1, there exists a positive constant c_6 such that $u_{\lambda_1} \geq c_6 d_{\Omega}$ in Ω, and so, for some positive constant c_7,

$$0 \leq m \leq c_7 d_{\Omega}^{-(1+\alpha)} \text{ in } \Omega. \tag{3.9}$$

As in the case $1 \leq \alpha < 3$, we have (3.7), and so, taking into account (3.8), (3.9) and (3.7), Lemma 2.9 gives a positive constant c such that $w \geq c d_{\Omega}$ in Ω. □

Let $g : \Omega \times (0, \infty) \to \mathbb{R}$ be a Carathéodory function (i.e., $g(x, .)$ is continuous for *a.e.* $x \in \Omega$ and $g(., s)$ is measurable for any $s \in [0, \infty)$). We say that $w \in W^{1,2}_{loc}(\Omega)$ is a subsolution (respectively a supersolution), in the sense of distributions, of the singular problem (without boundary condition)

$$-\Delta z = az^{-\alpha} + g(., z) \text{ in } \Omega \tag{3.10}$$

if $w > 0$ *a.e.* in Ω, $aw^{-\alpha} + g(., w) \in L^1_{loc}(\Omega)$, and for all nonnegative $\varphi \in C^\infty_c(\Omega)$, the following holds:

$$\int_\Omega \langle \nabla w, \nabla \varphi \rangle \leq (\text{resp. } \geq) \int_\Omega \left(aw^{-\alpha} + g(., w)\right) \varphi.$$

We say that $z \in W^{1,2}_{loc}(\Omega)$ is a solution, in the sense of distributions, of (3.10) if $z > 0$ *a.e.* in Ω, and, for all $\varphi \in C^\infty_c(\Omega)$, the following holds:

$$\int_\Omega \langle \nabla z, \nabla \varphi \rangle = \int_\Omega \left(az^{-\alpha} + g(., z)\right) \varphi. \tag{3.11}$$

Remark 3.1. *Let $g : \Omega \times (0, \infty) \to \mathbb{R}$ be a Carathéodory function, and assume that (3.10) has a subsolution $\underline{z}$ and a supersolution $\overline{z}$, in the sense of distributions, both in $L^\infty_{loc}(\Omega)$, and satisfying $0 < \underline{z} \leq \overline{z}$ a.e. in Ω. If, in addition, there exists $k \in L^\infty_{loc}(\Omega)$ such that $|a(x) s^{-\alpha} + g(x, s)| \leq k(x)$ a.e. $x \in \Omega$ for all $s \in \left[\underline{z}(x), \overline{z}(x)\right]$; then Theorem 2.4 in [34] says that (3.10) has a solution $z \in W^{1,2}_{loc}(\Omega)$ in the sense of distributions, satisfying $\underline{z} \leq z \leq \overline{z}$ a.e. in Ω.*

Lemma 3.2. *Let $\lambda \geq 0$, and suppose that $u \in W^{1,2}_{loc}(\Omega) \cap L^\infty(\Omega)$ is a solution, in the sense of distributions, of problem (1.1), and that one of the following two conditions holds:*
i) $0 < \alpha \leq 1$, and there exist positive constants c_1, c_2 and γ such that $c_1 d_\Omega \leq u \leq c_2 d^\gamma_\Omega$ a.e. in Ω.
ii) $1 < \alpha < 3$, and there exist positive constants c_1, c_2 and γ such that $c_1 d_\Omega^{\frac{2}{1+\alpha}} \leq u \leq c_2 d^\gamma_\Omega$ a.e. in Ω.
Then $u \in H^1_0(\Omega) \cap C^1(\Omega) \cap C\left(\overline{\Omega}\right)$, and u is a weak solution of (1.1).

Proof. For each $j \in \mathbb{N}$, let $h_j : \mathbb{R} \to \mathbb{R}$ be defined by $h_j(s) := 0$ if $s \leq \frac{1}{j}$, $h_j(s) := -3j^2 s^3 + 14js^2 - 19s + \frac{8}{j}$ if $\frac{1}{j} < s < \frac{2}{j}$, and $h_j(s) := s$ if $\frac{2}{j} \leq s$. Then $h_j \in C^1(\mathbb{R})$, $h'_j(s) = 0$ for $s < \frac{1}{j}$, $h'_j(s) \geq 0$ for $\frac{1}{j} < s < \frac{2}{j}$ and $h'_j(s) = 1$ for $\frac{2}{j} < s$. Also, $h_j(s) < s$ for all $s \in \left(0, \frac{2}{j}\right)$.
Let $h_j(u) := h_j \circ u$. Then, for all j, $\nabla\left(h_j(u)\right) = h'_j(u) \nabla u$. Since $u \in W^{1,2}_{loc}(\Omega)$, it follows that $h_j(u) \in W^{1,2}_{loc}(\Omega)$. Since $h_j(u)$ has compact support we have $h_j(u) \in H^1_0(\Omega)$. Therefore, for all j,

$$\int_\Omega \left\langle \nabla u, \nabla\left(h_j(u)\right)\right\rangle = \int_\Omega \left(au^{-\alpha} + f(\lambda, ., u)\right) h_j(u)$$

i.e.,

$$\int_{\{u>0\}} h'_j(u) |\nabla u|^2 = \int_\Omega \left(au^{-\alpha} + f(\lambda, ., u)\right) h_j(u). \tag{3.12}$$

Now, $h'_j(u) |\nabla u|^2$ is nonnegative and $\lim_{j\to\infty} h'_j(u) |\nabla u|^2 = |\nabla u|^2$ *a.e.* in Ω, and so, from (3.12) and Fatou's lemma, we have

$$\int_\Omega |\nabla u|^2 \leq \underline{\lim}_{j\to\infty} \int_\Omega \left(au^{-\alpha} + f(\lambda, ., u)\right) h_j(u).$$

Note that $au^{1-\alpha} \in L^1(\Omega)$. Indeed, this is clear when $0 < \alpha \leq 1$ (because $u \in L^\infty(\Omega)$). If $1 < \alpha < 3$, then $-2\frac{\alpha-1}{\alpha+1} > -1$, and so, from the assumption *ii)* of the lemma, $0 \leq u^{1-\alpha} \leq c_1^{1-\alpha} d_\Omega^{-\frac{2(\alpha-1)}{1+\alpha}}$ in Ω, which implies $au^{1-\alpha} \in L^1(\Omega)$. On the other hand, clearly $f(\lambda, ., u)u \in L^1(\Omega)$. Now, $\lim_{j\to\infty}(au^{-\alpha} + f(\lambda, ., u))h_j(u) = (au^{-\alpha} + f(\lambda, ., u))u$ and, for any $j \in \mathbb{N}$,

$$0 \leq (au^{-\alpha} + f(\lambda, ., u))h_j(u) \leq (au^{-\alpha} + f(\lambda, ., u))u \in L^1(\Omega).$$

Then, Lebesgue's dominated convergence theorem gives

$$\lim_{j\to\infty} \int_\Omega (au^{-\alpha} + f(\lambda, ., u))h_j(u) = \int_\Omega (au^{-\alpha} + f(\lambda, ., u))u < \infty.$$

Thus $\int_\Omega |\nabla u|^2 < \infty$, and so $u \in H^1(\Omega)$. Now, $-\Delta u = au^{-\alpha} + f(\lambda, ., u)$ in $D'(\Omega)$, also $u \in L^\infty(\Omega)$, therefore $f(\lambda, ., u) \in L^\infty(\Omega)$; and the assumptions *i)* and *ii)* of the lemma imply that $au^{-\alpha} \in L^\infty_{loc}(\Omega)$; thus $au^{-\alpha} + f(\lambda, ., u) \in L^\infty_{loc}(\Omega)$. Now, the inner elliptic estimates in ([27], Theorem 8.24) give that $u \in C(\Omega)$ and, from *i)* and *ii),* u is continuous on $\partial\Omega$, and so $u \in C\left(\overline{\Omega}\right)$.Thus, since $u \in H^1(\Omega)$, $u \in C\left(\overline{\Omega}\right)$ and $u = 0$ on $\partial\Omega$, we conclude that $u \in H_0^1(\Omega)$.

Let $\varphi \in H_0^1(\Omega)$. If $0 < \alpha < 1$, from *i),* we have

$$\left|au^{-\alpha}\varphi\right| = \left|au^{-\alpha} d_\Omega \frac{\varphi}{d_\Omega}\right| \leq c_1^{-\alpha} \|a\|_\infty d_\Omega^{1-\alpha} \left|\frac{\varphi}{d_\Omega}\right| \quad \text{in } \Omega,$$

and so, taking into account that $d_\Omega^{1-\alpha} \in L^\infty(\Omega)$, from the Hölder and the Hardy inequalities, we have $\|au^{-\alpha}\varphi\|_1 \leq c\|\varphi\|_{H_0^1(\Omega)}$ for some positive constant c independent of φ. If $1 \leq \alpha < 3$, *ii)* gives

$$\begin{aligned} \left|au^{-\alpha}\varphi\right| &= \left|au^{-\alpha} d_\Omega \frac{\varphi}{d_\Omega}\right| \leq c_1^{-\alpha} \|a\|_\infty d_\Omega^{1-\frac{2\alpha}{1+\alpha}} \left|\frac{\varphi}{d_\Omega}\right| \\ &= c_1^{-\alpha} \|a\|_\infty d_\Omega^{-\frac{\alpha-1}{\alpha+1}} \left|\frac{\varphi}{d_\Omega}\right| \quad \text{in } \Omega. \end{aligned} \tag{3.13}$$

Notice that $1 \leq \alpha < 3$ implies $2\frac{\alpha-1}{\alpha+1} < 1$, and then, from (3.13), Hölder's and Hardy's inequalities give $\|au^{-\alpha}\varphi\|_1 \leq c\|\varphi\|_{H_0^1(\Omega)}$ for some positive constant c independent of φ. Also, from *H3)*, and taking into account the Poincaré inequality, and that $u \in L^\infty(\Omega)$, we have, for any $\alpha \in (0, 3)$, $\|f(\lambda, ., u)\varphi\|_1 \leq c'\|\varphi\|_{H_0^1(\Omega)}$ for some constant c' independent of φ; then the maps $\varphi \to \int_\Omega au^{-\alpha}\varphi$ and $\varphi \to \int_\Omega f(\lambda, ., u)\varphi$ are continuous on $H_0^1(\Omega)$; since $u \in H_0^1(\Omega)$, also the map $\varphi \to \int_\Omega \langle\nabla u, \nabla\varphi\rangle$ is continuous on $H_0^1(\Omega)$.

Therefore, since $C_c^\infty(\Omega)$ is dense in $H_0^1(\Omega)$, and

$$\int_\Omega \langle\nabla u, \nabla\varphi\rangle = \int_\Omega (au^{-\alpha} + f(\lambda, ., u))\varphi \qquad \text{for any } \varphi \in C_c^\infty(\Omega); \tag{3.14}$$

we conclude that (3.14) holds for any $\varphi \in H_0^1(\Omega)$. Thus u is a weak solution of (1.1). □

Let us recall the following result from [1]:

Remark 3.3. *(See [1], Theorem 1.17): Let E be an ordered Banach space, let $P := \{\zeta \in E : \zeta \geq 0\}$) be its positive cone, and let $T : [0, \infty) \times P \to P$ be a continuous and compact map. Suppose that $T(0, 0) = 0$, and that 0 is the only fixed point of $T(0, .)$. Suppose, in addition, that there exists a positive number ρ such that $T(0, \zeta) \neq \sigma\zeta$ for all $\zeta \in S_\rho^+ := \{\zeta \in P : \|\zeta\|_E = \rho\}$ and all $\sigma \geq 1$. Then the set $\Sigma := \{(\lambda, \zeta) \in [0, \infty) \times P : T(\lambda, \zeta) = \zeta\}$ includes an unbounded subcontinuum (i.e. an unbounded closed and connected subset) that contains* $(0, 0)$.

We will also need the following result from [31]:

Lemma 3.4. *(See [31], Lemma 3.4) Assume the hypothesis H1)-H5) of Theorem 1.2, and that* $\lambda_0 > 0$. *Then there exists* $c_{\lambda_0} > 0$ *such that* $\|u\|_\infty < c_{\lambda_0}$ *whenever* $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ *is a weak solution, for some* $\varepsilon \in [0,1]$ *and* $\lambda \geq \lambda_0$, *of the problem*

$$\begin{cases} -\Delta u = a(u+\varepsilon)^{-\alpha} + f(\lambda, ., u) \ \text{in } \Omega, \\ u = 0 \ \text{on } \partial\Omega, \\ u > 0 \ \text{in } \Omega. \end{cases} \tag{3.15}$$

Proof of Theorem 1.3. By way of contradiction let us assume that there exists $\overline{\lambda} \in (0, \Lambda)$ such that, for $\lambda = \overline{\lambda}$, problem (1.1) has a unique weak solution $\overline{u} \in H_0^1(\Omega) \cap C\left(\overline{\Omega}\right)$. Thus $f\left(\overline{\lambda}, ., \overline{u}\right) \in C\left(\overline{\Omega}\right)$. Define the operator $T : [0, \infty) \times P \to P$ by $T(\mu, v) := S_0\left(f\left(\overline{\lambda} + \mu, ., \overline{u} + v\right)\right) - \overline{u}$, and let

$$\Sigma := \{(\lambda, \zeta) \in [0, \infty) \times P : T(\lambda, \zeta) = \zeta\}.$$

¿From Lemma 2.2, T is a continuous and compact operator. Since $\overline{u} = S_0\left(f\left(\overline{\lambda}, ., \overline{u}\right)\right)$ we have $T(0,0) = S_0\left(f\left(\overline{\lambda}, ., \overline{u}\right)\right) - \overline{u} = 0$. Furthermore,

$$0 \text{ is the only fixed point of } T(0, .). \tag{3.16}$$

Indeed, if $v \in P$ and $T(0, v) = v$, then

$$S_0\left(f\left(\overline{\lambda}, ., \overline{u} + v\right)\right) - \overline{u} = v,$$

i.e., $\overline{u} + v$ satisfies $-\Delta(\overline{u} + v) = a(\overline{u} + v)^{-\alpha} + f\left(\overline{\lambda}, ., \overline{u} + v\right)$ in Ω, $\overline{u} + v = 0$ on $\partial\Omega$, $\overline{u} + v > 0$ in Ω, which, by our contradiction assumption, implies $\overline{u} + v = \overline{u}$, i.e., $v = 0$. Then (3.16) holds.
Now, the following two possibilities arise:
a) There exists a positive number ρ such that $T(0, v) \neq \sigma v$ for all $v \in S_\rho^+ := \{v \in P : \|v\|_\infty = \rho\}$ and all $\sigma \geq 1$.
b) For any $\rho > 0$ there exist a number $\sigma \geq 1$ and $v \in P$ such that $\|v\|_\infty = \rho$ and $T(0, v) = \sigma v$.
If a) holds, then, by Remark 3.3, there exists an unbounded subcontinuum $C \subset \Sigma$ such that $(0,0) \in C$. Since $(\mu, w) \in \Sigma$ if and only if $\overline{u} + v$ satisfies $-\Delta(\overline{u} + w) = a(\overline{u} + w)^{-\alpha} + f\left(\overline{\lambda} + \mu, ., \overline{u} + w\right)$ in Ω, $\overline{u} + w = 0$ on $\partial\Omega$. Then $(\mu, w) \in \Sigma$ implies $\overline{\lambda} + \mu \leq \Lambda$ and $\|\overline{u} + w\|_\infty \leq c_{\overline{\lambda}}$, with $c_{\overline{\lambda}}$ as given by Lemma 3.4, which contradicts the fact that C is unbounded.
If b) holds, then, for each $j \in \mathbb{N}$, there exists $v_j \in P$, and a number $\sigma_j \geq 1$, such that $\left\|v_j\right\|_\infty = \frac{1}{j}$ and $T\left(0, v_j\right) = \sigma_j v_j$, i.e.,

$$\overline{u} + \sigma_j v_j = S_0\left(f\left(\overline{\lambda}, ., \overline{u} + v_j\right)\right). \tag{3.17}$$

Now, $\lim_{j\to\infty}\left(\overline{u} + v_j\right) = \overline{u}$ with convergence in $C\left(\overline{\Omega}\right)$, and so $f\left(\overline{\lambda}, ., \overline{u} + v_j\right)$ converges to $f\left(\overline{\lambda}, ., \overline{u}\right)$ in $C\left(\overline{\Omega}\right)$. By Lemma 2.2, $S_0 : C\left(\overline{\Omega}\right) \to C\left(\overline{\Omega}\right)$ is continuous, and so, from (3.17), $\lim_{j\to\infty}\left(\overline{u} + \sigma_j v_j\right) = \overline{u}$ with convergence in $C\left(\overline{\Omega}\right)$, i.e., $\lim_{j\to\infty} \sigma_j v_j = 0$ with convergence in $C\left(\overline{\Omega}\right)$.

Let us see that

$$\lim_{j\to\infty}\left\|\frac{\sigma_j v_j}{d_\Omega}\right\|_\infty = 0. \tag{3.18}$$

Indeed, let $M := 1 + \|\overline{u}\|_\infty$ and let $\varepsilon_j := \left\| f\left(\overline{\lambda}, ., \overline{u} + v_j\right) - f\left(\overline{\lambda}, ., \overline{u}\right)\right\|_\infty$. Since f is uniformly continuous on $[0, \Lambda] \times \overline{\Omega} \times [0, M]$, we have $\lim_{j\to\infty} \varepsilon_j = 0$. Since

$$\begin{aligned} -\Delta\left(\sigma_j v_j\right) &= a\left(\overline{u} + \sigma_j v_j\right)^{-\alpha} - a\left(\overline{u}\right)^{-\alpha} + f\left(\overline{\lambda}, ., \overline{u} + v_j\right) - f\left(\overline{\lambda}, ., \overline{u}\right) \\ &\le f\left(\overline{\lambda}, ., \overline{u} + v_j\right) - f\left(\overline{\lambda}, ., \overline{u}\right) \le \varepsilon_j \qquad \text{in } \Omega, \end{aligned}$$

we have $0 \le \sigma_j v_j \le \varepsilon_j (-\Delta)^{-1}(1) \le c\varepsilon_j d_\Omega$. Then (3.18) holds. Consequently there exists a sequence $\left\{\delta_j\right\}_{j\in\mathbb{N}}$ such that $\sigma_j v_j \le \delta_j d_\Omega$ in Ω, with $\lim_{j\to\infty} \delta_j = 0$. Since, by (3.17) and *H6)*, in weak sense,

$$\begin{cases} -\Delta\left(\overline{u} + \sigma_j v_j\right) \le a\left(\overline{u} + \sigma_j v_j\right)^{-\alpha} + f\left(\overline{\lambda}, ., \overline{u} + \sigma_j v_j\right) \text{ in } \Omega, \\ \overline{u} + \sigma_j v_j = 0 \text{ on } \partial\Omega, \end{cases}$$

we have that $\overline{u} + \sigma_j v_j$ is a subsolution, in the sense of the distributions, of the problem

$$\begin{cases} -\Delta u = au^{-\alpha} + f\left(\overline{\lambda}, ., u\right) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{3.19}$$

Also, $-\Delta u_\Lambda = au_\Lambda^{-\alpha} + f(\Lambda, ., u_\Lambda) \ge au_\Lambda^{-\alpha} + f\left(\overline{\lambda}, ., u_\Lambda\right)$ in Ω and so u_Λ is a supersolution of (3.19). On the other hand, by Theorem 1.2, we have, for some positive constant c, $\overline{u} + cd_\Omega = u_{\overline{\lambda}} + cd_\Omega \le u_\Lambda$ in Ω. Thus, for j large enough, $\overline{u} + \sigma_j v_j = u_{\overline{\lambda}} + \sigma_j v_j \le u_\Lambda - cd_\Omega + \delta_j d_\Omega \le u_\Lambda$. Moreover, since $\overline{u} \ge c' d_\Omega$ in Ω, there exists $k \in L^\infty_{loc}(\Omega)$ such that $|a(x)s^{-\alpha} + f(\lambda, x, s)| \le k(x)$ for all $s \in \left[\overline{u}(x) + cd_\Omega(x), u_\Lambda\right]$ *a.e.* $x \in \Omega$. Then, by Remark 3.1, there exists a solution z, in the sense of distributions, to (3.19) that satisfies $\overline{u} + \sigma_j v_j \le z \le u_\Lambda$ in Ω, and so, for j large enough, $z \ge \overline{u} + \sigma_j v_j > \overline{u}$ in Ω. Observe that, by Theorem 1.2, $u_\Lambda \in C\left(\overline{\Omega}\right)$, and so $f(\Lambda, ., u_\Lambda) \in L^\infty(\Omega)$. Now, $u_\Lambda = S_0(f(\Lambda, ., u_\Lambda))$, and then, by Lemma 2.2 *vii)*, there exist positive constants c and γ such that $u_\Lambda \le cd_\Omega^\gamma$ in Ω. Then $z \le cd_\Omega^\gamma$ in Ω. Also $\overline{u} \in L^\infty(\Omega)$, and so $f\left(\overline{\lambda}, ., \overline{u}\right) \in L^\infty(\Omega)$. Since $\overline{u} = S_0\left(f\left(\overline{\lambda}, ., \overline{u}\right)\right)$, Lemma 2.2 says that there exists a positive constant c' such that $\overline{u} \ge c' d_\Omega^\tau$ in Ω, with $\tau = 1$ if $0 < \alpha < 1$ and $\tau = \frac{2}{1+\alpha}$ if $1 \le \alpha < 3$. Then, for such τ and c', we have $z \ge c' d_\Omega^\tau$ in Ω, and so, by Lemma 3.2, z is a weak solution of (3.19), and it belongs to $H_0^1(\Omega) \cap C^1(\Omega) \cap L^\infty(\Omega)$, which contradicts our initial assumption that for $\lambda = \overline{\lambda}$ (1.1) has a unique weak solution. □

Proof of Corollaries 1 and 2. The corollaries follow from Theorems 1.2 and 1.3, taking $f(\lambda, x, s) := \lambda g(x, s)$ for corollary 1, and taking $f(\lambda, x, s) := g(x, \lambda s)$ for corollary 2. □

Conflict of interest

All authors declare no conflicts of interest in this paper

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