Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 2(1): 70-80
DOI:10.3934/Math.2017.1.70
Received: 4 December 2016
Accepted: 20 December 2016
Published: 5 January 2017

Research article

A Probabilistic Characterization of g-Harmonic Functions

Liang Cai[1,*] **Huan-Huan Zhang**[1] **and Li-Yun Pan**[2,3]

[1] School of Mathematics and Statistics, Beijing Institute of Technology, Beijing 100081, China
[2] Department of Basic Science, Beijing Institute of Graphic Communication, Beijing 102600, China
[3] Beijing Institute of Education, Beijing 100120, China

[*] **Correspondence:** Email: cailiang@bit.edu.cn

Abstract: Associated with a quasi-linear generator function g, we give a definition of g-harmonic functions. The relation between the g-harmonic functions and g-martingales will be delineated. It is direct to construct such relation for smooth case, but for continuous case we need the theory of viscosity solution. Under the nonlinear expectation mechanism, we can also get the similar relation between harmonic functions and martingales. The strict converse problem of mean value property of g-harmonic functions are discussed finally.

Keywords: BSDE; g-martingale; g-harmonic function; nonlinear Feynman-Kac formula; viscosity solution

1. Introduction and Preliminary

Harmonic function ($\Delta u = 0$) has a probabilistic interpretation as that if $\Delta u = 0$ on R^n, then $u(B_t^x)$ is a martingale for any $x \in R^n$ (see for example [6]). This relation between martingale and harmonic function connects probability with potential analysis. It helps us to give probabilistic characterization for harmonic function and more generalized X-harmonic function [6]. In 1997, Peng [9] introduced the notions of g-expectation and conditional g-expectation via backward stochastic differential equations (BSDE) with quasi-linear generator function g. Further, Peng [10] introduced the notion of g-martingale. Thanks to these works, we will give a probabilistic characterization of the g-harmonic functions which have quasi-linear generator function g.

Now we state our problem in detail. Let $(\Omega, \mathcal{F}, P)$ be a probability space endowed with the natural filtration $\{\mathcal{F}_t\}_{t\geq 0}$ generated by an n-dimensional Brownian motion $\{B_t\}_{t\geq 0}$, i.e.

$$\mathcal{F}_t = \sigma\{B_s : s \leq t\}.$$

Then we can define a g-martingale by an $\mathcal{F}_t$-adapted process $\{y_t\}_{t\geq 0}$ which satisfies the following BSDE for any $0 \leq s \leq t$:

$$y_s = y_t + \int_s^t g(y_r, z_r)dr - \int_s^t z_r dB_r. \tag{1.1}$$

Here $g : R \times R^n \longrightarrow R$, satisfies the conditions:
(H1). $g(y, 0) \equiv 0$ and the Lipschitz condition: $\exists C > 0$, for any $(y_1, z_1), (y_2, z_2) \in R \times R^n$ we have

$$|g(y_1, z_1) - g(y_2, z_2)| \leq C(|y_1 - y_2| + |z_1 - z_2|).$$

And the equality (1.1) can also be formulated simply as [11]:

$$\mathcal{E}^g_{s,t}(y_t) := y_s.$$

Then we can also get the definition of g-super(sub)martingale when

$$\mathcal{E}^g_{s,t}(y_t) \leq (\geq)\ y_s.$$

This definition derives from the definition of g-expectation in the beginning paper Peng [9]. When $g(y, z) \equiv 0$ the g-expectation is actually the classical expectation. Except that g-expectation is nonlinear in general, it holds many other important properties as its classical counterpart [2,4,10,12].

Given an n-dimensional Itô's diffusion process $\{X^x_t\}_{t\geq 0}$:

$$\begin{aligned} dX^x_t &= b(X^x_t)dt + \sigma(X^x_t)dB_t, \\ X^x_0 &= x \in R^n, \end{aligned} \tag{1.2}$$

where $b(x) : R^n \longrightarrow R^n$, $\sigma(x) : R^n \longrightarrow R^{n\times n}$ satisfy the Lipschitz condition: $\exists C > 0$ s.t.

$$|b(x_1) - b(x_2)| + |\sigma(x_1) - \sigma(x_2)| \leq C|x_1 - x_2|, \quad \forall x_1, x_2 \in R^n,$$

our problem is that: what kind of function $u(x) : R^n \longrightarrow R$ satisfies that $u(X^x_t)$ is a g-martingale for any $x \in R^n$?

This problem also has its classical counterpart:

First if $\{X^x_t\}$ is just the Brownian motion $\{B^x_t\}$, then we have the result that when $u(x)$ is harmonic on R^n i.e.

$$\Delta u = \sum_i \frac{\partial^2 u}{\partial x_i^2} = 0, \quad \text{for any } x \in R^n,$$

the process $u(B^x_t)$ is a martingale for any x. And conversely if $u(x)$ satisfies that $u(B^x_t)$ is a martingale for any x, then $u(x)$ must be harmonic on R^n. The proof may have many editions, here we can give a sketch of one which may induce the extension to g-martingale case.

If $u(x)$ is harmonic on R^n, then we use Itô's formula to $u(B^x_t)$ and get

$$du(B^x_t) = \sum_i \frac{\partial u}{\partial x_i}(B^x_t)dB_{i,t} + \frac{1}{2}\sum_i \frac{\partial^2 u}{\partial x_i^2}(B^x_t)dt = \sum_i \frac{\partial u}{\partial x_i}(B^x_t)dB_{i,t}.$$

Then we get $u(B_t^x)$ is a martingale for any $x \in R^n$. Conversely if $u(x)$ is continuous on R^n and for any $x \in R^n$, $u(B_t^x)$ is a martingale, then we have $E[u(B_\tau^x)] = u(x)$ for any stopping time τ. Particularly for any sphere $S(x,r) = \{y \in R^n : |y - x| < r\}$, we have

$$u(x) = E[u(B^x_{\tau_{S(x,r)}})] = \int_{\partial S(x,r)} u(y) d\sigma_y,$$

where $\tau_{S(x,r)}$ is the exit time of $\{B_t^x\}$ from the sphere $S(x,r)$, i.e.

$$\tau_{S(x,r)} = \inf\{t > 0 : |B_t^x - x| \geq r\},$$

and σ_y is the harmonic measure on the $\partial S(x,r)$. Then from the familiar converse of the mean value property for harmonic function, we can get $u(x)$ must be harmonic function.

Further we can extend the Brownian motion $\{B_t^x\}$ to the general diffusion process $\{X_t^x\}$:

If $u(x) \in C_0^2(R^n)$ and satisfies

$$\sum_i b_i \frac{\partial u}{\partial x_i}(x) + \frac{1}{2}\sum_{i,j}(\sigma\sigma^\tau)_{i,j}\frac{\partial^2 u}{\partial x_i \partial x_j}(x) = 0, \tag{1.3}$$

then we have $u(X_t^x)$ is a martingale for any x. The proof also uses the Itô's formula. But conversely if $u(X_t^x)$ is a martingale for any x, we can't conclude that $u(x)$ is smooth. Then with additional assumption $u(x) \in C_0^2(R^n)$ we can get that $u(x)$ satisfies the PDE (1.3) [6].

Then naturally we will ask that what happens when we substitute the expectation mechanism by the g-expectation mechanism. First we will define the infinitesimal generator:

Definition 1. *Let*

$$\mathcal{A}_g^X f(x) := \lim_{t \downarrow 0} \frac{\mathcal{E}_{0,t}^g[f(X_t^x)] - f(x)}{t}, \tag{1.4}$$

then we call $\mathcal{A}_g^X$ the infinitesimal generator of a diffusion process $\{X_t^x\}$ under g-expectations.

Thanks to the celebrating nonlinear Feynman-Kac formula [8], we can get the explicit form of $\mathcal{A}_g^X$ when $f \in C_0^2(R^n)$ by considering the following type of quasilinear parabolic PDE:

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) - \mathcal{L}u(t,x) - g(u(t,x), u_x(t,x)\sigma(x)) = 0, \\ u(0,x) = f(x). \end{cases} \tag{1.5}$$

where

$$\mathcal{L}u(t,x) = \sum_i b_i \frac{\partial u}{\partial x_i}(t,x) + \frac{1}{2}\sum_{i,j}(\sigma\sigma^\tau)_{i,j}\frac{\partial^2 u}{\partial x_i \partial x_j}(t,x). \tag{1.6}$$

When $f \in C_0^2(R^n)$, we assert that

$$u(t,x) = \mathcal{E}_{0,t}^g[f(X_t^x)] \tag{1.7}$$

is the solution of PDE (1.5). Then under the case $t = 0$, we get

$$\mathcal{A}_g^X f(x) = \mathcal{L}f(x) + g(f(x), f_x(x)\sigma(x)). \tag{1.8}$$

Then we finish the preliminary and we can introduce our main results. In section 2, we give a characterization of g-harmonic function under smooth case. In section 3, we characterize it under continuous case, where the differential operator is interpreted as viscosity solution. In section 4, we will investigate the strict converse problem of mean value property of g-harmonic function evoked by its classical counterpart [7].

2. Smooth Case

The equality (1.8) implies the relation between the g-martingales and the g-harmonic functions when $f \in C_0^2(R^n)$. In fact, the left side of (1.8) is related to a g-martingale and the right side is related to a harmonic PDE. At first we will give the definition of g-harmonic functions:

Definition 2. *Let $f \in C_0^2(R^n)$. We call it a g-(super)harmonic function w.r.t. $\{X_t^x\}$ if it satisfies*

$$\mathcal{A}_g^X f(x)(\leq) = 0, \qquad \text{for any} \quad x \in R^n. \tag{2.1}$$

Then we suffice to construct the relation between the g-supermartingales and the g-superharmonic functions.

Theorem 1. *If $f(x) \in C_0^2(R^n)$, then the following assertions are equivalent:*
(1)$f(x)$ is a g-superharmonic function.
(2)$\{f(X_t^x)\}$ is a g-supermartingale for any $x \in R^n$.

Proof. (i) (1) $\Rightarrow$ (2):

For any $f \in C^2(R^n)$, by Itô's formula, we can get $f(X_t^x)$ is still an Itô's diffusion process:

$$f(X_t^x) = f(X_s^x) + \int_s^t \mathcal{L}f(X_r^x)dr + \int_s^t f_x(X_r^x)\sigma(X_r^x)dB_r, \quad 0 \leq s \leq t.$$

and then we insert the term $g(f(X_r^x), f_x(X_r^x)\sigma(X_x^r))$ and get

$$\begin{aligned} f(X_s^x) &= f(X_t^x) - \int_s^t \mathcal{L}f(X_r^x)dr - \int_s^t f_x(X_r^x)\sigma(X_r^x)dB_r \\ &= f(X_t^x) + \int_s^t g(f(X_r^x), f_x(X_r^x)\sigma(X_x^r))dr - \int_s^t f_x(X_r^x)\sigma(X_r^x)dB_r \\ &\quad - \int_s^t [\mathcal{L}f(X_r^x) + g(f(X_r^x), f_x(X_r^x)\sigma(X_x^r))]dr. \end{aligned}$$

$f(x)$ is a g-superharmonic function, so

$$\mathcal{L}f(X_r^x) + g(f(X_r^x), f_x(X_r^x)\sigma(X_r^x)) = \mathcal{A}_g^X f(X_r^x) \leq 0.$$

And then according to the comparison theory of BSDE [10], we can get $\{f(X_t^x)\}$ is a g-supermartingale.

(ii) (2) ⇒ (1):

By the definition of the $\mathcal{A}_g^X$:

$$\mathcal{A}_g^X f(x) = \lim_{t\downarrow 0} \frac{\mathcal{E}_{0,t}^g[f(X_t^x)] - f(x)}{t}.$$

$\{f(X_t^x)\}$ is a g-supermartingale, so

$$\mathcal{E}_{0,t}^g[f(X_t^x)] - f(x) \le 0,$$

then

$$\mathcal{A}_g^X f(x) \le 0.$$

So we get $f(x)$ is a g-superharmonic function. □

3. Continuous Case

If we generalize the requirement of function $f(x)$ to be only continuous on R^n, how we get a function f which satisfies that $f(X_t^x)$ is a g-martingale for any $x \in R^n$? With the help of viscosity solution [3], we can also refer to the quasi-linear second order PDEs. Here we need a lemma due to Peng [8].

Lemma 1. *Let* $0 \le t \le T$ *and*

$$u(t,x) = \mathcal{E}_{0,T-t}^g[f(X_{T-t}^x)].$$

Then $u(t,x)$ *is the viscosity solution of the following PDE on* $(0,T)\times R^n$*:*

$$\begin{cases} \frac{\partial u}{\partial t} + \mathcal{L}u(t,x) + g(u(t,x), u_x(t,x)\sigma(x)) = 0, \\ u(T,x) = f(x). \end{cases} \tag{3.1}$$

Here $g(y,z)$ *and* $f(x)$ *satisfy:*
(H2) Let $F(u,p) = g(u, p\sigma(x))$*, then* $\exists C > 0$ *s.t.*

$$|F(u,p)| \le C(1 + |u| + |p|);$$
$$|D_u F(u,p)|, |D_p F(u,p)| \le C;$$

and (H3) $f(x)$ *is a continuous function with a polynomial growth at infinity.*

Definition 3. *Let* $u(t,x) \in C(R\times R^n)$*.* $u(t,x)$ *is said to be a viscosity super-solution (resp. sub-solution) of the following PDE (3.2):*

$$\frac{\partial u}{\partial t} + \mathcal{L}u(t,x) + g(u(t,x), u_x(t,x)\sigma(x)) = 0, \tag{3.2}$$

if for any $(t,x) \in R\times R^n$ *and* $\varphi \in C^{1,2}(R\times R^n)$ *such that* $\varphi(t,x) = u(t,x)$ *and* (t,x) *is a maximum (resp. minimum) point of* $\varphi - u$*,*

$$\frac{\partial \varphi}{\partial t}(t,x) + \mathcal{L}\varphi(t,x) + g(\varphi(t,x), \varphi_x(t,x)\sigma(x)) \le 0.$$

$$(resp. \quad \frac{\partial \varphi}{\partial t}(t,x) + \mathcal{L}\varphi(t,x) + g(\varphi(t,x), \varphi_x(t,x)\sigma(x)) \geq 0.)$$

$u(t,x)$ is said to be a viscosity solution of PDE (3.2) if it is both a viscosity super- and sub-solution of (3.2).

We also consider the viscosity solution of the following type of quasilinear elliptic PDE (3.3):

$$\mathcal{L}u(x) + g(u(x), u_x(x)\sigma(x)) = 0. \tag{3.3}$$

We can directly get an relation between the two solutions of (3.2) and (3.3):

Lemma 2. *Let $\tilde{u}(t,x) = u(x)$ for all $(t,x) \in R \times R^n$, then we have:*
$\tilde{u}(t,x)$ is the viscosity super-(sub-)solution of PDE (3.2) $\Leftrightarrow$ $u(x)$ is the viscosity super-(sub-)solution of PDE (3.3).

Proof. We suffice to prove the case of viscosity super-solution.
(i) "$\Rightarrow$ ":
For any $(t_0, x_0) \in R \times R^n$, and a function $\varphi(x) \in C^2(R^n)$ which satisfies $\varphi(x) \leq u(x), \varphi(x_0) = u(x_0)$, we define $\tilde{\varphi}(t,x) = \varphi(x)$ for all $(t,x) \in R \times R^n$. Then

$$\frac{\partial \tilde{\varphi}}{\partial t} = 0, \quad \tilde{\varphi}(t_0, x_0) = \tilde{u}(t_0, x_0), \quad \tilde{\varphi}(t,x) \leq \tilde{u}(t,x),$$

and due to the assumption that $\tilde{u}(t,x)$ is the viscosity super-solution of PDE (3.2), we have

$$\frac{\partial \tilde{\varphi}}{\partial t}(t_0, x_0) + \mathcal{L}\tilde{\varphi}(t_0, x_0) + g(\tilde{\varphi}(t_0, x_0), \tilde{\varphi}_x(t_0, x_0)\sigma(x_0)) \leq 0,$$

i.e.

$$\mathcal{L}\varphi(x_0) + g(\varphi(x_0), \varphi_x(x_0)\sigma(x_0)) \leq 0.$$

So $u(x)$ is the viscosity super-solution of PDE (3.3).
(ii). " $\Leftarrow$ ":
For any $(t_0, x_0) \in R \times R^n$, and a function $\varphi(t,x) \in C^2(R \times R^n)$ which satisfies

$$\varphi(t,x) \leq \tilde{u}(t,x) \quad \text{and} \quad \varphi(t_0, x_0) = \tilde{u}(t_0, x_0),$$

then

$$\frac{\partial \varphi}{\partial t}(t_0, x_0) = 0, \tag{3.4}$$

and due to the assumption that $u(x)$ is the viscosity super-solution of PDE (3.3), we have

$$\mathcal{L}\varphi(t_0, x_0) + g(\varphi(t_0, x_0), \varphi_x(t_0, x_0)\sigma(x_0)) \leq 0.$$

Combined with (3.4), we get

$$\frac{\partial \varphi}{\partial t}(t_0, x_0) + \mathcal{L}\varphi(t_0, x_0) + g(\varphi(t_0, x_0), \varphi_x(t_0, x_0)\sigma(x_0)) \leq 0.$$

So $\tilde{u}(t,x)$ is the viscosity super-solution of PDE (3.2). □

Then we can introduce our main result of this section:

Theorem 2. *We have the following two consequences:*
(i) For any $f(x) \in C(R^n)$, and $g(y,z)$ satisfying (H1), if $\forall x \in R^n$, $f(X_t^x)$ is a g-supermartingale, then $f(x)$ is a viscosity super-solution of PDE (3.3).
(ii) For any $f(x)$ satisfying (H3), and $g(y,z)$ satisfying (H1) and (H2), let $f(x)$ is a viscosity super-solution of PDE (3.3), then $\{f(X_t^x)\}$ is a g-supermartingale for all $x \in R^n$.

Actually, the consequence (ii) is the answer of our main problem and the consequence (i) is the converse of it. But (i) is easier to be proved, so we are going to prove (i) at first.

Proof. (i) For any $x \in R^n$, let $\varphi \in C^2(R^n)$, $\varphi(x) = f(x)$ where x is a maximum point of $\varphi - f$. It means $\forall \tilde{x} \in R^n$, we have $\varphi(\tilde{x}) \leq f(\tilde{x})$. Then from (1.8), we get

$$\begin{aligned}\mathcal{L}\varphi(x) + g(\varphi(x), \varphi_x(x)\sigma(x)) &= \mathcal{A}_g^X \varphi(x) \\ &= \lim_{t \downarrow 0} \frac{\mathcal{E}_t^g[\varphi(X_t^x)] - \varphi(x)}{t} \\ &= \lim_{t \downarrow 0} \frac{\mathcal{E}_t^g[\varphi(X_t^x)] - f(x)}{t}.\end{aligned}$$

According to the comparison theory of BSDE, we get

$$\mathcal{E}_t^g[\varphi(X_t^x)] \leq \mathcal{E}_t^g[f(X_t^x)],$$

and with the assumption $\{f(X_t^x)\}$ is a g-supermartingale, we can get

$$\mathcal{E}_t^g[\varphi(X_t^x)] - f(x) \leq \mathcal{E}_t^g[f(X_t^x)] - f(x) \leq 0.$$

Then

$$\mathcal{A}_g^X \varphi(x) = \lim_{t \downarrow 0} \frac{\mathcal{E}_t^g[\varphi(X_t^x)] - f(x)}{t} \leq 0,$$

i.e.

$$\mathcal{L}\varphi(x) + g(\varphi(x), \varphi_x(x)\sigma(x)) \leq 0.$$

By definition, it means $f(x)$ is a viscosity super-solution of PDE (3.3).
(ii) We want to prove $\{f(X_t^x)\}$ is a g-supermartingale for any $x \in R^n$. It means that we need to prove $\forall x \in R^n$ *and* $\forall 0 \leq s \leq t$, we have

$$\mathcal{E}_{s,t}^g[f(X_t^x)] \leq f(X_s^x).$$

Under the assumption, in fact $b(x), \sigma(x)$ and $g(y,z)$ are all independent of time t, so we can get the Markovian property of $\mathcal{E}_{s,t}^g$, i.e.

$$\mathcal{E}_{s,t}^g[f(X_t^x)] = \mathcal{E}_{t-s}^g[f(X_{t-s}^y)]|_{y=X_s^x}.$$

Then we get an equivalence relation:

$$\{f(X_t^x)\} \text{ is a g-(super)martingale for any } x \in R^n \Leftrightarrow$$

$$\mathcal{E}_t^g[f(X_t^x)] = (\leq) f(x) \text{ for any } t \geq 0 \text{ and } x \in R^n. \tag{3.5}$$

So we suffice to prove the latter assertion.

According to lemma 2, for any $T \geq 0$, the assumption $f(x)$ is a viscosity super-solution of PDE (3.3) implies that $\tilde{f}(t,x) := f(x)$ is a viscosity super-solution to the following PDE:

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) + \mathcal{L}u(t,x) + g(u(t,x), u_x(t,x)\sigma(x)) = 0, \\ u(T,x) = f(x). \end{cases} \tag{3.6}$$

And with the help of lemma 1,

$$u(t,x) = \mathcal{E}_{0,T-t}^g[f(X_{T-t}^x)]$$

is actually the viscosity solution of PDE (3.6). Moreover by the maximum principle of the viscosity solution [1], we can get

$$u(t,x) \leq \tilde{f}(t,x), \quad \text{for any } 0 \leq t \leq T.$$

Especially, we have

$$u(0,x) \leq \tilde{f}(0,x),$$

i.e.

$$\mathcal{E}_T^g[f(X_T^x)] \leq f(x).$$

□

Corollary 1. *(i) For any $f(x) \in C(R^n)$, and $g(y,z)$ satisfying (H1), if $\forall x \in R^n$, $f(X_t^x)$ is a g-martingale, then $f(x)$ is a viscosity solution of PDE (3.3).*
(ii) For any $f(x)$ satisfying (H3), and $g(y,z)$ satisfying (H1) and (H2), let $f(x)$ is a viscosity solution of PDE (3.3), then $\{f(X_t^x)\}$ is a g-martingale for all $x \in R^n$.

It is an immediate consequence from the theorem 2.

4. Strict Converse of Mean Value Property

For classical harmonic function, many generalized results of the converse problem of mean value property have been investigated [5,7]. In [7], Øksendal and Stroock give a technique to solve a strict converse of the mean value property for harmonic functions. Now we will generalize it to the case of g-harmonic function. Here the strictness means that for each $x \in R^n$ we don't need justify that for any stopping time τ whether $\mathcal{E}_{0,\tau}^g(f(X_\tau^x))$ equals $f(x)$. We only need to justify one appropriate stopping time of each x.

In the sequel we put $\Delta(x,r) = \{y \in R^n; |y-x| < r\}$ for any $x \in R^n$ and $r > 0$. Let $\tau_U = \inf\{t > 0; X_t^x \in U^c\}$ for any open set U. And we suppose the operator (1.6) is elliptic on R^n.

Theorem 3. *$f(x)$ is a local bounded continuous function on R^n. If for any $x \in R^n$, there exists a radius $r(x)$, the mean value property holds:*

$$\mathcal{E}_{0,\tau_x}^g[f(X_{\tau_x}^x)] = f(x), \quad \textit{here} \quad \tau_x = \tau_{\Delta(x,r(x))}. \tag{4.1}$$

And $r(x)$ is a measurable function of x and satisfies that for each x, there exists a bounded open set U_x, $x \in U_x$ and moreover $r(y)$, $y \in U_x$ should satisfy the following two conditions:

$$0 \le r(y) \le dist(y, \partial U_x), \tag{4.2}$$

and

$$inf\{r(y); y \in K\} > 0 \tag{4.3}$$

for all closed subsets K of U_x with $dist(K, \partial U_x) > 0$. Then we can get

(i) For each $y \in U_x$ the mean value property holds on the boundary:

$$\mathcal{E}^g_{0,\tau_y}[f(X^y_{\tau_y})] = f(y), \quad here \ \ \tau_y = inf\{t > 0; X^y_t \in U^c_x\}.$$

and furthormore

(ii) $f(x)$ is the viscosity solution of PDE (12).

Proof. $(i) \Rightarrow (ii)$ is also based on the nonlinear Feynman-Kac formula for elliptic PDE [8]. So we sufficiently prove the first conclusion.

For each $y \in U_x$, we define a sequence of stopping times τ_k for $\{X^y_t\}$ by induction as follows:

$$\begin{aligned} &\tau_0 \equiv 0 \\ &\tau_k = \inf\{t \ge \tau_{k-1}; |X^y_t - X^y_{\tau_{k-1}}| \ge r(X^y_{\tau_{k-1}})\}, \ \ k \ge 1. \end{aligned}$$

By the mean property (4.1), and the strong markovian property we can get

$$\begin{aligned} \mathcal{E}^g_{0,\tau_k}[f(X^y_{\tau_k})] &= \mathcal{E}^g_{0,\tau_{k-1}}[\mathcal{E}^g_{\tau_{k-1},\tau_k}[f(X^y_{\tau_k})]] \\ &= \mathcal{E}^g_{0,\tau_{k-1}}[\mathcal{E}^g_{0,\tau_k-\tau_{k-1}}[f(X^{X^y_{\tau_{k-1}}}_{\tau_k-\tau_{k-1}})]] \\ &= \mathcal{E}^g_{0,\tau_{k-1}}[f(X^y_{\tau_{k-1}})], \end{aligned}$$

then by induction we get

$$\mathcal{E}^g_{0,\tau_k}[f(X^y_{\tau_k})] = f(y).$$

In the following we will prove $\tau_k \to \tau_y$ *a.e.* when $k \to \infty$. Obviously

$$\tau_k \ge \tau_{k-1},$$

so there exists a stopping time τ s.t. $\tau_k \uparrow \tau$. If $\tau \ne \tau_y$, then there exists $\epsilon > 0$ s.t.

$$\text{dist}(X^y_{\tau_k}, \partial U_x) \ge \epsilon, \quad \text{for any } k.$$

Let $r_k = r(X^y_{\tau_k})$, according to the condition (4.3), we get there exists $r > 0$,

$$r_k \ge r, \quad \text{ for any } \ k.$$

It means

$$\text{dist}(X^y_{\tau_k}, X^y_{\tau_{k-1}}) \ge r.$$

And since X_t^y is continuous, then $\tau_k \to \infty$, which implies $\tau_y = \infty$. So

$$P(\tau_k \text{ don't converge to } \tau_y) \leq P(\tau_y = \infty).$$

But for (1.6) is elliptic and U_x is bounded, we have $P(\tau_y < \infty) = 1$. So

$$P(\tau_k \text{ converge to } \tau_y) = 1.$$

Then we get

$$\begin{aligned} f(y) &= \mathcal{E}^g_{0,\tau_k}[f(X^y_{\tau_k})] \\ &= \lim_{k\uparrow\infty} \mathcal{E}^g_{0,\tau_k}[f(X^y_{\tau_k})] \\ &= \mathcal{E}^g_{0,\tau_y}[f(X^y_{\tau_y})]. \end{aligned}$$

So we have finished the proof. □

Acknowledgement

The first author is supported by the National Natural Science Foundation of China (11026125) ,and the third author is supported by the BIGC Key Project (Ea201606).

Conflict of Interest

All authors declare no conflicts of interest in this paper.

References

1. G. Barles and E. Lesigne, *SDE, BSDE and PDE*. Pitman Research Notes in Mathematics Series, **364**, Backward Stochastic Differential Equation, Ed. by N. El Karoui and L.Mazliak (1997), 47-80.
2. Z. Chen and S. Peng, *Continuous properties of g-martingales*. Chin. Ann. of Math, **22** (2001), 115-128.
3. M. G. Crandall, H. Ishii and P. L. Lions, *User's guide to viscosity solutions of second order Partial differential equations*. Bull. Amer. Math. Soc., **27** (1992), 1-67.
4. L. Jiang, *Convexity, translation invariance and subadditivity for g-expectations and related risk measures*. The Annals of Applied Probability, **18** (2008), 245-258.
5. O. D. Kellogg, *Converses of Gauss' theorem on the arithmetic mean*. Tran. Amer. Math. Soc., **36** (1934), 227-242.
6. B. Øksendal, Stochastic differential Equations, Sixth Edition, Springer, Berlin, 2003.
7. B. Øksendal and D. W. Stroock, *A characterization of harmonic measure and markov processs whose hitting distritributions are preserved by rotations, translations and dilatations*. Ann. Inst. Fourier. **32** (1982), 221-232.
8. S. Peng, *A generalized dynamic programming principle and Hamilton-Jacobi-Bellman equation.* Stochastics and Stochastic Reports, **38** (1992), 119-134.

9. S. Peng, *BSDE and related g-expectation*. Pitman Research Notes in Mathematics Series, **364**, Backward Stochastic Differential Equation, Ed. by N. El Karoui and L.Mazliak (1997), 141-159.

10. S. Peng, *Monotonic limit theorem of BSDE and nonlinear decomposition theorem of Doob-Meyer's type*. Prob. Theory Rel. Fields, **113** (1999), 473-499.

11. S. Peng, *Nonlinear expectations, nonlinear evaluations and risk measures.* Stochastic Methods in Finance. Lecture Notes in Mathematics Series. **1856** (2004), 165-253.

12. W. Wang, *Maximal inequalities for g-martingales*. Statist. Probab. Lett., **79** (2009), 1169-1174.