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Research article

On the Caginalp phase-field system based on the Cattaneo law with nonlinear coupling

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Abstract: We focus in this paper on a Caginalp phase-field system based on the Cattaneo law with nonlinear coupling. We start our analysis by establishing existence, uniqueness and regularity based on Moser's iterations. We finish with the study of the spatial behavior of the solutions in a semi-infinite cylinder, assuming the existence of such solutions.

Keywords: Caginalp phase field system; Cattaneo law; nonlinear coupling; well-posedness; regularity; spatial behavior

1. Introduction

The Caginalp phase-field model

$$\frac{\partial u}{\partial t} - \Delta u + f(u) = \theta \tag{1.1}$$

$$\frac{\partial \theta}{\partial t} - \Delta \theta = -\frac{\partial u}{\partial t} \tag{1.2}$$

as described in [1] has been the subject of numerous studies in recent years [2, 3, 4, 7, 11, 13, 35]. This model describes the behavior of certain materials in their stages of melting and solidification. In this case θ and u can represent respectively the temperature and the order parameter.

Using Fourier's law to the aforementioned model, one can observe a disparity between the observed results and the expected outcome. One of them is known as "paradox of heat conduction" [9]. In order to make the model more realistic by adjusting the latter, some alternative laws have been proposed, the

Maxwell-Cattaneo law [25] or the Gurtin-Pipkin law [21, 22]. Furthermore, in [18, 19, 20] Green and Naghdi proposed an alternative theory based on a thermomechanical theory of deformable media to obtain very rational models.

In recent years, the study of models derived from these new laws have been the subject of particular attention, especially with regard to the qualitative study of solutions [14, 15, 16, 17, 23, 27, 29, 30, 31, 32].

The purpose of our study is the following model

$$\frac{\partial u}{\partial t} - \Delta u + f(u) = g(u)\frac{\partial \alpha}{\partial t} \tag{1.3}$$

$$\eta\frac{\partial^2 \alpha}{\partial t^2} + \frac{\partial \alpha}{\partial t} - \Delta\alpha = -\eta g(u)\frac{\partial u}{\partial t} - G(u) + h, \quad \eta > 0 \tag{1.4}$$

$$u|_{\partial\Omega} = \alpha|_{\partial\Omega} = 0 \tag{1.5}$$

$$u|_{t=0} = u_0, \quad \alpha|_{t=0} = \alpha_0, \quad \left.\frac{\partial \alpha}{\partial t}\right|_{t=0} = \alpha_1, \tag{1.6}$$

where Ω is a bounded and regular domain of $\mathbb{R}^n$ with $n = 2$ or 3. This model is motivated by the recent works of Miranville and Quintanilla [24, 25, 26].

This paper is organized as follows. In Section 2, we give a rigorous derivation of our model using Cattaneo's law and a nonlinear coupling. Then, in Section 3 we prove existence, uniqueness and regularity results. We finish, in Section 4, by the study of the spatial behavior of the solutions in a semi-infinite cylinder, assuming that such solutions exist.

Throughout this paper, the same letters c, c' and c'' denote constants which may change from line to line.

2. Derivation of the model

Our equations (1.3)-(1.6) modeling phase transition are derived as follows.

Let Ψ be the total energy of the system defined as

$$\Psi(u,\theta) = \int_\Omega \Big(\frac{1}{2}|\nabla u|^2 + F(u) - G(u)\theta - \frac{1}{2}\theta^2\Big)dx,$$

with $G' = g$ and $F' = f$. Let H be the enthalpy satisfying

$$H = \partial_\theta \Psi = G(u) + \theta. \tag{2.1}$$

Furthermore,

$$\tfrac{\partial u}{\partial t} = -\partial_u \Psi \tag{2.2}$$

$$\tfrac{\partial H}{\partial t} + \mathrm{divq} = 0. \tag{2.3}$$

In particular, considering the Maxwell-Cattaneo law

$$(1 + \eta\frac{\partial}{\partial t})q = -\nabla\theta, \quad \eta > 0, \tag{2.4}$$

we get (using (2.1), (2.2) and (2.4))

$$(1+\eta\frac{\partial}{\partial t})\frac{\partial\theta}{\partial t}-\Delta\theta=(1+\eta\frac{\partial}{\partial t})\frac{\partial G(u)}{\partial t}. \tag{2.5}$$

Setting

$$\alpha=\int_0^t\theta(\tau)d\tau+\alpha_0, \quad \theta=\frac{\partial\alpha}{\partial t}, \tag{2.6}$$

we have, after integrating (2.5) over $[0,t]$,

$$\eta\frac{\partial^2\alpha}{\partial t^2}+\frac{\partial\alpha}{\partial t}-\Delta\alpha=-\eta g(u)\frac{\partial u}{\partial t}-G(u)+h, \tag{2.7}$$

with

$$h=\eta\frac{\partial^2\alpha}{\partial t^2}(0)+\frac{\partial\alpha}{\partial t}(0)-\Delta\alpha(0)-\eta g(u)\frac{\partial u}{\partial t}(0)-G(u)(0) \tag{2.8}$$

and

$$\frac{\partial G(u)}{\partial t}=g(u)\frac{\partial u}{\partial t}. \tag{2.9}$$

This leads to the above system (1.3)-(1.6).

3. Existence and uniqueness of solutions

We start by giving an existence result, the assumptions for the proof being the following: f and g are of class C^1 and

$$|G(s)|^2\le c_1\,F(s)+c_2,\ c_1,c_2\ge 0, \tag{3.1}$$

$$|g(s)s|\le c_3(|G(s)|+1),\ c_3\ge 0, \tag{3.2}$$

$$c_4\,s^{k+2}-c_5\le c_0F(s)-c_0'\le f(s)s\le c_6\,s^{k+2}-c_7,\ c_0,c_4,c_6>0,\ c_0',c_5,c_7\ge 0, \tag{3.3}$$

$$|g(s)|\le c_8(|s|+1), \quad |g'(s)|\le c_9,\ c_8,c_9\ge 0, \tag{3.4}$$

$$|f'(s)|\le c_{10}(|s|^k+1),\ c_{10}\ge 0, \tag{3.5}$$

where k is an integer.

We have the

Theorem 3.1. *We assume that (3.1)-(3.3) hold true. If in addition* $(u_0,\alpha_0,\alpha_1)\in H_0^1(\Omega)\cap L^{k+2}(\Omega)\times H_0^1(\Omega)\times L^2(\Omega)$*, then (1.3)-(1.6) admits a solution* (u,α) *such that* $u\in L^\infty(0,T;H_0^1(\Omega)\cap L^{k+2}(\Omega))$, $\frac{\partial u}{\partial t}\in L^2(0,T;L^2(\Omega))$, $\alpha\in L^\infty(0,T;H_0^1(\Omega))$ *and* $\frac{\partial\alpha}{\partial t}\in L^\infty(0,T;L^2(\Omega))$, $\forall T>0$.

Proof. We will focus on the priori estimates. The proof of existence follows from these estimates and a proper Galerkin scheme [12] and [34].

Multiplying (1.3) by $\frac{\partial u}{\partial t}$ and integrating over Ω, we have

$$\left\|\frac{\partial u}{\partial t}\right\|_2^2+\frac{1}{2}\frac{d}{dt}\|\nabla u\|_2^2+\frac{d}{dt}\int_\Omega F(u)dx=\int_\Omega g(u)\frac{\partial\alpha}{\partial t}\frac{\partial u}{\partial t}dx, \tag{3.6}$$

where $\|.\|_p$ denotes the usual L^p norm and $(.\,,\,.)$ the usual L^2 scalar product; more generally, we denote by $\|.\|_X$ the norm in the Banach space X.

Similarly, multiplying (1.4) by $\frac{\partial \alpha}{\partial t}$, we obtain

$$\begin{aligned}\frac{\eta}{2}\frac{d}{dt}\left\|\frac{\partial \alpha}{\partial t}\right\|_2^2+\left\|\frac{\partial \alpha}{\partial t}\right\|_2^2+\frac{1}{2}\frac{d}{dt}\|\nabla\alpha\|_2^2 &= -\eta\int_\Omega g(u)\frac{\partial u}{\partial t}\frac{\partial \alpha}{\partial t}dx\\ &\quad -\int_\Omega G(u)\frac{\partial \alpha}{\partial t}dx+\int_\Omega h\frac{\partial \alpha}{\partial t}dx.\end{aligned} \tag{3.7}$$

Summing η(3.6) and (3.7), we find

$$\begin{aligned}\eta\left\|\frac{\partial u}{\partial t}\right\|_2^2+\frac{\eta}{2}\frac{d}{dt}\|\nabla u\|_2^2+\eta\frac{d}{dt}\int_\Omega F(u)dx+\frac{\eta}{2}\frac{d}{dt}\left\|\frac{\partial \alpha}{\partial t}\right\|_2^2+\left\|\frac{\partial \alpha}{\partial t}\right\|_2^2\\ +\frac{1}{2}\frac{d}{dt}\|\nabla\alpha\|_2^2=-\int_\Omega G(u)\frac{\partial \alpha}{\partial t}dx+\int_\Omega h\frac{\partial \alpha}{\partial t}dx.\end{aligned} \tag{3.8}$$

We thus obtain a differential inequality of the form

$$\frac{d}{dt}E_1+\eta\left\|\frac{\partial u}{\partial t}\right\|_2^2+\left\|\frac{\partial \alpha}{\partial t}\right\|_2^2=-\int_\Omega G(u)\frac{\partial \alpha}{\partial t}dx+\int_\Omega h\frac{\partial \alpha}{\partial t}dx, \tag{3.9}$$

with $E_1=\frac{\eta}{2}\|\nabla u\|_2^2+\eta\int_\Omega F(u)dx+\frac{\eta}{2}\left\|\frac{\partial \alpha}{\partial t}\right\|_2^2+\frac{1}{2}\|\nabla\alpha\|_2^2$.

Multiplying (1.3) by u, we find

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2+\|\nabla u\|_2^2+(f(u),u)=\int_\Omega g(u)\frac{\partial \alpha}{\partial t}u dx. \tag{3.10}$$

We have, owing to (3.2), (3.3) and (3.10),

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2+\|\nabla u\|_2^2+c_0\int_\Omega F(u)dx\le c\int_\Omega |G(u)|^2dx+\frac{1}{2}\left\|\frac{\partial \alpha}{\partial t}\right\|_2^2+c''. \tag{3.11}$$

From (3.9) and (3.11), we obtain

$$\begin{aligned}\frac{d}{dt}\Big(E_1+\frac{1}{2}\|u\|_2^2\Big)+\eta\left\|\frac{\partial u}{\partial t}\right\|_2^2+\frac{1}{2}\left\|\frac{\partial \alpha}{\partial t}\right\|_2^2+\|\nabla u\|_2^2+c_0\int_\Omega F(u)dx\le\\ -\int_\Omega G(u)\frac{\partial \alpha}{\partial t}dx+\int_\Omega h\frac{\partial \alpha}{\partial t}dx+c\int_\Omega |G(u)|^2dx+c''.\end{aligned} \tag{3.12}$$

Multiplying (1.4) by α, we get

$$\begin{aligned}\eta\frac{d}{dt}\Big(\frac{\partial \alpha}{\partial t},\alpha\Big)+\Big(\frac{\partial \alpha}{\partial t},\alpha\Big)+\|\nabla\alpha\|_2^2 &= -\eta\int_\Omega g(u)\frac{\partial u}{\partial t}\alpha dx\\ &\quad +\int_\Omega (h-G(u))\alpha dx+\eta\left\|\frac{\partial \alpha}{\partial t}\right\|_2^2.\end{aligned} \tag{3.13}$$

Adding δ(3.13) and (3.12) with $\delta > 0$, we find

$$\begin{aligned}
&\frac{d}{dt}\Big(E_1 + \frac{1}{2}\|u\|_2^2\Big) + \eta\Big\|\frac{\partial u}{\partial t}\Big\|_2^2 + \frac{1}{2}\Big\|\frac{\partial \alpha}{\partial t}\Big\|_2^2 + \|\nabla u\|_2^2 + c_0\int_\Omega F(u)dx \\
&+ \eta\delta\frac{d}{dt}\Big(\frac{\partial \alpha}{\partial t}, \alpha\Big) + \delta\Big(\frac{\partial \alpha}{\partial t}, \alpha\Big) + \delta\|\nabla\alpha\|_2^2 \le \\
&- \int_\Omega G(u)\frac{\partial \alpha}{\partial t}dx + \int_\Omega h\frac{\partial \alpha}{\partial t}dx + c\int_\Omega |G(u)|^2dx + c'' \\
&- \delta\eta\int_\Omega g(u)\frac{\partial u}{\partial t}\alpha dx + \delta\int_\Omega (h - G(u))\alpha dx + \delta\eta\Big\|\frac{\partial \alpha}{\partial t}\Big\|_2^2.
\end{aligned} \tag{3.14}$$

Since

$$\int_\Omega g(u)\frac{\partial u}{\partial t}\alpha dx = \frac{d}{dt}\int_\Omega G(u)\alpha dx - \int_\Omega G(u)\frac{\partial \alpha}{\partial t}dx, \tag{3.15}$$

i.e.,

$$-\eta\delta\int_\Omega g(u)\frac{\partial u}{\partial t}\alpha dx = -\eta\delta\frac{d}{dt}\int_\Omega G(u)\alpha dx + \eta\delta\int_\Omega G(u)\frac{\partial \alpha}{\partial t}dx, \tag{3.16}$$

we get, owing to (3.14) and (3.16),

$$\begin{aligned}
&\frac{d}{dt}E_2 + \eta\Big\|\frac{\partial u}{\partial t}\Big\|_2^2 + \frac{1}{2}\Big\|\frac{\partial \alpha}{\partial t}\Big\|_2^2 + \|\nabla u\|_2^2 + c_0\int_\Omega F(u)dx + \delta\|\nabla\alpha\|_2^2 \le \\
&\int_\Omega (h - G(u))\frac{\partial \alpha}{\partial t}dx + c\int_\Omega |G(u)|^2dx + c'' \\
&+ \delta\eta\int_\Omega G(u)\frac{\partial \alpha}{\partial t}dx + \delta\int_\Omega (h - G(u))\alpha dx + \delta\eta\Big\|\frac{\partial \alpha}{\partial t}\Big\|_2^2,
\end{aligned} \tag{3.17}$$

with

$$\begin{aligned}
E_2 =& \frac{1}{2}\|u\|_2^2 + \frac{\eta}{2}\|\nabla u\|_2^2 + \eta\int_\Omega F(u)dx + \frac{\eta}{2}\Big\|\frac{\partial \alpha}{\partial t}\Big\|_2^2 + \frac{1}{2}\|\nabla\alpha\|_2^2 \\
&+ \eta\delta\int_\Omega G(u)\alpha dx + \frac{\delta}{2}\|\alpha\|_2^2 + \eta\delta\Big(\frac{\partial \alpha}{\partial t}, \alpha\Big).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\frac{d}{dt}E_2 + \eta\Big\|\frac{\partial u}{\partial t}\Big\|_2^2 + \frac{1}{2}\Big\|\frac{\partial \alpha}{\partial t}\Big\|_2^2 + \|\nabla u\|_2^2 + c_0\int_\Omega F(u)dx + \delta\|\nabla\alpha\|_2^2 \le \\
&\int_\Omega h\frac{\partial \alpha}{\partial t}dx + c\int_\Omega |G(u)|^2dx + c'' + \delta\eta\Big\|\frac{\partial \alpha}{\partial t}\Big\|_2^2 \\
&+ (\delta\eta - 1)\int_\Omega G(u)\frac{\partial \alpha}{\partial t}dx - \delta\int_\Omega G(u)\alpha dx + \delta\int_\Omega h\alpha dx.
\end{aligned} \tag{3.18}$$

Noting that

$$\begin{aligned}
(\delta\eta - 1)\int_\Omega G(u)\frac{\partial \alpha}{\partial t}dx &\le \frac{1}{8}\Big\|\frac{\partial \alpha}{\partial t}\Big\|_2^2 + c\int_\Omega |G(u)|^2dx, \\
\delta\int_\Omega G(u)\alpha dx &\le c\int_\Omega |G(u)|^2dx + \frac{\delta}{4}\|\nabla\alpha\|_2^2,
\end{aligned} \tag{3.19}$$

$$\delta\int_\Omega h\alpha dx \le c\|h\|_2^2+\frac{\delta}{4}\|\nabla\alpha\|_2^2$$
$$\int_\Omega h\,\frac{\partial\alpha}{\partial t}dx \le \frac{1}{8}\left\|\frac{\partial\alpha}{\partial t}\right\|_2^2+2\|h\|_2^2, \tag{3.20}$$

we obtain, owing to (3.18), (3.19) and (3.20)

$$\begin{aligned}&\frac{d}{dt}E_2+\eta\left\|\frac{\partial u}{\partial t}\right\|_2^2+\frac{5}{8}\left\|\frac{\partial\alpha}{\partial t}\right\|_2^2+\|\nabla u\|_2^2+c_0\int_\Omega F(u)dx\\&+\frac{\delta}{2}\|\nabla\alpha\|_2^2\le c\int_\Omega|G(u)|^2dx+c''.\end{aligned} \tag{3.21}$$

Choosing δ such that

$$\frac{\eta}{2}\left\|\frac{\partial\alpha}{\partial t}\right\|_2^2+\delta\Big(\frac{\partial\alpha}{\partial t},\alpha\Big)+\frac{1}{2}\|\nabla\alpha\|_2^2\ge c\Big(\left\|\frac{\partial\alpha}{\partial t}\right\|_2^2+\|\nabla\alpha\|_2^2\Big), \tag{3.22}$$

and using (3.1), we have

$$\eta\int_\Omega F(u)dx+\frac{1}{2}\|\nabla\alpha\|_2^2+\eta\delta\int_\Omega G(u)\alpha dx\ge c\Big(\int_\Omega F(u)dx+\|\nabla\alpha\|_2^2\Big)-\delta\, c_2. \tag{3.23}$$

We have, taking into account (3.3), (3.22) and (3.23),

$$E_2\le c\Big(\|u\|_{H^1(\Omega)}^2+\|u\|_{k+2}^{k+2}+\left\|\frac{\partial\alpha}{\partial t}\right\|_2^2+\|\alpha\|_{H^1(\Omega)}^2\Big)+k_1\quad k_1>0. \tag{3.24}$$

Similarly

$$E_2\ge c\Big(\|u\|_{H^1(\Omega)}^2+\|u\|_{k+2}^{k+2}+\left\|\frac{\partial\alpha}{\partial t}\right\|_2^2+\|\alpha\|_{H^1(\Omega)}^2\Big)-k_1\quad k_1>0. \tag{3.25}$$

There holds owing to (3.1) and(3.21)

$$\frac{d}{dt}E_2+c\left\|\frac{\partial u}{\partial t}\right\|_2^2\le c'\,E_2+c''. \tag{3.26}$$

Finally the proof is deduced from (3.24)-(3.26). □

Let us consider a more restrictive assumption on G as follows:

$$\forall\epsilon>0,\quad |G(s)|^2\le\epsilon F(s)+c_\epsilon,\quad s\in\mathbb{R}. \tag{3.27}$$

We also have the

Theorem 3.2. *We assume that (3.2),(3.3) hold true and* $(u_0,\alpha_0,\alpha_1)\in H_0^1(\Omega)\cap L^{k+2}(\Omega)\times H_0^1(\Omega)\times L^2(\Omega)$. *If in addition we consider (3.27), then* $u\in L^\infty(\mathbb{R}^+;H_0^1(\Omega)$
$)\cap L^\infty(\mathbb{R}^+;L^{k+2}(\Omega))$, $\alpha\in L^\infty(\mathbb{R}^+;H_0^1(\Omega))$ *and* $\frac{\partial\alpha}{\partial t}\in L^\infty(\mathbb{R}^+;L^2(\Omega))$, $\forall T>0$.

Proof. From (3.21), we had

$$\begin{aligned} &\frac{d}{dt}E_2 + \eta\Big\|\frac{\partial u}{\partial t}\Big\|_2^2 + \frac{5}{8}\Big\|\frac{\partial \alpha}{\partial t}\Big\|_2^2 + \|\nabla u\|_2^2 + c_0\int_\Omega F(u)dx \\ &+ \frac{\delta}{2}\|\nabla\alpha\|_2^2 \le c\int_\Omega |G(u)|^2 dx + c'', \end{aligned} \tag{3.28}$$

with

$$\begin{aligned} E_2 =& \frac{1}{2}\|u\|_2^2 + \frac{\eta}{2}\|\nabla u\|_2^2 + \eta\int_\Omega F(u)dx + \frac{\eta}{2}\Big\|\frac{\partial\alpha}{\partial t}\Big\|_2^2 + \frac{1}{2}\|\nabla\alpha\|_2^2 \\ &+ \eta\delta\int_\Omega G(u)\alpha dx + \frac{\delta}{2}\|\alpha\|_2^2 + \eta\delta\Big(\frac{\partial\alpha}{\partial t}, \alpha\Big). \end{aligned}$$

Using (3.27), we obtain

$$\begin{aligned} &\frac{d}{dt}E_2 + \eta\Big\|\frac{\partial u}{\partial t}\Big\|_2^2 + \frac{5}{8}\Big\|\frac{\partial \alpha}{\partial t}\Big\|_2^2 + \|\nabla u\|_2^2 + (c_0 - k_\epsilon)\int_\Omega F(u)dx \\ &+ \frac{\delta}{2}\|\nabla\alpha\|_2^2 \le h_\epsilon, \end{aligned} \tag{3.29}$$

with $k_\epsilon = c.\epsilon$ and $h_\epsilon = c.c_\epsilon|\Omega| + c''$.

We also get by using Young's inequality ($|G(u)\alpha| \le \frac{1}{\delta}|G(u)|^2 + \frac{\delta}{4}|\alpha|^2$) and (3.27)

$$\eta\int_\Omega F(u)dx + \eta\delta\int_\Omega G(u)\alpha dx \ge \eta(1-\epsilon)\int_\Omega F(u)dx - \frac{\eta\delta^2}{4}\|\alpha\|_2^2 - p_\epsilon, \tag{3.30}$$

with $p_\epsilon = \eta|\Omega|c_\epsilon$.

In addition, choosing δ such that

$$\frac{\eta}{2}\Big\|\frac{\partial\alpha}{\partial t}\Big\|_2^2 + \delta\Big(\frac{\partial\alpha}{\partial t}, \alpha\Big) + \frac{1}{2}\|\nabla\alpha\|_2^2 + \frac{\delta}{2}\|\alpha\|_2^2 - \frac{\eta\delta^2}{4}\|\alpha\|_2^2 \ge c\Big(\Big\|\frac{\partial\alpha}{\partial t}\Big\|_2^2 + \|\nabla\alpha\|_2^2\Big),\ c > 0, \tag{3.31}$$

and $\epsilon < 1$ such that $c_0 - k_\epsilon > 0$, we deduce from (3.29), (3.30) and (3.31) that

$$\frac{d}{dt}E_2 + c\Big(E_2 + \Big\|\frac{\partial u}{\partial t}\Big\|_2^2\Big) \le c'',\ c > 0.$$

The proof follows from Gronwall's lemma. □

Remark 3.3. *The previous theorem proves that the system is dissipative in* $L^{k+2}(\Omega)$ $\cap\, H_0^1(\Omega) \times H_0^1(\Omega) \times L^2(\Omega)$.

We give in what follows a regularity result of the solution which is based on Moser's iterations. We will use a restriction on k, in particular k should be an even integer.

Theorem 3.4. *We assume that the assumptions of theorem 3.2 hold and that* $n = 3$. *Let* u *be a classical solution to (1.3)-(1.6) defined in* $[0, T]$ *and* k *be an even integer. We consider for all* $q > 1$, $U_q = \sup_{t\le T}\|u(t)\|_q < \infty$. *Then* $U_\infty < \infty$.

The proof is based on the following lemma.

Lemma 3.5. *Let u be a classical solution to (1.3)-(1.6) defined in $[0,T]$ and k be an even integer. Given $r>1$ such that $\tilde{U}_r = \max\{1, \|u_0\|_\infty, U_r = sup_{t\le T}\|u(t)\|_r\}$, then there exists a constant $C_3 = C_3(\left\|\frac{\partial\alpha}{\partial t}\right\|_{L^\infty(0,\infty;\,L^2(\Omega))})$ such that*

$$\tilde{U}_{2r} \le (C_3)^{\sigma(r)} r^{\sigma(r)} \tilde{U}_r,$$

with

$$\sigma(r) = \frac{5q}{r(5q-6)}, \quad q > \frac{3}{2}. \tag{3.32}$$

Proof. Multiplying (1.3) by u^{2r-1} with (3.3) and (3.4), we get

$$\begin{aligned}&\frac{1}{2r}\frac{d}{dt}\int_\Omega u^{2r}dx + \frac{2r-1}{r^2}\int_\Omega |\nabla(u^r)|^2dx + c_4\int_\Omega u^{k+2r}dx\\ &- c_5\int_\Omega u^{2r-2}dx \le c_8\int_\Omega |u|^{2r}\frac{\partial\alpha}{\partial t}dx + c_8\int_\Omega u^{2r-1}\frac{\partial\alpha}{\partial t}dx,\end{aligned} \tag{3.33}$$

and using $c_4\int_\Omega u^{k+2r}dx \ge 0$, we have

$$\begin{aligned}&\frac{1}{2r}\frac{d}{dt}\int_\Omega u^{2r}dx + \frac{2r-1}{r^2}\int_\Omega |\nabla(u^r)|^2dx\\ &- c_5\int_\Omega u^{2r-2}dx \le c_8\int_\Omega |u|^{2r}\frac{\partial\alpha}{\partial t}dx + c_8\int_\Omega u^{2r-1}\frac{\partial\alpha}{\partial t}dx.\end{aligned} \tag{3.34}$$

Let $p>1$ be such that $\frac{1}{p}+\frac{1}{q}=1$. It is clear that condition $q>\frac{3}{2}$ is equivalent to $p<3$. Taking $w=u^r$ in (3.34), we obtain after some calculations

$$\frac{1}{2r}\frac{d}{dt}\left\|w\right\|_2^2 + \frac{2r-1}{r^2}\|\nabla w\|_2^2 \le \lambda_1\|w\|_{\kappa p}^{\kappa} + \lambda_2\|w\|_{\kappa p}^{\kappa-\frac{1}{r}} \tag{3.35}$$

with $\kappa = (2r-1)/r$, $\lambda_1 = \lambda_1(\left\|\frac{\partial\alpha}{\partial t}\right\|_2)$ and $\lambda_2 = \lambda_2(\left\|\frac{\partial\alpha}{\partial t}\right\|_2)$.

Let β be such that

$$\frac{1}{\kappa p} = \beta + \frac{1-\beta}{6}. \tag{3.36}$$

Since

$$\beta = \frac{6r - p(2r-1)}{5p(2r-1)}$$

we claim that $\beta \in (0,1)$. In fact, from $p < \frac{6r}{2r-1}$ it follows that $\beta > 0$. Moreover

$$6r < 6p(2r-1),$$

i.e.

$$6r - p(2r-1) < 5p(2r-1),$$

proves that $\beta < 1$. This leads to $\beta \in (0,1)$. Using proper interpolation inequalities, there holds

$$\begin{aligned}&\frac{1}{2r}\frac{d}{dt}\left\|w\right\|_2^2 + \frac{2r-1}{r^2}\|\nabla w\|_2^2 \le\\ &\lambda_1\Big(\|w\|_1^{\beta}\|w\|_{2^\star}^{1-\beta}\Big)^2 + \lambda_2\Big(\|w\|_1^{\beta}\|w\|_{2^\star}^{1-\beta}\Big)^{\kappa-\frac{1}{r}}\end{aligned} \tag{3.37}$$

and using proper Sobolev's injections, there holds

$$
\begin{aligned}
&\frac{1}{2r}\frac{d}{dt}\|w\|_2^2+\frac{2r-1}{r^2}\|\nabla w\|_2^2\leq\\
&\lambda_1\Big[\|w\|_1^{2\beta}C^{2(1-\beta)}(4r)^{(1-\beta)}\Big]\Big[(\frac{1}{4r})^{(1-\beta)}\|\nabla w\|_2^{2(1-\beta)}\Big]\\
&+\lambda_2\Big[\|w\|_1^{\beta(\kappa-\frac{1}{r})}C^{(1-\beta)(\kappa-\frac{1}{r})}(4r)^{\frac{(\kappa-\frac{1}{r})(1-\beta)}{2}}\Big]\\
&\Big[(\frac{1}{4r})^{\frac{(\kappa-\frac{1}{r})(1-\beta)}{2}}\|\nabla w\|_2^{(1-\beta)(\kappa-\frac{1}{r})}\Big].
\end{aligned}
\tag{3.38}
$$

Using Young's inequality, we find

$$
\begin{aligned}
&\frac{1}{2r}\frac{d}{dt}\|w\|_2^2+\frac{2r-1}{r^2}\|\nabla w\|_2^2\leq\beta\Big[\lambda_1^{1/\beta}\|w\|_1^2\,C^{\frac{2(1-\beta)}{\beta}}(4r)^{\frac{(1-\beta)}{\beta}}\Big]\\
&+(1-\beta)\Big[(\frac{1}{4r})\|\nabla w\|_2^2\Big]+\frac{(\kappa-\frac{1}{r})(1-\beta)}{2}\Big[(\frac{1}{4r})\|\nabla w\|_2^2\Big]\\
&+\delta_1\Big[\lambda_2^{1/\delta_1}\|w\|_1^{\beta(\kappa-\frac{1}{r})/\delta_1}C^{\frac{(1-\beta)(\kappa-\frac{1}{r})}{\delta_1}}(4r)^{\frac{(\kappa-\frac{1}{r})(1-\beta)}{2\delta_1}}\Big]
\end{aligned}
\tag{3.39}
$$

with $\delta_1=1-\frac{(\kappa-\frac{1}{r})(1-\beta)}{2}$, $\delta_1\in(0,1)$. Hence

$$
\begin{aligned}
&\frac{1}{2r}\frac{d}{dt}\|w\|_2^2+\frac{6r-4}{4r^2}\|\nabla w\|_2^2\leq\Big[\lambda_1^{1/\beta}\|w\|_1^2\,C^{\frac{2(1-\beta)}{\beta}}(4r)^{\frac{(1-\beta)}{\beta}}\Big]\\
&+\Big[\lambda_2^{1/\delta_1}\|w\|_1^{\beta(\kappa-\frac{1}{r})/\delta_1}C^{\frac{(1-\beta)(\kappa-\frac{1}{r})}{\delta_1}}(4r)^{\frac{(\kappa-\frac{1}{r})(1-\beta)}{2\delta_1}}\Big].
\end{aligned}
\tag{3.40}
$$

Since $\frac{6r-4}{2r}>1$,

$$
\begin{aligned}
&\frac{d}{dt}\|w\|_2^2+\|\nabla w\|_2^2\leq\lambda_1^{1/\beta}\|w\|_1^2\,C^{\frac{2(1-\beta)}{\beta}}(4r)^{\frac{(1-\beta)}{\beta}+1}\\
&+\lambda_2^{1/\delta_1}\|w\|_1^{\beta(\kappa-\frac{1}{r})/\delta_1}C^{\frac{(1-\beta)(\kappa-\frac{1}{r})}{\delta_1}}(4r)^{\frac{(\kappa-\frac{1}{r})(1-\beta)}{2\delta_1}+1}.
\end{aligned}
\tag{3.41}
$$

Setting

$$
\begin{aligned}
&2r\sigma_1(r)-1=\frac{(1-\beta)}{\beta},\ 2r\sigma_2(r)-1=\frac{(\kappa-\frac{1}{r})(1-\beta)}{2\delta_1}\\
&2\rho_2(r)=\frac{\beta(\kappa-\frac{1}{r})}{\delta_1},
\end{aligned}
\tag{3.42}
$$

we have owing to Poincaré's inequality

$$
\begin{aligned}
\frac{d}{dt}\|w\|_2^2+C_0\|w\|_2^2\leq{}&\lambda_1^{1/\beta}\|w\|_1^2\,C^{4r\sigma_1(r)-2}(4r)^{2r\sigma_1(r)}\\
&+\lambda_2^{1/\delta_1}\|w\|_1^{2\rho_2(r)}C^{4r\sigma_2(r)-2}(4r)^{2r\sigma_2(r)},
\end{aligned}
\tag{3.43}
$$

with

$$\rho_2(r) = \frac{(r-1)[6r - p(2r-1)]}{5rp(2r-1) - 6(r-1)[p(2r-1) - r]}.$$

We claim that $\rho_2(r) \in (0,1)$. In fact, since $\beta > 0$, $\kappa - \frac{1}{r} > 0$ and $\delta_1 > 0$ it follows that $\rho_2(r) > 0$. In addition, from

$$\begin{aligned}
5(r-1)p(2r-1) &< 5rp(2r-1),\\
(r-1)[6r - p(2r-1)] &< 5rp(2r-1)\\
&\qquad - 6p(2r-1)(r-1) + 6r(r-1),
\end{aligned}$$

we see that

$$\rho_2(r) = \frac{(r-1)[6r - p(2r-1)]}{5rp(2r-1) - 6(r-1)[p(2r-1) - r]} < 1.$$

Hence $\rho_2(r) \in (0,1)$. Integrating (3.43) over $[0,t)$, we get

$$\begin{aligned}
\|w(t)\|_2^2 \le \|w(0)\|_2^2 &+ C_1^{1/\beta}\|w\|_1^2\, C^{4r\sigma_1(r)-2}(4r)^{2r\sigma_1(r)}\\
&+ C_2^{1/\delta_1}\|w\|_1^2\, C^{4r\sigma_2(r)-2}(4r)^{2r\sigma_2(r)},
\end{aligned} \tag{3.44}$$

with

$$C_1 = C_1\Big(\Big\|\frac{\partial\alpha}{\partial t}\Big\|_{L^\infty(0,\infty;\,L^2(\Omega))}\Big) \quad C_2 = C_2\Big(\Big\|\frac{\partial\alpha}{\partial t}\Big\|_{L^\infty(0,\infty;\,L^2(\Omega))}\Big).$$

In addition, we note that

$$\|w(0)\|_2^2 = \int_\Omega w(0)^2 dx = \int_\Omega u(0)^{2r} dx \le |\Omega|\|u(0)\|_\infty^{2r} \le |\Omega|\tilde{U}_r^{2r}. \tag{3.45}$$

It follows from (3.44) and (3.45) that

$$\begin{aligned}
\tilde{U}_{2r}^{2r} \le |\Omega|\tilde{U}_r^{2r} &+ \frac{1}{C^2}C_1^{1/\beta}\, C^{4r\sigma_1(r)}(4r)^{2r\sigma_1(r)}\tilde{U}_r^{2r}\\
&+ \frac{1}{C^2}C_2^{1/\delta_1}\, C^{4r\sigma_2(r)}(4r)^{2r\sigma_2(r)} + \tilde{U}_r^{2r}.
\end{aligned} \tag{3.46}$$

We also get from (3.42)

$$\sigma_1(r) = \frac{1}{2r\beta},\ \sigma_2(r) = \frac{1}{2r\delta_1},\ \ \delta_1 > \beta, \tag{3.47}$$

with

$$\sigma_1(r) = \frac{5p(2r-1)}{2r(6r - (2r-1)p)}.$$

This implies

$$\sigma_2(r) < \sigma_1(r) \quad \forall r > 1. \tag{3.48}$$

Setting $\sigma(r) = \frac{5q}{r\left(5q-6\right)} = \frac{5p}{r(6-p)}$, it is not difficult to see that

$$\sigma_2(r) < \sigma_1(r) \le \sigma(r). \tag{3.49}$$

We obtain from (3.46), (3.47) and (3.49) that

$$\tilde{U}_{2r} \leq C_3^{\sigma(r)}\, r^{\sigma(r)} \tilde{U}_r, \tag{3.50}$$

with

$$C_3 = C_3\Big(\Big\|\frac{\partial \alpha}{\partial t}\Big\|_{L^\infty(0,\infty;\,L^2(\Omega))}\Big).$$

This achieves the proof of the lemma.

We now turn to the proof of Theorem 3.2. By Lemma 3.5, we had

$$\tilde{U}_{2r} \leq C_3^{\sigma(r)}\, r^{\sigma(r)} \tilde{U}_r. \tag{3.51}$$

Using Moser's iterations with $r = h$, $r = 2h$, $r = 2^2h$, etc, we get

$$\tilde{U}_{2^{n+1}h} \leq (C_3)^{\kappa_1} 2^{\kappa_2} h^{\kappa_1}\, \tilde{U}_h, \tag{3.52}$$

with

$$\kappa_1 := \sigma(h) + \sigma(2h) + \sigma(2^2h) + \cdots + \sigma(2^{n-1}h) + \sigma(2^n r),$$
$$\kappa_2 := \sigma(2h) + 2\sigma(2^2h) + 3\sigma(2^3h) + \cdots + (n-1)\sigma(2^{n-1}h) + n\sigma(2^n r).$$

Since $\sigma(2^{n+1}h) = \frac{1}{2^{n+1}}\sigma(h)$, a direct computation gives

$$\kappa_1 := \sum_{k=0}^{n} \sigma(2^k h) \leq \sum_{k=0}^{+\infty} \frac{1}{2^k}\sigma(h) = 2\sigma(h),$$
$$\kappa_2 := \sum_{k=1}^{n} k\sigma(2^k h) \leq \sum_{k=1}^{+\infty} \frac{k}{2^k}\sigma(h) = 4\sigma(2h).$$

This proves that $\kappa_1, \kappa_2 < +\infty$ at infinity and achieves the proof of the theorem. □

Remark 3.6. *The case where k is an even integer is more relevant in the sense that it allows us to consider physically realistic problems. In fact, we can already take the usual cubic nonlinear term $f(u) = u^3 - u$.*

Remark 3.7. *It is also possible to treat in a similar way the case $n = 2$ by choosing β such that $\frac{1}{\kappa p} = \beta + \frac{1-\beta}{6}$.*

We finally give a uniqueness result.

Theorem 3.8. *Let (u_1, α_1) and (u_2, α_2) be two solutions to (1.3)-(1.6). We assume that (3.4), (3.5) and the assumptions of Theorem 3.1 and Theorem 3.2 are satisfied. Then problem (1.3)-(1.6) admits a unique solution.*

Proof. We have

$$\frac{\partial u}{\partial t} - \Delta u + f(u_1) - f(u_2) = g(u_1)\frac{\partial \alpha_1}{\partial t} - g(u_2)\frac{\partial \alpha_2}{\partial t}, \tag{3.53}$$

$$\eta\frac{\partial^2\alpha}{\partial t^2}+\frac{\partial\alpha}{\partial t}-\Delta\alpha=-\eta\Big(g(u_1)\frac{\partial u_1}{\partial t}-g(u_2)\frac{\partial u_2}{\partial t}\Big)-G(u_1)+G(u_2), \tag{3.54}$$

with $u=u_1-u_2$ and $\alpha=\alpha_1-\alpha_2$. We also write

$$\frac{\partial u}{\partial t}-\Delta u+f(u_1)-f(u_2)=\Big(g(u_1)-g(u_2)\Big)\frac{\partial\alpha_1}{\partial t}+g(u_2)\frac{\partial\alpha}{\partial t}, \tag{3.55}$$

$$\eta\frac{\partial^2\alpha}{\partial t^2}+\frac{\partial\alpha}{\partial t}-\Delta\alpha=-\eta\Big(g(u_1)-g(u_2)\Big)\frac{\partial u_1}{\partial t}-\eta g(u_2)\frac{\partial u}{\partial t}-G(u_1)+G(u_2). \tag{3.56}$$

Multiplying (3.55) by $\frac{\partial u}{\partial t}$ and integrating over Ω, we obtain

$$\begin{aligned}\left\|\frac{\partial u}{\partial t}\right\|_2^2+\frac{1}{2}\frac{d}{dt}\|\nabla u\|_2^2+\Big(f(u_1)-f(u_2),\frac{\partial u}{\partial t}\Big)=&\int_\Omega\Big(g(u_1)-g(u_2)\Big)\frac{\partial\alpha_1}{\partial t}\frac{\partial u}{\partial t}\,dx\\&+\int_\Omega g(u_2)\frac{\partial\alpha}{\partial t}\frac{\partial u}{\partial t}dx.\end{aligned} \tag{3.57}$$

Similarly, multiplying (3.56) by $\frac{\partial\alpha}{\partial t}$, we have

$$\begin{aligned}&\frac{\eta}{2}\frac{d}{dt}\left\|\frac{\partial\alpha}{\partial t}\right\|_2^2+\left\|\frac{\partial\alpha}{\partial t}\right\|_2^2+\frac{1}{2}\frac{d}{dt}\|\nabla\alpha\|_2^2=-\eta\int_\Omega(g(u_1)-g(u_2))\frac{\partial u_1}{\partial t}\frac{\partial\alpha}{\partial t}dx\\&-\eta\int_\Omega g(u_2)\frac{\partial u}{\partial t}\frac{\partial\alpha}{\partial t}dx-\int_\Omega(G(u_1)-G(u_2))\frac{\partial\alpha}{\partial t}dx,\end{aligned} \tag{3.58}$$

and, adding (3.57) multiplied by η and (3.58), we obtain

$$\begin{aligned}&\frac{dE}{dt}+\eta\left\|\frac{\partial u}{\partial t}\right\|_2^2+\left\|\frac{\partial\alpha}{\partial t}\right\|_2^2+\leq-\eta\Big(f(u_1)-f(u_2),\frac{\partial u}{\partial t}\Big)\\&-\eta\int_\Omega(g(u_1)-g(u_2))\frac{\partial u_1}{\partial t}\frac{\partial\alpha}{\partial t}dx-\int_\Omega(G(u_1)-G(u_2))\frac{\partial\alpha}{\partial t}dx\\&+\eta\int_\Omega(g(u_1)-g(u_2))\frac{\partial\alpha_1}{\partial t}\frac{\partial u}{\partial t}\,dx\end{aligned} \tag{3.59}$$

with

$$E=\frac{\eta}{2}\|\nabla u\|_2^2+\frac{\eta}{2}\left\|\frac{\partial\alpha}{\partial t}\right\|_2^2+\frac{1}{2}\|\nabla\alpha\|_2^2.$$

Considering (3.1)-(3.3), we see that

$$\begin{aligned}\eta\int_\Omega|f(u_1)-f(u_2)|\left|\frac{\partial u}{\partial t}\right|dx\leq&\ c_{10}\eta\int_\Omega(|u_2|^k+1)\,|u|\left|\frac{\partial u}{\partial t}\right|dx\\&\leq c(\big\|u_2\big\|_{H^1(\Omega)}^{2k}+1)\|\nabla u\|_2^2+\frac{\eta}{2}\left\|\frac{\partial u}{\partial t}\right\|_2^2,\end{aligned} \tag{3.60}$$

$$\begin{aligned}\eta\int_\Omega(g(u_1)-g(u_2))\frac{\partial\alpha_1}{\partial t}\frac{\partial u}{\partial t}\,dx\leq&\ \eta\,c_9\int_\Omega|u|\left|\frac{\partial\alpha_1}{\partial t}\right|\left|\frac{\partial u}{\partial t}\right|dx\\&\leq c\Big(\left\|\nabla\frac{\partial\alpha_1}{\partial t}\right\|_2^2\|\nabla u\|_2^2\Big)+\frac{\eta}{2}\left\|\frac{\partial u}{\partial t}\right\|_2^2,\end{aligned} \tag{3.61}$$

$$\int_{\Omega}|G(u_1)-G(u_2)|\left|\frac{\partial\alpha}{\partial t}\right|dx \le c(\left\|u_2\right\|^2_{H^1(\Omega)}+1)\|\nabla u\|_2^2+\frac{1}{2}\left\|\frac{\partial\alpha}{\partial t}\right\|_2^2, \tag{3.62}$$

and

$$\begin{aligned}\eta\int_{\Omega}|g(u_1)-g(u_2)|\left|\frac{\partial u_1}{\partial t}\right|\left|\frac{\partial\alpha}{\partial t}\right|dx &\le \eta c_9\|u\|_\infty\int_{\Omega}\left|\frac{\partial u_1}{\partial t}\right|\left|\frac{\partial\alpha}{\partial t}\right|dx\\ &\le c\left\|\frac{\partial u_1}{\partial t}\right\|_2^2\|\nabla u\|_2^2+\frac{1}{2}\left\|\frac{\partial\alpha}{\partial t}\right\|_2^2.\end{aligned} \tag{3.63}$$

From (3.59)-(3.63), we deduce that

$$\frac{dE}{dt}\le c\|\nabla u\|_2^2\Big(\left\|u_2\right\|^{2k}_{H^1(\Omega)}+2+\left\|\nabla\frac{\partial\alpha_1}{\partial t}\right\|_2^2+\left\|u_2\right\|^2_{H^1(\Omega)}+\left\|\frac{\partial u_1}{\partial t}\right\|_2^2\Big). \tag{3.64}$$

The proof follows from Gronwall's lemma. □

4. Spatial behavior of the solutions

To study the spatial behavior of the solutions in a semi-infinite cylinder we need to add some assumptions. We first assume that such solutions exist. We then consider the boundary conditions

$$u=\alpha=0 \quad \text{on} \quad (0,+\infty)\times\partial D\times(0,T), \tag{4.1}$$

$$u(0,x_2,x_3,t)=h_1(x_2,x_3,t), \tag{4.2}$$

$$\alpha(0,x_2,x_3,t)=h_2(x_2,x_3,t) \quad \text{on} \quad \{0\}\times D\times(0,T) \tag{4.3}$$

and the initial conditions

$$u|_{t=0}=\alpha|_{t=0}=\left.\frac{\partial\alpha}{\partial t}\right|_{t=0}=0 \quad \text{on} \quad R. \tag{4.4}$$

Here D denotes a two dimensional bounded domain and R a semi-infinite cylinder $(0,+\infty)\times D$. We will sometimes use some assumptions on the functions F and G; these will be specified later on. We further assume that $h=0$.

We consider the function

$$F_\omega(z,t)=\int_0^t\int_{D(z)}e^{-(w\,s)}[\alpha_s\alpha_{,1}+u_{,1}(\gamma u+\eta u_s)]da\,ds \tag{4.5}$$

where $D(z)=\{x\in R, \quad x_1=z\}$, $u_{,1}=\frac{\partial u}{\partial x_1}$, $u_s=\frac{\partial u}{\partial s}$ and w is a positive constant. By a differentiation of F_ω, we get

$$\begin{aligned}\frac{\partial F_\omega(z,t)}{\partial z}=&\int_0^t\int_{D(z)}e^{-(w\,s)}\Big(\nabla\alpha\nabla\alpha_s+\eta\alpha_s\alpha_{ss}+|\alpha_s|^2+\eta g(u)u_s\alpha_s\\ &+G(u)\alpha_s+\eta\nabla u\nabla u_s+\eta|u_s|^2+\eta f(u)u_s\\ &-\eta g(u)u_s\alpha_s+\gamma|\nabla u|^2+\gamma u\,u_s+\gamma f(u)u-\gamma g(u)\,u\alpha_s\Big)da\,ds\end{aligned} \tag{4.6}$$

which yields after simplification

$$\frac{\partial F_\omega(z,t)}{\partial z} = \int_0^t \int_{D(z)} e^{-(w\,s)}\Big(|\alpha_s|^2 + \eta|u_s|^2 + \gamma|\nabla u|^2\Big)da\,ds$$
$$+ \int_0^t \int_{D(z)} e^{-(w\,s)}\Big(\nabla\alpha\nabla\alpha_s + \eta\alpha_s\alpha_{ss} + \eta\nabla u\nabla u_s + \eta f(u)u_s + \gamma u\,u_s\Big)da\,ds \tag{4.7}$$
$$+ \int_0^t \int_{D(z)} e^{-(w\,s)}\Big((G(u) - \gamma g(u)u)\alpha_s + \gamma f(u)\,u\Big)da\,ds.$$

We also have

$$\frac{d}{dt}\int_{D(z)} e^{-(\omega\,s)}\Big(|\nabla\alpha|^2 + \eta|\alpha_s|^2 + \eta|\nabla u|^2 + 2\eta F(u) + \gamma|u|^2\Big)da =$$
$$-\,\omega\int_{D(z)} e^{-(\omega\,s)}\Big(|\nabla\alpha|^2 + \eta|\alpha_s|^2 + \eta|\nabla u|^2 + 2\eta F(u) + \gamma|u|^2\Big)da \tag{4.8}$$
$$+ \int_{D(z)} e^{-(w\,s)}\Big(\nabla\alpha\nabla\alpha_s + \eta\alpha_s\alpha_{ss} + \eta\nabla u\nabla u_s + \eta f(u)u_s + \gamma u\,u_s\Big)da.$$

In other words

$$\int_{D(z)} e^{-(w\,s)}\Big(\nabla\alpha\nabla\alpha_s + \eta\alpha_s\alpha_{ss} + \eta\nabla u\nabla u_s + \eta f(u)u_s + \gamma u\,u_s\Big)da =$$
$$\frac{1}{2}\frac{d}{dt}\int_{D(z)} e^{-(\omega\,s)}\Big(|\nabla\alpha|^2 + \eta|\alpha_s|^2 + \eta|\nabla u|^2 + 2\eta F(u) + \gamma|u|^2\Big)da \tag{4.9}$$
$$+ \frac{\omega}{2}\int_{D(z)} e^{-(\omega\,s)}\Big(|\nabla\alpha|^2 + \eta|\alpha_s|^2 + \eta|\nabla u|^2 + 2\eta F(u) + \gamma|u|^2\Big)da.$$

We deduce from (4.7) and (4.9) that

$$\frac{\partial F_\omega(z,t)}{\partial z} = \int_0^t \int_{D(z)} e^{-(\omega\,s)}\Big((|\alpha_s|^2 + \eta|u_s|^2 + \gamma|\nabla u|^2\Big)da\,ds$$
$$+ e^{-(\omega\,t)}\int_{D(z)} (|\nabla\alpha|^2 + \eta|\alpha_s|^2 + \eta|\nabla u|^2 + 2\eta F(u) + \gamma|u|^2)da$$
$$+ \int_0^t \int_{D(z)} e^{-(w\,s)}\Big[(G(u) - \gamma g(u)u)\alpha_s + \gamma f(u)\,u \tag{4.10}$$
$$+ \frac{\omega}{2}(|\nabla\alpha|^2 + \eta|\alpha_s|^2 + \eta|\nabla u|^2 + 2\eta F(u) + \gamma|u|^2)\Big]da\,ds.$$

We assume that, for γ large enough, $2\eta F(s) + \gamma|s|^2 \geq K_1(|s|^2 + |s|^{k+2})$, k integer, $K_1 > 0$. Then, there exists a constant $K_2 > 0$ such that

$$|\nabla\alpha|^2 + \eta|\alpha_s|^2 + \eta|\nabla u|^2 + 2\eta F(u) + \gamma|u|^2 \geq K_2(|\nabla\alpha|^2 + |\alpha_s|^2 + |\nabla u|^2 + |u|^2 + |u|^{k+2}). \tag{4.11}$$

We further assume that $(G(s)-\gamma g(s)s)^2 \leq K_3(|s|^2+|s|^{k+2})$, $K_3 > 0$, and that there exist positive constants κ_1 and κ_2 such that $f(s)s + \kappa_1|s|^2 \geq \kappa_2|s|^2$. Then, for ω large enough (here, ω depends on γ), there holds

$$\begin{aligned}&(G(u) - \gamma g(u)u)\alpha_s + \gamma f(u)\, u + \frac{\omega}{2}(|\nabla\alpha|^2 + \eta|\alpha_s|^2 \\ &+ \eta|\nabla u|^2 + 2\eta F(u) + \gamma|u|^2) \geq K_4(|\nabla\alpha|^2 + |\alpha_s|^2 + |\nabla u|^2 + |u|^2),\end{aligned} \tag{4.12}$$

where K_4 is a positive constant. Note that g having at most a linear growth, $|g(s)| \leq c|s|$, $G(0) = 0$, and $F(s) = c's^4 + c''s^2$ (having in mind the usual cubic nonlinear term $f(s) = s^3 - s$), $c' > 0$, satisfy the above assumptions. We finally deduce from (4.10)-(4.12) the existence of $K_5 > 0$ such that

$$\frac{\partial F_\omega(z,t)}{\partial z} \geq K_5 \int_0^t \int_{D(z)} e^{-(\omega\, s)}\Big[|\nabla\alpha|^2 + |\alpha_s|^2 + |\nabla u|^2 + |u|^2 + |u_s|^2\Big] da\, ds. \tag{4.13}$$

We now give a spatial derivative estimate on $|F_\omega|$. Using Cauchy-Schwarz's inequality in (4.5), we obtain

$$\begin{aligned}|F_\omega| \leq &\Big(\int_0^t \int_{D(z)} e^{-(w\, s)}\alpha_s^2\, da\, ds\Big)^{\frac{1}{2}}\Big(\int_0^t \int_{D(z)} e^{-(w\, s)}\alpha_{,1}^2\, da\, ds\Big)^{\frac{1}{2}} \\ &+ \Big(\int_0^t \int_{D(z)} e^{-(w\, s)}u_{,1}^2\, da\, ds\Big)^{\frac{1}{2}}\Big(\int_0^t \int_{D(z)} \gamma^2 e^{-(w\, s)}u^2\, da\, ds\Big)^{\frac{1}{2}} \\ &+ \Big(\int_0^t \int_{D(z)} e^{-(w\, s)}u_{,1}^2\, da\, ds\Big)^{\frac{1}{2}}\Big(\int_0^t \int_{D(z)} \eta^2 e^{-(w\, s)}u_s^2\, da\, ds\Big)^{\frac{1}{2}}.\end{aligned} \tag{4.14}$$

Hence

$$|F_\omega| \leq K_6 \int_0^t \int_{D(z)} e^{-(\omega\, s)}\Big[|\nabla\alpha|^2 + |\alpha_s|^2 + |\nabla u|^2 + |u|^2 + |u_s|^2\Big] da\, ds. \tag{4.15}$$

Choosing $K^\star = \frac{K_6}{K_5}$, there holds

$$|F_\omega| \leq K^\star \frac{\partial F_\omega}{\partial z}. \tag{4.16}$$

Due to (4.16), we arrive at a Phragmén-Lindelöf alternative (see [10], [33]) namely, either there exists $z_0 \geq 0$ such that $F(z_0, t) > 0$ or $F(z_0, t) \leq 0$ for all $z \geq 0$. In the first case our solution satisfies

$$F_\omega(z,t) \geq e^{\left(K^{\star^{-1}}(z-z_0)\right)} F_\omega(z_0, t), \quad z \geq z_0, \tag{4.17}$$

and, in the latter one $F(z_0, t) \leq 0$ for all $z \geq 0$, in which case our solution satisfies

$$-F_\omega(z,t) \leq -e^{\left(-K^{\star^{-1}} z\right)} F_\omega(0,t), \quad z \geq 0. \tag{4.18}$$

Inequality (4.17) shows that $F_\omega(z,t)$ tends exponentially fast to infinity.

On the contrary inequality (4.18) shows that $F_\omega(z,t)$ tends to 0 and

$$G_\omega(z,t) \leq e^{\left(-K^{\star^{-1}} z\right)} G_\omega(0,t), \quad z \geq 0, \tag{4.19}$$

where

$$G_\omega(z,t) = \int_0^t \int_{R(z)} e^{-(\omega\, s)}\Big((|\alpha_s|^2 + \eta|u_s|^2 + \gamma|\nabla u|^2\Big)da\, ds$$
$$+ e^{-(\omega\, s)} \int_{R(z)} (|\nabla\alpha|^2 + \eta|\alpha_s|^2 + \eta|\nabla u|^2 + 2\eta F(u) + \gamma|u|^2)da$$
$$+ \int_0^t \int_{R(z)} e^{-(w\, s)}\Big[(G(u) - \gamma g(u)u)\alpha_s + \gamma f(u)\, u$$
$$+ \frac{\omega}{2}(|\nabla\alpha|^2 + \eta|\alpha_s|^2 + \eta|\nabla u|^2 + 2\eta F(u) + \gamma|u|^2)\Big]da\, ds \tag{4.20}$$

where $R(z) = \{x \in R, \quad z < x_1\}$. Setting

$$\mathcal{E}_\omega(z,t) = \int_0^t \int_{R(z)} \Big((|\alpha_s|^2 + \eta|u_s|^2 + \gamma|\nabla u|^2\Big)da\, ds$$
$$+ \int_{R(z)} (|\nabla\alpha|^2 + \eta|\alpha_s|^2 + \eta|\nabla u|^2 + 2\eta F(u) + \gamma|u|^2)da$$
$$+ \int_0^t \int_{R(z)} \Big[(G(u) - \gamma g(u)u)\alpha_s + \gamma f(u)\, u$$
$$+ \frac{\omega}{2}(|\nabla\alpha|^2 + \eta|\alpha_s|^2 + \eta|\nabla u|^2 + 2\eta F(u) + \gamma|u|^2)\Big]da\, ds \tag{4.21}$$

we get

$$\mathcal{E}_\omega(z,t) \le e^{\left(\omega t - K^{\star^{-1}} z\right)} G_\omega(0,t), \quad z \ge 0. \tag{4.22}$$

We give in what follows the main result of this section

Theorem 4.1. *Let (u,α) be a solution to problem (1.3)-(1.6) with the boundary conditions (4.1)-(4.4). Then, either this solution satisfies (4.17) or it satisfies (4.22).*

Remark 4.2. *Estimates (4.17) and (4.22) are known respectively as growth and decay estimates.*

Remark 4.3. *It is possible due to (4.17) to specify the rate of growth of our solutions to infinity. In fact, if (4.17) is satisfied, then*

$$\int_0^t \int_{R(0,z)} e^{-(\omega\, s)}\Big((|\alpha_s|^2 + \eta|u_s|^2 + \gamma|\nabla u|^2\Big)da\, ds$$
$$+ e^{-(\omega\, s)} \int_{R(0,z)} (|\nabla\alpha|^2 + \eta|\alpha_s|^2 + \eta|\nabla u|^2 + 2\eta F(u) + \gamma|u|^2)da$$
$$+ \int_0^t \int_{R(0,z)} e^{-(w\, s)}\Big[(G(u) - \gamma g(u)u)\alpha_s + \gamma f(u)\, u$$
$$+ \frac{\omega}{2}(|\nabla\alpha|^2 + \eta|\alpha_s|^2 + \eta|\nabla u|^2 + 2\eta F(u) + \gamma|u|^2)\Big]da\, ds \tag{4.23}$$

where $R(0,z) = \{x \in R, \quad 0 < x_1 < z\}$, tends exponentially fast to infinity.

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