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Research article

The Cahn–Hilliard equation and some of its variants

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Abstract: Our aim in this article is to review and discuss the Cahn–Hilliard equation, as well as some of its variants. Such variants have applications in, e.g., biology and image inpainting.

Keywords: Cahn–Hilliard equation; Cahn–Hilliard–Oono equation; proliferation term; fidelity term; well-posedness; logarithmic nonlinear terms
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1. Introduction

The Cahn–Hilliard system

$$\begin{cases} \frac{\partial u}{\partial t} = \kappa \Delta \mu, \ \kappa > 0, \\ \mu = -\alpha \Delta u + f(u), \ \alpha > 0, \end{cases} \tag{1.1}$$

and, equivalently, the Cahn–Hilliard equation

$$\frac{\partial u}{\partial t} + \alpha \kappa \Delta^2 u - \kappa \Delta f(u) = 0 \tag{1.2}$$

play an essential role in materials science as they describe important qualitative features of two-phase systems related with phase separation processes, assuming isotropy and a constant temperature. This can be observed, e.g., when a binary alloy is cooled down sufficiently. One then observes a partial nucleation (i.e., the apparition of nuclides in the material) or a total nucleation, known as spinodal decomposition: the material quickly becomes inhomogeneous, resulting in a very finely dispersed microstructure. In a second stage, which is called coarsening, occurs at a slower time scale and is less understood, these microstructures coarsen. Such phenomena play an essential role in the mechanical properties of the material, e.g., strength, hardness, fracture toughness and ductility. We refer the reader to, e.g., [58, 60, 255, 259, 276, 277, 311, 313] for more details.

Here, u is the order parameter (one usually considers a rescaled density of atoms or concentration of one of the material's components which takes values between -1 and 1, -1 and 1 corresponding to the pure states; the density of the second component is $1 - u$, meaning that the total density is a conserved quantity) and μ is the chemical potential. Furthermore, f is the derivative of a double-well potential F whose wells correspond to the phases of the material. A thermodynamically relevant potential F is the following logarithmic function which follows from a mean-field model:

$$F(s) = \frac{\theta_c}{2}(1 - s^2) + \frac{\theta}{2}[(1 - s)\ln(\frac{1 - s}{2}) + (1 + s)\ln(\frac{1 + s}{2})], \ s \in (-1, 1), \ 0 < \theta < \theta_c, \tag{1.3}$$

i.e.,

$$f(s) = -\theta_c s + \frac{\theta}{2} \ln \frac{1 + s}{1 - s}, \tag{1.4}$$

although such a function is very often approximated by regular ones, typically, $F(s) = \frac{1}{4}(s^2 - 1)^2$, i.e., $f(s) = s^3 - s$; more generally, one can take $F(s) = \frac{1}{4}(s^2 - \beta^2)^2, \beta \in \mathbb{R}$. The logarithmic terms in (1.3) correspond to the entropy of mixing and θ and θ_c are proportional to the absolute temperature (assumed constant during the process) and a critical temperature, respectively; the condition $\theta < \theta_c$ ensures that F has indeed a double-well form and that phase separation can occur. Also note that the polynomial approximation is reasonable when the quench is shallow, i.e., when the absolute temperature is close to the critical one. Finally, κ is the mobility and α is related to the surface tension at the interface.

We assume in this article that the mobility is a strictly positive constant. Actually, κ is often expected to depend on the order parameter and to degenerate at the singular points of f in the case of a logarithmic nonlinear term (see [59, 138, 139, 178, 370]; see also [373] for a discussion in the context

of immiscible binary fluids). Note however that this essentially restricts the diffusion process to the interfacial region and is observed, typically, when the movements of atoms are confined to this region (see [330]). In that case, the first equation of (1.1) reads

$$\frac{\partial u}{\partial t} = \mathrm{div}(\kappa(u)\nabla\mu),$$

where, typically, $\kappa(s) = 1 - s^2$. In particular, the existence of solutions to the Cahn–Hilliard equation with degenerate mobilities and logarithmic nonlinearities is proved in [138]. The asymptotic behavior, and, more precisely, the existence of attractors, of the Cahn–Hilliard equation with nonconstant mobilities is studied in [342, 343].

From a phenomenological point of view, the Cahn–Hilliard system can be derived as follows. One considers the following (total) free energy, called Ginzburg–Landau free energy:

$$\Psi_\Omega(u, \nabla u) = \int_\Omega (\frac{\alpha}{2}|\nabla u|^2 + F(u))\, dx, \tag{1.5}$$

where $|\cdot|$ denotes the usual Euclidean norm and $\Omega \subset \mathbb{R}^n$, $n = 1$, 2 or 3, is the domain occupied by the material.

We can note that the gradient term in the Ginzburg–Landau free energy accounts for the fact that the interactions between the material's components are assumed to be short-ranged. Actually, this term is obtained by approximation of a nonlocal term which also accounts for long-ranged interactions (see [60]). The Cahn–Hilliard equation, with a nonlocal term, was derived rigorously by G. Giacomin and J.L. Lebowitz in [188, 189], based on stochastic arguments, by considering a lattice gas with long range Kac potentials (i.e., the interaction energy between two particles at x and y ($x, y \in \mathbb{Z}^n$) is given by $\gamma^n\mathcal{K}(\gamma|x - y|)$, $\gamma > 0$ being sent to 0 and $\mathcal{K}$ being a smooth function). In that case, the (total) free energy reads

$$\Psi_\Omega(u) = \int_{\mathbb{T}^n} [f(u(x)) + u(x)\int_{\mathbb{T}^n} \mathcal{K}(|x - y|)(1 - u(y))\, dy]\, dx, \tag{1.8}$$

where $\mathbb{T}^n$ is the n-dimensional torus. Furthermore, rewriting the total free energy in the form

$$\Psi_\Omega(u) = \int_{\mathbb{T}^n} [f(u(x)) + k_1(x)u(x)(1 - u(x)) + \frac{1}{2}\int_{\mathbb{T}^n} \mathcal{K}(|x - y|)|u(x) - u(y)|^2\, dy]\, dx,$$

where $k_1(x) = \int_{\mathbb{T}^n} \mathcal{K}(|x-y|)\, dy$, one can, by expanding the last term and keeping only some terms in the expansion, recover the Ginzburg–Landau free energy (this is reasonable when the scale on which the free energy varies is large compared with γ^{-1}; the macroscopic evolution is observed here on the spatial scale γ^{-1} and time scale γ^{-2}). Such models were studied, e.g., in [2,23,162,168] (see also [72,216–218] for the numerical analysis and simulations).

One then has the mass balance

$$\frac{\partial u}{\partial t} = -\mathrm{div} h, \tag{1.6}$$

where h is the mass flux which is related to the chemical potential μ by the following (postulated) constitutive equation which resembles the Fick's law:

$$h = -\kappa \nabla \mu. \tag{1.7}$$

The usual definition of the chemical potential is that it is the derivative of the free energy with respect to the order parameter. Here, such a definition is incompatible with the presence of ∇u in the free energy. Instead, μ is defined as a variational derivative of the free energy with respect to u, which yields (assuming proper boundary conditions)

$$\mu = -\alpha \Delta u + f(u), \tag{1.8}$$

whence the Cahn–Hilliard system.

The Cahn–Hilliard system/equation is now well understood, at least from a mathematical point of view. In particular, one has a rather complete picture as far as the existence, the uniqueness and the regularity of solutions and the asymptotic behavior of the associated dynamical system are concerned. We refer the reader to (among a huge literature), e.g., [4, 31, 53, 88, 91, 116, 123, 132, 134, 138, 140, 143, 163, 164, 172, 245, 258, 264, 274, 301, 302, 305, 307–311, 313, 326, 329, 335, 349, 368, 376]. As far as the asymptotic behavior of the system is concerned, one has, in particular, the existence of finite-dimensional attractors. Such sets give information on the global/ all possible dynamics of the system. Furthermore, the finite dimensionality means, very roughly speaking, that, even though the initial phase space is infinite-dimensional, the limit dynamics can be described by a finite number of parameters. We refer the interested reader to, e.g., [15, 92, 125, 303, 349] for more details and discussions on this. One also has the convergence of single trajectories to steady states.

Now, it is interesting to note that the Cahn–Hilliard equation and some of its variants are also relevant in other phenomena than phase separation in binary alloys. We can mention, for instance, dealloying (this can be observed in corrosion processes, see [145]), population dynamics (see [95]), tumor growth (see [14, 244]), bacterial films (see [253]), thin films (see [315, 350]), chemistry (see [354]), image processing (see [24, 25, 66, 73, 124]) and even the rings of Saturn (see [353]) and the clustering of mussels (see [271]).

In particular, several such phenomena can be modeled by the following generalized Cahn–Hilliard equation:

$$\frac{\partial u}{\partial t} + \alpha\kappa\Delta^2 u - \kappa\Delta f(u) + g(x,u) = 0,\ \alpha,\ \kappa > 0 \tag{1.9}$$

(here, α and κ do not necessarily have the same physical meaning as in the original Cahn–Hilliard equation). The above general equation contains, in particular, the following models:

(i) Mixed Allen–Cahn/Cahn–Hilliard system. In that case, we consider the following system of equations:

$$\begin{cases} \frac{\partial u}{\partial t} = \varepsilon^2 D\Delta\mu - \mu,\ D,\ \varepsilon > 0, \\ \mu = -\Delta u + \frac{f(u)}{\varepsilon^2}, \end{cases}$$

which can be rewritten, equivalently, as

$$\frac{\partial u}{\partial t} + \varepsilon^2 D\Delta^2 u - \Delta(Df(u) + u) + \frac{f(u)}{\varepsilon^2} = 0$$

and is indeed of the form above. In particular, without the term $\varepsilon^2 D\Delta\mu$ in the first equation, we have the Allen–Cahn equation and, without the term $-\mu$, we have the Cahn–Hilliard equation. These equations were proposed in order to account for microscopic mechanisms such as surface diffusion and adsorption/desorption (see [238, 240, 241, 281]) and were studied in [232–235, 239].

(ii) Cahn–Hilliard–Oono equation (see [287, 314, 355]). In that case,

$$g(x, s) = g(s) = \beta s,\ \beta > 0.$$

This function was proposed in [314] in order to account for long-ranged (i.e., nonlocal) interactions in phase separation, but also to simplify numerical simulations, due to the fact that we do not have to account for the conservation of mass (see below), although it seems that this equation has never been considered in simulations. A variant of this model, proposed in [90] to model microphase separation of diblock copolymers, consists in taking

$$g(x, s) = g(s) = \beta(s - \frac{1}{\mathrm{Vol}(\Omega)} \int_\Omega u_0(x)\, dx),\ \beta > 0,$$

where u_0 is the initial condition. In that case, we have the conservation of mass and efficient simulations, based on multigrid solvers, were performed in [13]. This variant of the Cahn–Hilliard–Oono equation can also be coupled with the incompressible Navier–Stokes equations to model a chemically reacting binary fluid (see [228, 229]; see also [43] for the mathematical analysis).

(iii) Proliferation term. In that case,

$$g(x, s) = g(s) = \lambda s(s - 1),\ \lambda > 0.$$

This function was proposed in [244] in view of biological applications and, more precisely, to model wound healing and tumor growth (in one space dimension) and the clustering of brain tumor cells (in two space dimensions); see also [354] for other quadratic functions with chemical applications and [14] for other polynomials with biological applications.

(iv) Fidelity term. In that case,

$$g(x, s) = \lambda_0 \chi_{\Omega\setminus D}(x)(s - h(x)),\ \lambda_0 > 0,\ D \subset \Omega,\ h \in L^2(\Omega),$$

where χ denotes the indicator function, and we consider the following equation:

$$\frac{\partial u}{\partial t} + \varepsilon\Delta^2 u - \frac{1}{\varepsilon}\Delta f(u) + g(x, u) = 0,\ \varepsilon > 0.$$

Written in this way, ε corresponds to the interface thickness. This function g was proposed in [24, 25] in view of applications to image inpainting. Here, h is a given (damaged) image and D is the inpainting (i.e., damaged) region. Furthermore, the fidelity term $g(x, u)$ is added in order to keep the solution close to the image outside the inpainting region. The idea in this model is to solve the equation up to steady state to obtain an inpainted (i.e., restored) version $u(x)$ of $h(x)$.

The generalized equation (1.9) was studied in [288, 292] (see also [148]) under very general assumptions on the additional term g, when endowed with Dirichlet boundary conditions. In that case, one essentially recovers the results (well-posedness, regularity and existence of finite-dimensional attractors) known for the original Cahn–Hilliard equation. The case of Neumann boundary conditions

is much more involved, due to the fact that one no longer has the conservation of mass, i.e., of the spatial average of the order parameter, when compared with the original Cahn–Hilliard equation with Neumann boundary conditions (see [73, 74, 85, 148, 149]).

Another variant of the Cahn–Hilliard equation, which we will not address in this review, is concerned with higher-order Cahn–Hilliard models. More precisely, G. Caginalp and E. Esenturk recently proposed in [57] (see also [70]) higher-order phase-field models in order to account for anisotropic interfaces (see also [254, 348, 363] for other approaches which, however, do not provide an explicit way to compute the anisotropy). More precisely, these authors proposed the following modified free energy, in which we omit the temperature:

$$\Psi_{\mathrm{HOGL}} = \int_\Omega (\frac{1}{2}\sum_{i=1}^{M}\sum_{|k|=i} a_k|\mathcal{D}^k u|^2 + F(u))\,dx,\ M \in \mathbb{N}, \tag{1.10}$$

where, for $k = (k_1, k_2, k_3) \in (\mathbb{N} \cup \{0\})^3$,

$$|k| = k_1 + k_2 + k_3$$

and, for $k \neq (0, 0, 0)$,

$$\mathcal{D}^k = \frac{\partial^{|k|}}{\partial x_1^{k_1}\partial x_2^{k_2}\partial x_3^{k_3}}$$

(we agree that $\mathcal{D}^{(0,0,0)}v = v$). The corresponding higher-order Cahn–Hilliard equation then reads

$$\frac{\partial u}{\partial t} - \Delta\sum_{i=1}^{M}(-1)^i\sum_{|k|=i} a_k\mathcal{D}^{2k}u - \Delta f(u) = 0. \tag{1.11}$$

For $M = 1$ (anisotropic Cahn–Hilliard equation), we have an equation of the form

$$\frac{\partial u}{\partial t} + \Delta\sum_{i=1}^{3} a_i\frac{\partial^2 u}{\partial x_i^2} - \Delta f(u) = 0$$

and, for $M = 2$ (sixth-order anisotropic Cahn–Hilliard equation), we have an equation of the form

$$\frac{\partial u}{\partial t} - \Delta\sum_{i,j=1}^{3} a_{ij}\frac{\partial^4 u}{\partial x_i^2\partial x_j^2} + \Delta\sum_{i=1}^{3} b_i\frac{\partial^2 u}{\partial x_i^2} - \Delta f(u) = 0.$$

We studied in [81] the corresponding higher-order isotropic model, namely,

$$\frac{\partial u}{\partial t} - \Delta P(-\Delta)u - \Delta f(u) = 0, \tag{1.12}$$

where

$$P(s) = \sum_{i=1}^{M} a_i s^i,\ a_k > 0,\ M \geq 1,\ s \in \mathbb{R},$$

and, in [82], the anisotropic higher-order model (1.11) (there, numerical simulations were also performed to illustrate the effects of the higher-order terms and of the anisotropy). Furthermore, these models contain sixth-order Cahn–Hilliard models. We can note that there is currently a strong interest in the study of sixth-order Cahn–Hilliard equations. Such equations arise in situations such as strong anisotropy effects being taken into account in phase separation processes (see [352]), atomistic models of crystal growth (see [24, 25, 144, 173]), the description of growing crystalline surfaces with small slopes which undergo faceting (see [341]), oil-water-surfactant mixtures (see [200, 201]) and mixtures of polymer molecules (see [115]). We refer the reader to [68, 208, 213–215, 227, 256, 257, 278, 279, 287, 289–291, 293, 316, 317, 322, 323, 356, 357, 367] for the mathematical and numerical analysis of such models.

We can also note that the variant (1.9) can be relevant in the context of higher-order models (we can mention, for instance, anisotropic effects in tumor growth). We refer the reader to [83] for the analysis and numerical simulations of such models.

Our aim in this article is to review and discuss some of the aforementioned Cahn–Hilliard models (1.9). More precisely, we will focus on the last three examples mentioned above. We also discuss the original Cahn–Hilliard equation, with an emphasis on the thermodynamically relevant logarithmic nonlinear terms.

2. The Cahn–Hilliard and Cahn–Hilliard–Oono equations

The Cahn–Hilliard system, in a bounded and regular domain Ω of $\mathbb{R}^n$, $n = 1$, 2 or 3, usually is endowed with Neumann boundary conditions, namely,

$$\frac{\partial \mu}{\partial \nu} = 0, \text{ on } \Gamma, \tag{2.1}$$

meaning that there is no mass flux at the boundary (note that $h.\nu = -\kappa \frac{\partial \mu}{\partial \nu}$), and

$$\frac{\partial u}{\partial \nu} = 0, \text{ on } \Gamma, \tag{2.2}$$

which is a natural variational boundary condition (it also yields that the interface is orthogonal to the boundary). Here, $\Gamma = \partial\Omega$ and ν is the unit outer normal to the boundary. In particular, it follows from the first boundary condition that we have the conservation of mass, i.e., of the spatial average of the order parameter, obtained by (formally) integrating the first equation of (1.1) over Ω,

$$\langle u(t) \rangle = \frac{1}{\mathrm{Vol}(\Omega)} \int_\Omega u(t,x)\, dx = \langle u(0) \rangle, \ \forall t \geq 0. \tag{2.3}$$

If we have in mind the fourth-order in space Cahn–Hilliard equation, we can rewrite these boundary conditions, equivalently, as

$$\frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0, \text{ on } \Gamma. \tag{2.4}$$

Remark 2.1. We can also consider periodic boundary conditions (in which case $\Omega = \Pi_{i=1}^n (0, L_i),\ L_i > 0, i = 1, \ldots, n$).

As mentioned in the introduction, the Cahn–Hilliard equation is now well understood from a mathematical point of view. This is in particular the case for the usual cubic nonlinear term $f(s) = s^3 - s$, but also for more general regular nonlinear terms.

Now, the case of the thermodynamically relevant logarithmic nonlinear terms is more difficult. Indeed, in order to prove the existence of a solution, one generally approximates the singular nonlinear term by regular ones (e.g., by polynomials as in [116]; we also mention [4,301] for different approaches, based on semigroup theory and a regularization by the viscous Cahn–Hilliard equation proposed in [310], respectively) and one then passes to the limit. But then, when passing to the limit, one must make sure that the order parameter stays in the physically relevant interval ($(-1, 1)$ in our case); otherwise the equations would not make sense. From a physical point of view, this separation property says that, in the phase separation process, one never completely reaches the pure states.

Remark 2.2. It would be interesting to see whether, for a regular nonlinear term and, in particular, for the usual cubic one, the order parameter also remains in the physically relevant interval. This is however not the case and one can construct simple counterexamples, already in one space dimension (see [324]).

We now give a proof of existence of a solution which is based on proper approximations of the logarithmic nonlinear term and which can be easily extended to more general singular nonlinear terms (see also [140]).

We actually consider the following more general initial and boundary value problem:

$$\frac{\partial u}{\partial t} + \beta u = \Delta\mu, \ \beta \geq 0, \tag{2.5}$$

$$\mu = -\Delta u + f(u), \tag{2.6}$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial \mu}{\partial \nu} = 0, \text{ on } \Gamma, \tag{2.7}$$

$$u|_{t=0} = u_0. \tag{2.8}$$

When $\beta = 0$, we recover the original Cahn–Hilliard equation, and, when $\beta > 0$, we have the Cahn–Hilliard–Oono equation (here, we have set the other parameters equal to 1).

Remark 2.3. (i) As mentioned in the introduction, the term βu, $\beta > 0$, models nonlocal interactions. In particular, short-ranged interactions tend to homogenize the system, whereas long-ranged ones forbid the formation of too large structures; the competition between these two effects translates into the formation of a micro-separated state (also called super-crystal) with a spatially modulated order parameter, defining structures with a uniform size (see [355] for more details and references).

(ii) Actually, it can be surprising that nonlocal interactions can be described by such a simple linear term. This can be seen by noting that the equations are obtained by considering the free energy

$$\psi = |\nabla u|^2 + F(u) + \int_\Omega u(y)g(y, x)u(x)\, dy, \tag{2.9}$$

where the function g describes the long-ranged interactions. In particular, in Oono's model and, e.g., in three space dimensions, one takes

$$g(y,x) = \frac{4\pi\beta}{|y-x|},\ \beta > 0. \tag{2.10}$$

Note that the long-ranged interactions are repulsive when $u(y)$ and $u(x)$ have opposite signs and thus favor the formation of interfaces (see [355] and the references therein). We finally write, as in the derivation of the classical Cahn–Hilliard equation,

$$\frac{\partial u}{\partial t} = \partial_u \psi, \tag{2.11}$$

where ∂_u denotes the variational derivative with respect to u. Noting that $-\frac{1}{|y-x|}$ is the Green function associated with the Laplace operator and defining μ as above, we obtain (2.5)-(2.6) (see [355] and the references therein for more details).

Remark 2.4. It is easier to prove the existence of a weak solution to the Cahn–Hilliard system with the degenerate mobility $\kappa(s) = 1 - s^2$ and the thermodynamically relevant logarithmic nonlinear term (1.4). Indeed, one uses the fact that $\kappa(s)f'(s)$ is bounded (see [138] for details). Here, we can adapt the techniques in [138] to the Cahn–Hilliard–Oono equation.

As far as the nonlinear term f is concerned, we assume more generally that

$$f \in C^1(-1,1),\ f(0) = 0, \tag{2.12}$$

$$\lim_{s\to\pm 1} f(s) = \pm\infty,\ \lim_{s\to\pm 1} f'(s) = +\infty. \tag{2.13}$$

In particular, it follows from these assumptions that

$$f' \geq -c_0,\ c_0 \geq 0, \tag{2.14}$$

$$-c_1 \leq F(s) \leq f(s)s + c_2,\ c_1,\ c_2 \geq 0,\ s \in (-1,1), \tag{2.15}$$

where $F(s) = \int_0^s f(r)\,dr$ (in particular, in order to obtain the second of (2.15), we can study the variations of the function $s \mapsto f(s)s - F(s) + \frac{c_0}{2}s^2$, whose derivative has, owing to (2.14), the sign of s).

Remark 2.5. In particular, the thermodynamically relevant logarithmic functions (1.4) satisfy the above assumptions.

Next, we define, for $N \in \mathbb{N}$, the approximated functions $f_N \in C^1(\mathbb{R})$ by

$$f_N(s) = \begin{cases} f(-1+\frac{1}{N}) + f'(-1+\frac{1}{N})(s+1-\frac{1}{N}),\ s < -1+\frac{1}{N}, \\ f(s),\ |s| \leq 1-\frac{1}{N}, \\ f(1-\frac{1}{N}) + f'(1-\frac{1}{N})(s-1+\frac{1}{N}),\ s > 1-\frac{1}{N}. \end{cases}$$

We thus have

$$f_N' \geq -c_0 \tag{2.16}$$

and, setting $F_N(s) = \int_0^s f_N(r)\,dr$,

$$-c_3 \leq F_N(s) \leq c_4 f_N(s)s + c_5,\ c_4 > 0,\ c_3,\ c_5 \geq 0,\ s \in \mathbb{R}, \tag{2.17}$$

$$f_N(s)s \geq c_6|f_N(s)| - c_7,\ c_6 > 0,\ c_7 \geq 0, \tag{2.18}$$

where the constants c_i, $i = 3, ..., 7$, are independent of N (see [305]). Actually, there holds, more generally, for N large enough (see [301], Proposition A.1, and [305] for details),

$$f_N(s+m)s \geq c_m'(|f_N(s+m)| + F_N(s+m)) - c_m'', \tag{2.19}$$
$$c_m' > 0,\ c_m'' \geq 0,\ s \in \mathbb{R},\ m \in (-1, 1),$$

where the constants c_m' and c_m'' are independent of N and depend continuously and boundedly on m.

We finally introduce the approximated problems

$$\frac{\partial u_N}{\partial t} + \beta u_N = \Delta \mu_N, \tag{2.20}$$

$$\mu_N = -\Delta u_N + f_N(u_N), \tag{2.21}$$

$$\frac{\partial u_N}{\partial \nu} = \frac{\partial \mu_N}{\partial \nu} = 0,\ \text{on } \Gamma, \tag{2.22}$$

$$u_N|_{t=0} = u_0. \tag{2.23}$$

We denote by $((\cdot,\cdot))$ the usual L^2-scalar product, with associated norm $\|\cdot\|$. We further set $\|\cdot\|_{-1} = \|(-\Delta)^{-\frac{1}{2}}\cdot\|$, where $(-\Delta)^{-1}$ denotes the inverse minus Laplace operator associated with Neumann boundary conditions and acting on functions with null spatial average. More generally, $\|\cdot\|_X$ denotes the norm on the Banach space X.

We set, for $v \in L^1(\Omega)$,

$$\langle v \rangle = \frac{1}{\mathrm{Vol}(\Omega)} \int_\Omega v\,dx,$$

and, for $v \in H^1(\Omega)'$,

$$\langle v \rangle = \frac{1}{\mathrm{Vol}(\Omega)} \langle v, 1 \rangle_{H^1(\Omega)', H^1(\Omega)}.$$

We note that

$$v \mapsto (\|v - \langle v \rangle\|_{-1}^2 + \langle v \rangle^2)^{\frac{1}{2}}$$

is a norm on $H^1(\Omega)'$ which is equivalent to the usual norm. Similarly,

$$v \mapsto (\|v - \langle v\rangle\|^2 + \langle v\rangle^2)^{\frac{1}{2}}$$

and

$$v \mapsto (\|\nabla v\|^2 + \langle v\rangle^2)^{\frac{1}{2}}$$

are norms on $L^2(\Omega)$ and $H^1(\Omega)$, respectively, which are equivalent to their usual norms.

We further set

$$W = \{v \in H^1(\Omega),\ \langle v\rangle = 0\}$$

and note that, on W, the generalized Poincaré's inequality

$$\|v\| \leq c\|\nabla v\|$$

holds. Moreover, we have the continuous embedding $H^{-1}(\Omega) \subset W'$.

In what follows, the same letters c, c' and c'' denote (generally positive) constants which may vary from line to line and which are independent of N.

2.1. A priori estimates

Our aim in this subsection is to derive a priori estimates for the solutions u_N and μ_N to (2.20)-(2.23). These a priori estimates are independent of N and are formal, i.e., we assume that u_N and μ_N are as smooth as needed. The crucial step, to prove the existence of a solution, consists in deriving an a priori estimate independent of N on $f_N(u_N)$ in $L^2((0,T)\times\Omega)$, $T > 0$.

Classically, these a priori estimates allow us to obtain the existence of a solution to (2.20)-(2.23) by implementation of a Galerkin approximation (see, e.g., [287] for more details). This will also allow us to pass to the limit $N \to +\infty$ in the approximated system (2.20)-(2.23).

From now on, we assume that

$$\|u_0\|_{L^\infty(\Omega)} \leq 1 - \delta,\ \delta \in (0,1), \tag{2.24}$$

where δ is a fixed constant.

First, integrating (2.20) over Ω, we find

$$\frac{d\langle u_N\rangle}{dt} + \beta\langle u_N\rangle = 0, \tag{2.25}$$

which yields

$$\langle u_N(t)\rangle = e^{-\beta t}\langle u_0\rangle,\ t \geq 0. \tag{2.26}$$

We thus deduce from (2.26) that

$$|\langle u_N(t)\rangle| \leq |\langle u_0\rangle|,\ t \geq 0, \tag{2.27}$$

whence, in view of (2.24),

$$|\langle u_N(t)\rangle| \leq 1-\delta,\ t \geq 0, \tag{2.28}$$

i.e., $\langle u_N\rangle$ is strictly separated from the pure states ± 1.

Then, setting, for a function φ defined in Ω, $\overline{\varphi} = \varphi - \langle\varphi\rangle$, we can rewrite (2.20) in the equivalent form

$$\frac{\partial \overline{u}_N}{\partial t} + \beta \overline{u}_N = \Delta \mu_N, \tag{2.29}$$

$$\mu_N = -\Delta \overline{u}_N + f_N(u_N), \tag{2.30}$$

owing to (2.25).

In a next step, we multiply (2.29) by μ_N. Integrating over Ω and by parts, we obtain

$$\|\nabla \mu_N\|^2 + ((\frac{\partial \overline{u}_N}{\partial t}, \mu_N)) + \beta((\overline{u}_N, \mu_N)) = 0. \tag{2.31}$$

Furthermore, it follows from (2.30) that

$$\begin{aligned}((\frac{\partial \overline{u}_N}{\partial t}, \mu_N)) + \beta((\overline{u}_N, \mu_N)) &= \frac{1}{2}\frac{d}{dt}\|\nabla \overline{u}_N\|^2 + \frac{d}{dt}\int_\Omega F_N(u_N)\,dx \\ &-((\frac{d\langle u_N\rangle}{dt}, f_N(u_N))) + \beta\|\nabla \overline{u}_N\|^2 + \beta((\overline{u}_N, f_N(u_N))).\end{aligned} \tag{2.32}$$

Noting that it follows from (2.25) that

$$-((\frac{d\langle u_N\rangle}{dt}, f_N(u_N))) = \beta((\langle u_N\rangle, f_N(u_N))), \tag{2.33}$$

we finally deduce from (2.18) and (2.31)-(2.33) the differential inequality

$$\begin{aligned}&\frac{d}{dt}(\|\nabla \overline{u}_N\|^2 + 2\int_\Omega F_N(u_N)\,dx) \\ &+c(\|\nabla \overline{u}_N\|^2 + \|f_N(u_N)\|_{L^1(\Omega)} + \|\nabla \mu_N\|^2) \leq c',\ c > 0.\end{aligned} \tag{2.34}$$

We then multiply (2.30) by $\overline{u}_N$ and find, owing to (2.19) (taking $s = \overline{u}_N$ and $m = \langle u_N\rangle$) and to the Poincaré inequality on W,

$$\begin{aligned}\|\nabla \overline{u}_N\|^2 + c(\|f_N(u_N)\|_{L^1(\Omega)} + \int_\Omega F_N(u_N)\,dx) &\leq c' + ((\mu_N, \overline{u}_N)) = c' + ((\overline{\mu}_N, \overline{u}_N)) \\ &\leq c' + c''\|\nabla \overline{u}_N\|\|\nabla \mu_N\|,\end{aligned}$$

where the constants c, c' and c'' depend on δ, but are independent of N, at least for N large enough, whence

$$\|\nabla \overline{u}_N\|^2 + c(\|f_N(u_N)\|_{L^1(\Omega)} + \int_\Omega F_N(u_N)\,dx) \leq c' + c''\|\nabla \mu_N\|^2,\ c > 0. \tag{2.35}$$

Summing (2.34) and ξ_1 times (2.35), where $\xi_1 > 0$ is small enough, we have the differential inequality

$$\frac{dE_N}{dt} + c(E_N + \|f_N(u_N)\|_{L^1(\Omega)} + \|\nabla\mu_N\|^2) \leq c',\ c > 0, \tag{2.36}$$

where

$$E_N = \|u_N\|^2 + \|\nabla\overline{u}_N\|^2 + 2\int_\Omega F_N(u_N)\,dx. \tag{2.37}$$

We now rewrite (2.29) in the equivalent form

$$(-\Delta)^{-1}\frac{\partial\overline{u}_N}{\partial t} + \beta(-\Delta)^{-1}\overline{u}_N = -\overline{\mu}_N \tag{2.38}$$

(note indeed that $\langle\frac{\partial\overline{u}_N}{\partial t}\rangle = 0$). Multiplying (2.38) by $\frac{\partial\overline{u}_N}{\partial t}$, we obtain

$$\frac{\beta}{2}\frac{d}{dt}\|\overline{u}_N\|_{-1}^2 + \|\frac{\partial\overline{u}_N}{\partial t}\|_{-1}^2 = -((\overline{\mu}_N, \frac{\partial\overline{u}_N}{\partial t})). \tag{2.39}$$

We note that, thanks to the Poincaré inequality on W,

$$\begin{aligned} |((\overline{\mu}_N, \frac{\partial\overline{u}_N}{\partial t}))| &\leq c\|\frac{\partial\overline{u}_N}{\partial t}\|_{-1}\|\nabla\mu_N\| \\ &\leq \frac{1}{4}\|\frac{\partial\overline{u}_N}{\partial t}\|_{-1}^2 + c\|\nabla\mu_N\|^2. \end{aligned} \tag{2.40}$$

It thus follows from (2.39) that

$$\beta\frac{d}{dt}\|\overline{u}_N\|_{-1}^2 + \|\frac{\partial\overline{u}_N}{\partial t}\|_{-1}^2 \leq c\|\nabla\mu_N\|^2. \tag{2.41}$$

Next, we note that it follows from (2.30) that

$$\overline{\mu}_N = -\Delta\overline{u}_N + \overline{f_N(u_N)}. \tag{2.42}$$

Multiplying (2.42) by $-\Delta\overline{u}_N$, we find, owing to (2.16),

$$\begin{aligned} \|\Delta\overline{u}_N\|^2 &\leq c_0\|\nabla\overline{u}_N\|^2 - ((\overline{\mu}_N, \Delta\overline{u}_N)) \\ &\leq c_0\|\nabla\overline{u}_N\|^2 + \frac{1}{2}\|\Delta\overline{u}_N\|^2 + c\|\nabla\mu_N\|^2, \end{aligned}$$

which yields

$$\|\Delta\overline{u}_N\|^2 \leq 2c_0\|\nabla\overline{u}_N\|^2 + c\|\nabla\mu_N\|^2. \tag{2.43}$$

Summing (2.36) and ξ_2 times (2.43), where $\xi_2 > 0$ is small enough, we have the differential inequality

$$\frac{dE_N}{dt} + c(E_N + \|\overline{u}_N\|_{H^2(\Omega)}^2 + \|f_N(u_N)\|_{L^1(\Omega)} + \|\nabla\mu_N\|^2) \leq c',\ c > 0. \tag{2.44}$$

We also note that (2.42) implies

$$\|\overline{f_N(u_N)}\| \leq c(\|\overline{u}_N\|_{H^2(\Omega)} + \|\nabla\mu_N\|), \tag{2.45}$$

which, combined with (2.44), yields

$$\frac{dE_N}{dt} + c(E_N + \|\overline{u}_N\|^2_{H^2(\Omega)} + \|f_N(u_N)\|_{L^1(\Omega)} + \|\overline{f_N(u_N)}\|^2 + \|\nabla\mu_N\|^2) \leq c',\ c > 0. \tag{2.46}$$

Finally, taking $s = \overline{u}_N$ and $m = \langle u_N\rangle$ in (2.19) and integrating over Ω, we obtain, owing to (2.28),

$$\int_\Omega |f_N(u_N)|\,dx \leq c|\int_\Omega f_N(u_N)\overline{u}_N\,dx| + c',$$

where the constants c and c' only depend on δ (and are, in particular, independent of N), so that

$$|\langle f_N(u_N)\rangle| \leq c|\int_\Omega \overline{f_N(u_N)}\overline{u}_N\,dx| + c'$$

and

$$|\langle f_N(u_N)\rangle| \leq c\|\overline{u}_N\|\|\overline{f_N(u_N)}\| + c'. \tag{2.47}$$

Since

$$\|f_N(u)\|^2 \leq c(\|\overline{f_N(u_N)}\|^2 + |\langle f_N(u_N)\rangle|^2),$$

it follows from (2.47) that

$$\|f_N(u_N)\|_{L^2((0,T)\times\Omega)} \leq c(\|\overline{f_N(u_N)}\|_{L^2((0,T)\times\Omega)} + \|\overline{u}_N\|_{L^\infty(0,T;L^2(\Omega))}\|\overline{f_N(u_N)}\|_{L^2((0,T)\times\Omega)}) + c',\ T > 0. \tag{2.48}$$

Now, Poincaré's inequality and (2.46) imply that

$$\|\overline{f_N(u_N)}\|^2_{L^2((0,T)\times\Omega)} \leq cE_N(0) + c'T$$

and

$$\|\overline{u}_N\|^2_{L^\infty(0,T;L^2(\Omega))} \leq c\|\nabla\overline{u}_N\|^2_{L^\infty(0,T;L^2(\Omega)^n)} \leq c'E_N(0) + c''T.$$

It thus follows from (2.48) that

$$\|f_N(u_N)\|_{L^2((0,T)\times\Omega)} \leq c_{T,\delta}(\|u_0\|^2_{H^1(\Omega)} + 1),\ T > 0, \tag{2.49}$$

where the constant $c_{T,\delta}$ is independent of N, at least for N large enough. Here, we have used the fact that, owing to (2.24), if N is large enough, $F_N(u_0) = F(u_0)$, so that we can handle the term $\int_\Omega F_N(u_0)\,dx$ which appears in the right-hand side when integrating (2.46) with respect to time.

We also note that (2.46) and Gronwall's lemma imply the dissipative estimate

$$E_N(t) \leq e^{-ct}E_N(0) + c',\ c > 0,\ t \geq 0, \tag{2.50}$$

which yields

$$\|u_N(t)\|_{H^1(\Omega)} \leq c_\delta e^{-c't}(\|u_0\|_{H^1(\Omega)} + 1) + c'',\ c' > 0,\ t \geq 0. \tag{2.51}$$

Finally, noting that $\langle \mu_N \rangle = \langle f_N(u_N) \rangle$, it follows from (2.46) and (2.49) that

$$\|\mu_N\|_{L^2(0,T;H^1(\Omega))} \leq c_{T,\delta}(\|u_0\|^2_{H^1(\Omega)} + 1),\ T > 0. \tag{2.52}$$

2.2. Existence of solutions

We have the

Theorem 2.6. *We assume that u_0 is given such that $u_0 \in H^1(\Omega)$ and $\|u_0\|_{L^\infty(\Omega)} < 1$. Then,* (2.5)-(2.8) *possesses at least one (weak) solution such that, $\forall T > 0$,*

$$u \in C([0,T]; H^1(\Omega)_{\mathrm{w}}) \cap L^2(0,T; H^2(\Omega)),$$

$$\frac{\partial u}{\partial t} \in L^2(0,T; H^1(\Omega)'),$$

$$\mu \in L^2(0,T; H^1(\Omega)),$$

where the subscript w *stands for the weak topology, and*

$$\frac{d}{dt}((u,q)) + \beta((u,q)) = -((\nabla\mu, \nabla q)),$$

$$((\mu, \Xi)) = ((\nabla u, \nabla \Xi)) + ((f(u), \Xi)),$$

a.e. $t \in [0,T]$, $\forall q$, $\Xi \in C^\infty_{\mathrm{c}}(\Omega)$,

$$u(0) = u_0.$$

Furthermore, $u \in C([0,T]; H^{1-\eta}(\Omega))$, $\forall \eta > 0$, and $-1 < u(t,x) < 1$, a.e. (t,x).

Proof. We consider a solution (u_N, μ_N) to the approximated problem (2.20)-(2.23) (the proof of existence of such a solution having the above regularity can be obtained by a standard Galerkin scheme).

Furthermore, since the estimates derived in the previous section are independent of N, this solution converges, up to a subsequence which we do not relabel, to a limit function (u, μ) in the following sense:

$$u_N \to u \text{ in } L^\infty(0,T; H^1(\Omega)) \text{ weak star and in } L^2(0,T; H^2(\Omega)) \text{ weakly},$$

$$\frac{\partial u_N}{\partial t} \to \frac{\partial u}{\partial t} \text{ in } L^2(0,T; H^1(\Omega)') \text{ weakly},$$

$$u_N \to u \text{ a.e. } (t,x) \text{ and in } L^2((0,T) \times \Omega),$$

$$\mu_N \to \mu \text{ in } L^2(0,T;H^1(\Omega)) \text{ weakly.}$$

Here, we have used (2.41), (2.51)-(2.52) and classical Aubin–Lions compactness results.

The only difficulty, when passing to the limit, is to pass to the limit in the nonlinear term $f_N(u_N)$.

First, it follows from (2.46) that $f_N(u_N)$ is bounded, independently of N, in $L^1((0,T)\times\Omega)$. Then, it follows from the explicit expression of f_N that

$$\text{meas}(E_{N,M}) \leq \varphi(\frac{1}{N}),\ N \leq M,$$

where

$$E_{N,M} = \{(t,x) \in (0,T)\times\Omega,\ |u_M(t,x)| > 1 - \frac{1}{N}\}$$

and

$$\varphi(s) = \frac{c}{\max(|f(1-s)|, |f(s-1)|)}.$$

Here, the constant c is independent of N and M. Note indeed that there holds

$$\int_0^T \int_\Omega |f_M(u_M)|\,dx\,dt \geq \int_{E_{N,M}} |f_M(u_M)|\,dx\,dt \geq c'\text{meas}(E_{N,M})\frac{1}{\varphi(\frac{1}{N})}, \tag{2.53}$$

where the constant c' is independent of N and M. We can pass to the limit $M \to +\infty$ (employing Fatou's lemma, see (2.53)) and then $N \to +\infty$ (noting that $\lim_{s\to 0}\varphi(s) = 0$) to find

$$\text{meas}\{(t,x) \in (0,T)\times\Omega,\ |u(t,x)| \geq 1\} = 0,$$

so that

$$-1 < u(t,x) < 1,\ \text{a.e. } (t,x). \tag{2.54}$$

Next, it follows from the above almost everywhere convergence of u_N to u, from (2.54) and the explicit expression of f_N that

$$f_N(u_N) \to f(u),\ \text{a.e. } (t,x) \in (0,T)\times\Omega. \tag{2.55}$$

Finally, since, owing to (2.49), $f_N(u_N)$ is bounded, independently of N, in $L^2((0,T)\times\Omega)$, it follows from (2.55) that $f_N(u_N) \to f(u)$ in $L^2((0,T)\times\Omega)$ weakly, which finishes the proof of the passage to the limit.

□

Remark 2.7. We consider the particular case of the physically relevant nonlinear term (1.4). In that case, it is not difficult to see that the function $F(s) = \int_0^s f(r)\,dr$ is bounded on $(-1,1)$. Noting then that the function $F_N(s) = \int_0^s f_N(r)\,dr$ is given by

$$F_N(s) = \begin{cases} F(-1+\frac{1}{N}) + f(-1+\frac{1}{N})(s+1-\frac{1}{N}) \\ +\frac{1}{2}f'(-1+\frac{1}{N})(s+1-\frac{1}{N})^2, \ s < -1+\frac{1}{N}, \\ F(s), \ |s| \le 1-\frac{1}{N}, \\ F(1-\frac{1}{N}) + f(1-\frac{1}{N})(s-1+\frac{1}{N}) \\ +\frac{1}{2}f'(1-\frac{1}{N})(s-1+\frac{1}{N})^2, \ s > 1-\frac{1}{N}, \end{cases}$$

elementary computations show that F_N is also bounded on $(-1, 1)$. Therefore, we can relax the assumptions of Theorem 2.6 and assume that u_0 only satisfies $-1 < u_0(x) < 1$, a.e. $x \in \Omega$, and $|\langle u_0 \rangle| < 1$. Indeed, we deduce from the above that $\int_\Omega F_N(u_0)\, dx$ is bounded independently of N and we do not need the strict separation property (2.24) to derive the a priori estimates on u_N, namely, when integrating (2.46) with respect to time.

Remark 2.8. (i) It is not difficult to prove the uniqueness of solutions. Indeed, let u^1 and u^2 be two solutions with initial conditions u_0^1 and u_0^2, respectively, such that $\langle u_0^1 \rangle = \langle u_0^2 \rangle$. Then, we have, setting $u = u^1 - u^2$ and $u_0 = u_0^1 - u_0^2$ and noting that $\langle u \rangle = 0$,

$$(-\Delta)^{-1}\frac{\partial u}{\partial t} + \beta(-\Delta)^{-1}u - \Delta u + f(u^1) - f(u^2) - \langle f(u^1) - f(u^2) \rangle = 0, \tag{2.56}$$

$$\frac{\partial u}{\partial \nu} = 0, \ \text{on } \Gamma, \tag{2.57}$$

$$u|_{t=0} = u_0. \tag{2.58}$$

Multiplying (2.56) by u, we obtain, in view of (2.14),

$$\frac{1}{2}\frac{d}{dt}\|u\|_{-1}^2 + \beta\|u\|_{-1}^2 + \|\nabla u\|^2 \le c_0\|u\|^2. \tag{2.59}$$

Employing the interpolation inequality

$$\|u\|^2 \le c\|u\|_{-1}\|\nabla u\|, \tag{2.60}$$

we deduce that

$$\frac{d}{dt}\|u\|_{-1}^2 + c\|u\|_{H^1(\Omega)}^2 \le c'\|u\|_{-1}^2, \ c > 0. \tag{2.61}$$

It follows from (2.61) and Gronwall's lemma that

$$\|u(t)\|_{H^{-1}(\Omega)} \le ce^{c't}\|u_0\|_{H^{-1}(\Omega)}, \ t \ge 0, \tag{2.62}$$

whence the continuous dependence with respect to the initial conditions (in the H^{-1}-norm) and the uniqueness (for u; the uniqueness for μ is then straightforward).

(ii) We set

$$\Phi_m = \{v \in H^1(\Omega) \cap L^\infty(\Omega), \ -1 < v(x) < 1, \ \text{a.e. } x \in \Omega, \ \langle v \rangle = m\}, \ m \in (-1, 1).$$

It follows from the above that we can define the continuous (for the H^{-1}-norm) family of solving operators

$$S(t) : \Phi_m \cap \{v \in L^\infty(\Omega),\ \|v\|_{L^\infty(\Omega)} < 1\} \to \Phi_m,\ u_0 \mapsto u(t),\ t \geq 0.$$

Furthermore, it follows from (2.51) and Gronwall's lemma that $S(t)$ is dissipative on Φ_m, i.e., it possesses a bounded absorbing set $\mathcal{B}_0 \subset \Phi_m$ (in the sense that, $\forall B \subset \Phi_m$ bounded, $\exists t_0 = t_0(B)$ such that $t \geq t_0$ implies $S(t)B \subset \mathcal{B}_0$). Note that, in the case of the thermodynamically relevant logarithmic potentials, the family of solving operators $S(t)$, $t \geq 0$, forms a continuous (for the H^{-1}-norm) and dissipative semigroup on Φ_m. Furthermore, in all cases, it follows from (2.62) that we can extend (in a unique way and by continuity) $S(t)$ to a semigroup acting on the closure of Φ_m in the H^{-1}-topology, i.e., on $L_m = \{v \in L^\infty(\Omega),\ \|v\|_{L^\infty(\Omega)} \leq 1,\ \langle v \rangle = m\}$, meaning that we can now consider initial data which contain the pure states; note that $S(t) : L_m \to \Phi_m$, as soon as $t > 0$.

(iii) We can note that the pure states are not (weak) solutions to our problem. However, in [301], we were able to prove, by a careful study of the structure of attractors, that, in the case of the original Cahn–Hilliard equation, the pure states can indeed be considered as weak solutions, by setting $S(t)(\pm 1) = \pm 1$.

Remark 2.9. We can also study the limit β goes to 0. In particular, we proved in [287] that, for regular nonlinear terms, the asymptotic behavior of the Cahn–Hilliard–Oono equation and the limit Cahn–Hilliard equation are close in some proper sense when β is small. The case of logarithmic nonlinear terms is much more involved and we only proved in [300] the convergence of solutions on finite time intervals.

2.3. Further regularity and strict separation

A natural way to obtain further regularity is to first differentiate the equation for u with respect to time. However, for $\beta > 0$, such a technique cannot be applied in a direct way and is more involved. Indeed, we recall that we have the equation

$$(-\Delta)^{-1}\frac{\partial \overline{u}_N}{\partial t} + \beta(-\Delta)^{-1}\overline{u}_N = -\overline{\mu}_N. \tag{2.63}$$

Differentiating (2.63) with respect to time, we have

$$(-\Delta)^{-1}\frac{\partial}{\partial t}\frac{\partial \overline{u}_N}{\partial t} + \beta(-\Delta)^{-1}\frac{\partial \overline{u}_N}{\partial t} = -\frac{\partial \overline{\mu}_N}{\partial t}, \tag{2.64}$$

where

$$\frac{\partial \overline{\mu}_N}{\partial t} = -\Delta\frac{\partial \overline{u}_N}{\partial t} + \overline{f'_N(u_N)\frac{\partial u_N}{\partial t}}. \tag{2.65}$$

Multiplying (2.64) by $\frac{\partial \overline{u}_N}{\partial t}$, we obtain, owing to (2.65),

$$\frac{1}{2}\frac{d}{dt}\|\frac{\partial \overline{u}_N}{\partial t}\|_{-1}^2 + \beta\|\frac{\partial \overline{u}_N}{\partial t}\|_{-1}^2 + \|\nabla\frac{\partial \overline{u}_N}{\partial t}\|^2 + ((f'_N(u_N)\frac{\partial u_N}{\partial t}, \frac{\partial \overline{u}_N}{\partial t})) = 0. \tag{2.66}$$

We infer from (2.16) that

$$((f_N'(u_N)\frac{\partial u_N}{\partial t}, \frac{\partial \overline{u}_N}{\partial t})) \geq -c_0\|\frac{\partial \overline{u}_N}{\partial t}\|^2 + ((f_N'(u_N)\frac{\partial \langle u_N\rangle}{\partial t}, \frac{\partial \overline{u}_N}{\partial t})),$$

which yields, owing to a proper interpolation inequality,

$$\frac{d}{dt}\|\frac{\partial \overline{u}_N}{\partial t}\|_{-1}^2 + 2\beta\|\frac{\partial \overline{u}_N}{\partial t}\|_{-1}^2 + \|\nabla\frac{\partial \overline{u}_N}{\partial t}\|^2 \leq c\|\frac{\partial \overline{u}_N}{\partial t}\|_{-1}^2 - 2((f_N'(u_N)\frac{\partial \langle u_N\rangle}{\partial t}, \frac{\partial \overline{u}_N}{\partial t})). \tag{2.67}$$

The problem is that we now need to estimate the term

$$((f_N'(u_N)\frac{\partial \langle u_N\rangle}{\partial t}, \frac{\partial \overline{u}_N}{\partial t})) = -\beta \mathrm{Vol}(\Omega)\langle u_N\rangle\langle f_N'(u_N)\frac{\partial \overline{u}_N}{\partial t}\rangle$$

(when $\beta = 0$, this term vanishes) and we are not able to estimate it uniformly with respect to N. Note that, if $\langle u_0\rangle = 0$, then $\langle u(t)\rangle = 0$, $\forall t \geq 0$, and we do not have this problem.

In order to avoid dealing with this term, we write instead that

$$\begin{aligned}((f_N'(u_N)\frac{\partial u_N}{\partial t}, \frac{\partial \overline{u}_N}{\partial t})) &= ((f_N'(u_N)\frac{\partial u_N}{\partial t}, \frac{\partial u_N}{\partial t})) - \frac{\partial \langle u_N\rangle}{\partial t}((f_N'(u_N)\frac{\partial u_N}{\partial t}, 1))\\ &\geq -c_0\|\frac{\partial u_N}{\partial t}\|^2 - \frac{\partial \langle u_N\rangle}{\partial t}\frac{d}{dt}\int_\Omega F_N(u_N)\,dx\\ &= -c_0\|\frac{\partial u_N}{\partial t}\|^2 - \frac{d}{dt}(\frac{\partial \langle u_N\rangle}{\partial t}\int_\Omega F_N(u_N)\,dx) + \frac{\partial^2 \langle u_N\rangle}{\partial t^2}\int_\Omega F_N(u_N)\,dx.\end{aligned}$$

Setting

$$\Lambda = \frac{1}{2}\|\frac{\partial \overline{u}_N}{\partial t}\|_{-1}^2 - \frac{\partial \langle u_N\rangle}{\partial t}\int_\Omega F_N(u_N)\,dx,$$

we infer from (2.66) the differential inequality

$$\frac{d\Lambda}{dt} + \beta\|\frac{\partial \overline{u}_N}{\partial t}\|_{-1}^2 + \|\nabla\frac{\partial \overline{u}_N}{\partial t}\|^2 \leq c_0\|\frac{\partial u_N}{\partial t}\|^2 - \frac{\partial^2 \langle u_N\rangle}{\partial t^2}\int_\Omega F_N(u_N)\,dx,$$

which yields, employing a proper interpolation inequality,

$$\frac{d\Lambda}{dt} + \beta\|\frac{\partial \overline{u}_N}{\partial t}\|_{-1}^2 + \frac{1}{2}\|\nabla\frac{\partial \overline{u}_N}{\partial t}\|^2 \leq c(\|\frac{\partial \overline{u}_N}{\partial t}\|_{-1}^2 + |\frac{\partial \langle u_N\rangle}{\partial t}|^2) - \frac{\partial^2 \langle u_N\rangle}{\partial t^2}\int_\Omega F_N(u_N)\,dx. \tag{2.68}$$

Recalling (2.26) and (2.50), we can see that Λ is bounded from below,

$$\Lambda \geq \frac{1}{2}\|\frac{\partial \overline{u}_N}{\partial t}\|_{-1}^2 - c,$$

for some positive constant c which is independent of N. Similarly, we can easily prove that the last two terms in the right-hand side of (2.68) are bounded from above. We thus end up with a differential inequality of the form

$$\frac{d\Lambda}{dt} + \beta\|\frac{\partial \overline{u}_N}{\partial t}\|_{-1}^2 + \frac{1}{2}\|\nabla\frac{\partial \overline{u}_N}{\partial t}\|^2 \leq c(\|\frac{\partial \overline{u}_N}{\partial t}\|_{-1}^2 + \langle u_0\rangle^2 + E_N(0)^2 + 1), \tag{2.69}$$

where the positive constant c is independent of N, and we can conclude by using, e.g., the uniform Gronwall's lemma (see, e.g., [349]). We thus see that the additional term βu in the Oono's model brings essential difficulties.

Having this, we can go further and prove additional regularity on u_N (see, e.g., [132]). Note however that some corresponding constants may a priori depend on N.

A related question is the strict separation from the pure states, namely, an estimate of the form

$$\|u(t)\|_{L^\infty(\Omega)} \leq 1 - \delta,\ \delta \in (0,1).$$

From a physical point of view, this would mean that not only we never have the pure states during the phase separation process, but we also stay in some sense far from the pure states. From a mathematical point of view, this would mean that we actually have the same problem, but with a regular nonlinear term (actually, even better: with a bounded and globally Lipschitz continuous nonlinear term). Studying further regularity on u and the asymptotic behavior of the problem would then be straightforward tasks. This strict separation property is however related with the aforementioned question of the additional regularity; to be more precise, this requires proper estimates on $f'(u)$ (see [301]).

As mentioned above, in [301], one regularizes the Cahn–Hilliard equation by the viscous Cahn–Hilliard equation

$$\varepsilon\frac{\partial u^\varepsilon}{\partial t} + (-\Delta)^{-1}\frac{\partial u^\varepsilon}{\partial t} - \Delta u^\varepsilon + f(u^\varepsilon) - \langle f(u^\varepsilon)\rangle = 0,\ \varepsilon > 0, \tag{2.70}$$

$$\frac{\partial u^\varepsilon}{\partial \nu} = 0,\ \text{on}\ \Gamma, \tag{2.71}$$

$$u^\varepsilon|_{t=0} = u_0. \tag{2.72}$$

One advantage of the viscous Cahn–Hilliard equation is that one has the strict separation from the pure states. More precisely, one can prove the

Theorem 2.10. *We assume that $\varepsilon > 0$ and $u_0 \in D^\varepsilon_m$, $|m| \leq 1 - \eta$, $\eta \in (0,1)$, where*

$$D^\varepsilon_m = \{q \in H^2(\Omega),\ \frac{\partial q}{\partial \nu} = 0,\ \text{on}\ \Gamma,\ \|q\|_{L^\infty(\Omega)} \leq 1,\ \langle q\rangle = m,$$
$$f(q) \in L^2(\Omega),\ \sqrt{\varepsilon}\phi \in L^2(\Omega),\ \phi \in H^{-1}(\Omega),\ \phi = (\varepsilon I + (-\Delta)^{-1})^{-1}(\Delta q - f(q) + \langle f(q)\rangle)\}.$$

Then, for every $\xi > 0$, there holds

$$\|u^\varepsilon(t)\|_{L^\infty(\Omega)} \leq 1 - \delta_{\varepsilon,\eta,\xi},\ \forall t \geq \xi, \tag{2.73}$$

where the constant $\delta_{\varepsilon,\eta,\xi} \in (0,1)$ is independent of t and u^ε. Furthermore, if

$$\|u_0\|_{L^\infty(\Omega)} \leq 1 - \delta_0, \tag{2.74}$$

for some $\delta_0 \in (0,1)$, then

$$\|u^\varepsilon(t)\|_{L^\infty(\Omega)} \leq 1 - \delta'_{\varepsilon,\eta,\delta_0,\|u_0\|_{D^\varepsilon_m}},\ \forall t \geq 0, \tag{2.75}$$

where

$$\|q\|^2_{D^\varepsilon_m} = \|q\|^2_{H^2(\Omega)} + \|f(q)\|^2_{L^2(\Omega)} + \varepsilon\|\phi\|^2_{L^2(\Omega)} + \|\phi\|^2_{H^{-1}(\Omega)}$$

and the constant $\delta'_{\varepsilon,\eta,\delta_0,\|u_0\|_{D^\varepsilon_m}} \in (0,1)$ *is independent of* t *and* u^ε.

Remark 2.11. It follows from the above that, if $\|u_0\|_{L^\infty(\Omega)} \leq 1$, then any solution u^ε to the viscous Cahn–Hilliard equation is a priori strictly separated from the singularities of f as soon as $t > 0$. Furthermore, if the initial datum is strictly separated from ± 1, then u^ε remains uniformly a priori strictly separated from ± 1 for all times.

Remark 2.12. (i) Unfortunately, both constants $\delta_{\varepsilon,\eta,\xi}$ and $\delta'_{\varepsilon,\eta,\delta_0,\|u_0\|_{D^\varepsilon_m}}$ tend a priori to 0^+ as $\varepsilon \to 0^+$, so that the above theorem does not say anything on strict separation properties for the solutions to the Cahn–Hilliard equation.

(ii) Actually, we can prove similar strict separation properties for the solutions to the Cahn–Hilliard equation if we further assume that

$$|f'(s)| \leq c(|f(s)|^2 + 1), \ s \in (-1,1),$$

for some positive constant c. In particular, this inequality holds if f has a growth of the form $\frac{\pm 1}{(1-s^2)^p}$, $p \geq 1$ (we can actually improve this and take $p > \frac{3}{7}$, see [301]), close to ± 1, but does not hold for the physically relevant logarithmic potentials.

(iii) In one space dimension, owing to the continuous embedding $H^1(\Omega) \subset C(\overline{\Omega})$, we can easily prove the above strict separation properties for the Cahn–Hilliard equation. Furthermore, in two space dimensions, using the embedding of $H^1(\Omega)$ into a proper Orlicz space, we can prove these properties, provided that

$$|f'(s)| \leq e^{c|f(s)|+c'}, \ s \in (-1,1),$$

for some positive constants c and c' (see [301] for details). In particular, the physically relevant logarithmic potentials satisfy these assumptions. Now, in three space dimensions, the strict separation from the singularities is an open problem for the thermodynamically relevant logarithmic nonlinear terms.

An alternative proof, in two space dimensions, based again on the Orlicz embedding and also valid for the Cahn–Hilliard–Oono equation, was given in [192] (see also [119, 165, 193, 195]). There, in order to avoid dealing with the the viscous Cahn–Hilliard equation and differentiating the equation for u with respect to time, we used instead proper truncations and difference quotients. Having the strict separation property (for, say, $t \geq 2$ in our case), we can then prove the existence of finite-dimensional attractors and the convergence of single trajectories to steady states.

Remark 2.13. We refer the reader to [129] for the study of the Cahn–Hilliard system with a singular potential in unbounded cylindrical domains (in that case, the equations are endowed with Dirichlet boundary conditions).

2.4. Dynamic boundary conditions

The question of how the phase separation process (i.e., the spinodal decomposition) is influenced by the presence of walls has gained much attention (see [155, 156, 246] and the references therein). This problem has mainly been studied for polymer mixtures (although it should also be important for other systems, such as binary metallic alloys): from a technological point of view, binary polymer mixtures are particularly interesting, since the occurring structures during the phase separation process may be frozen by a rapid quench into the glassy state; microstructures at surfaces on very small length scales can be produced in this way.

We also recall that the usual variational boundary condition $\frac{\partial u}{\partial \nu} = 0$ on the boundary yields that the interface is orthogonal to the boundary, meaning that the contact line, when the interface between the two components meets the walls, is static, which is not reasonable in many situations. This is the case, e.g., for mixtures of two immiscible fluids: in that case, the contact angle should be dynamic, due to the movements of the fluids. This can also be the case in the context of binary alloys, whence the need to define dynamic boundary conditions for the Cahn–Hilliard equation.

In that case, we again write that there is no mass flux at the boundary (i.e., that (2.1) still holds). Then, in order to obtain the second boundary condition, following the phenomenological derivation of the Cahn–Hilliard system, we consider, in addition to the usual Ginzburg–Landau free energy and assuming that the interactions with the walls are short-ranged, a surface free energy of the form

$$\Psi_\Gamma(u, \nabla_\Gamma u) = \int_\Gamma (\frac{\alpha_\Gamma}{2}|\nabla_\Gamma u|^2 + G(u))\, d\sigma,\ \alpha_\Gamma > 0, \tag{2.76}$$

where ∇_Γ is the surface gradient and G is a surface potential. Thus, the total free energy of the system reads

$$\Psi = \Psi_\Omega + \Psi_\Gamma. \tag{2.77}$$

Writing finally that the system tends to minimize the excess surface energy, we are led to postulate the following boundary condition:

$$\frac{1}{d}\frac{\partial u}{\partial t} - \alpha_\Gamma \Delta_\Gamma u + g(u) + \alpha \frac{\partial u}{\partial \nu} = 0,\ \text{on } \Gamma, \tag{2.78}$$

i.e., there is a relaxation dynamics on the boundary (note that it follows from the boundary conditions that

$$\frac{d\Psi}{dt} = -\frac{1}{\kappa}\|\frac{\partial u}{\partial t}\|^2_{H^{-1}(\Omega)} - \frac{1}{d}\|\frac{\partial u}{\partial t}\|^2_{L^2(\Gamma)} \leq 0),$$

where Δ_Γ is the Laplace–Beltrami operator, $g = G'$ and $d > 0$ is some relaxation parameter, which is usually referred to as dynamic boundary condition, in the sense that the kinetics, i.e., $\frac{\partial u}{\partial t}$, appears explicitly. Furthermore, in the original derivation, one has $G(s) = \frac{1}{2}a_\Gamma s^2 - b_\Gamma s$, where $a_\Gamma > 0$ accounts for a modification of the effective interaction between the components at the walls and b_Γ characterizes the possible preferential attraction (or repulsion) of one of the components by the walls (when b_Γ vanishes, there is no preferential attraction). We also refer the reader to [26, 157] for other physical derivations of the dynamic boundary condition, obtained by taking the continuum limit of lattice models within a direct mean-field approximation and by applying a density functional theory, respectively, to [328] for

the derivation of dynamic boundary conditions in the context of two-phase fluid flows and to [337,341] for an approach based on concentrated capacity.

Remark 2.14. Actually, it would seem more reasonable, in the case of nonpermeable walls, to write the conservation of mass both in the bulk Ω and on the boundary Γ, i.e.,

$$\frac{d}{dt}(\int_\Omega u\,dx + \int_\Gamma u\,d\sigma) = 0.$$

Indeed, due to the interactions with the walls, one should expect some mass on the boundary. We assume that the first equation of (1.1) still holds. Then, writing that

$$\mu = \partial_u \Psi,$$

where ∂ is the variational derivative mentioned above (note that, in the original derivation, one has $\mu = \partial_u \Psi_\Omega$), we obtain the second equation of (1.1), together with the boundary condition

$$\mu = -\alpha_\Gamma \Delta_\Gamma u + g(u) + \alpha \frac{\partial u}{\partial \nu}, \text{ on } \Gamma.$$

We now note that, owing to the first equation of (1.1), the above mass conservation reads

$$\int_\Gamma (\frac{\partial u}{\partial t} + \kappa \frac{\partial \mu}{\partial \nu})\,d\sigma = 0.$$

A class of boundary conditions which ensure this mass conservation reads

$$\frac{\partial u}{\partial t} + \beta_\Gamma \Delta_\Gamma u + \kappa \frac{\partial \mu}{\partial \nu} = 0, \text{ on } \Gamma,\ \beta_\Gamma \geq 0.$$

We can thus see that, when $\beta_\Gamma > 0$, we also have a Cahn–Hilliard type system on the boundary. Note that, when $\beta_\Gamma = 0$, it follows from the above that

$$\frac{d\Psi}{dt} = -\kappa \|\nabla \mu\|^2_{L^2(\Omega)^n} \leq 0.$$

We refer the reader to [77, 197] for the study of this problem (see also [282, 291] for higher-order models and [163, 164, 172] for similar dynamic boundary conditions in the case of semipermeable walls).

Again, for regular nonlinear terms, the problem is well understood from a mathematical point of view (see, e.g., [88, 163, 164, 172, 302, 326, 329, 368]).

The first proof of existence of solutions to the Cahn–Hilliard equation with singular (and, in particular, logarithmic) potentials and dynamic boundary conditions is given in [190] (see also [191]), assuming that the (regular in our case; see [104, 190] for singular surface nonlinear terms) surface nonlinearity g has the right sign at the singular points of the bulk nonlinearity f, namely,

$$\pm\, g(\pm 1) > 0 \tag{2.79}$$

(see also [80] for a similar result for the Caginalp phase-field system; note that the Cahn–Hilliard equation can also be derived as a singular limit of the Caginalp phase-field system, see [54–56]).

Roughly speaking, these conditions force the order parameter to stay away from the pure states ± 1 on the boundary.

A first natural question is what happens when the above sign conditions are not satisfied.

An important feature of the Cahn–Hilliard equation with a singular potential and dynamic boundary conditions is that one can have nonexistence of classical (i.e., in the sense of distributions) solutions (of course, when the sign conditions (2.79) are not satisfied), already in one space dimension.

We will illustrate this by considering the following simple scalar ODE:

$$y'' - f(y) = 0, \text{ in } (-1, 1), \ y'(\pm 1) = K > 0. \tag{2.80}$$

We assume that f is odd and singular at ± 1 and that F has finite limits at ± 1 (here, F is any antiderivative of f). In particular, these assumptions are satisfied by the physically relevant logarithmic potentials.

Then, when K is small, one has the existence and uniqueness of a (classical) solution to (2.80) which is separated from the singular values of f (i.e., $\|y\|_{L^\infty(-1,1)} < 1$). Furthermore, it follows from standard interior regularity estimates that

$$|y'(x)| \leq c_0, \ x \in (-\frac{1}{2}, \frac{1}{2}), \tag{2.81}$$

where the positive constant c_0 is independent of K.

Multiplying (2.80) by y' and integrating over $(0, 1)$, we obtain

$$|\frac{1}{2}K^2 - F(y(1))| \leq c_1, \tag{2.82}$$

where, owing to (2.81), the positive constant c_1 is independent of K. Since F is bounded, independently of K, this shows that the above inequality cannot hold when K is large, meaning that there cannot be a classical solution to (2.80).

Finally, noting that y is odd, we can rewrite (2.80) in the equivalent form

$$y'' - f(y) = \langle y'' - f(y) \rangle, \text{ in } (-1, 1), \ y'(\pm 1) = K,$$

which corresponds to the one-dimensional stationary Cahn–Hilliard equation (with $\kappa = \alpha = 1$) with dynamic boundary conditions (with surface potential $g \equiv -K$; note that, in one space dimension, the Laplace–Beltrami operator does not make sense and does not appear).

Thus, when the sign conditions are not satisfied, we should expect to have nonexistence of classical solutions. However, we will see below that, approximating the singular nonlinear term by regular ones, we can prove that, at least for a subsequence, the corresponding solutions converge to some limit function.

We first rewrite the problem in the following form (where, for simplicity, we have set all physical constants equal to 1):

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta \mu, \ \frac{\partial \mu}{\partial \nu}|_\Gamma = 0, \\ \mu = -\Delta u + f_0(u) + \lambda u, \ \lambda \in \mathbb{R}, \\ \frac{\partial v}{\partial t} - \Delta_\Gamma v + g_0(v) + v + \frac{\partial u}{\partial \nu} = 0, \text{ on } \Gamma, \ v = u|_\Gamma, \end{cases} \tag{2.83}$$

where $f = f_0 + \lambda I$ and $g = g_0 + I$.

We then make the following assumptions:

$$f_0 \in C^2(-1,1),\ f_0(0) = 0,\ f_0' \geq 0,\ \mathrm{sgn}(s) f_0''(s) \geq 0,\ s \in (-1,1), \tag{2.84}$$

$$\lim_{s\to\pm 1} f_0(s) = \pm\infty,\ \lim_{s\to\pm 1} f_0'(s) = +\infty, \tag{2.85}$$

$$g_0 \in C^2(\mathbb{R}),\ \|g_0\|_{C^2(\mathbb{R})} < +\infty. \tag{2.86}$$

We can note that the physically relevant logarithmic functions f (1.4) can indeed be decomposed into a sum $f_0 + \lambda I$, where f_0 satisfies the above assumptions.

We now introduce, as above, the following regular approximations of f_0, for $N \in \mathbb{N}$:

$$f_{0,N}(s) = \begin{cases} f_0(s),\ |s| \leq 1 - \frac{1}{N}, \\ f_0(1-\frac{1}{N}) + f_0'(1-\frac{1}{N})(s-1+\frac{1}{N}),\ s > 1-\frac{1}{N}, \\ f_0(-1+\frac{1}{N}) + f_0'(-1+\frac{1}{N})(s+1-\frac{1}{N}),\ s < -1+\frac{1}{N}, \end{cases} \tag{2.87}$$

and we consider (2.83) in which f_0 is replaced by $f_{0,N}$, $N \in \mathbb{N}$.

The existence, uniqueness and regularity of the solution u_N to this regularized problem is clear, since the nonlinear term is now regular (see [302]). Furthermore, we have the following estimates, for N large enough:

$$\begin{aligned} &\|u_N(t)\|^2_{C^\eta(\Omega)} + \|u_N(t)\|^2_{H^2(\Gamma)} + \|u_N(t)\|^2_{H^2(\Omega_\varepsilon)} + \|u_N(t)\|^2_{H^1(\Omega)} + \|\frac{\partial u_N}{\partial t}(t)\|^2_{H^{-1}(\Omega)} \\ &+\|\frac{\partial u_N}{\partial t}(t)\|^2_{L^2(\Gamma)} + \|\nabla D_\tau u_N(t)\|^2_{L^2(\Omega)^{(n-1)n}} + \|f_{0,N}(u_N(t))\|_{L^1(\Omega)} \\ &+ \int_t^{t+1} (\|\frac{\partial u_N}{\partial t}(s)\|^2_{H^1(\Omega)} + \|\frac{\partial u_N}{\partial t}(s)\|^2_{H^1(\Gamma)})\, ds \\ &\leq c_1 e^{-c_2 t}(1 + \|u_N(0)\|^2_{H^1(\Omega)} + \|u_N(0)\|^2_{H^1(\Gamma)} + \|\frac{\partial u_N}{\partial t}(0)\|^2_{H^{-1}(\Omega)} + \|\frac{\partial u_N}{\partial t}(0)\|^2_{L^2(\Gamma)})^2 + c_3, \end{aligned} \tag{2.88}$$

where $\Omega_\varepsilon = \{x \in \Omega,\ \mathrm{dist}(x,\Gamma) > \varepsilon\}$, $\varepsilon > 0$, $D_\tau u_N = \nabla u_N - \frac{\partial u_N}{\partial \nu}\nu$ is the tangential part of the gradient (here, ν also denotes a smooth extension of the unit outer normal to the boundary in Ω) and the positive constants $\eta < \frac{1}{4}$ and c_i, $i = 1,\ ...,\ 3$, are independent of N.

The only difficulty, to derive these estimates, is to obtain the estimate on the tangential part of the gradient and the interior H^2-estimate. This is achieved by a proper variant of the nonlinear localization technique, see [305] for details.

Remark 2.15. Actually, u_N also belongs to $H^2(\Omega)$, but the H^2-norm of u_N depends a priori on N and such a regularity does not pass to the limit.

We also have the following (parabolic) regularization property on the solution u_N to the regularized problem:

$$\|\frac{\partial u_N}{\partial t}(t)\|^2_{H^{-1}(\Omega)} + \|\frac{\partial u_N}{\partial t}(t)\|^2_{L^2(\Gamma)} \tag{2.89}$$

$$\leq \frac{c}{t}(1 + \|u_N(0) - \langle u_N(0)\rangle\|^2_{H^{-1}(\Omega)} + \|u_N(0)\|^2_{L^2(\Gamma)}),\ t \in (0, 1],$$

where the positive constant c is independent of N.

Finally, for any two solutions u_1 and u_2 to the regularized problem (for simplicity, we omit the index N here), with initial data having the same average (note that we still have the conservation of the average when considering dynamic boundary conditions), there holds

$$\begin{aligned} &\|u_1(t) - u_2(t)\|_{H^{-1}(\Omega)} + \|u_1(t) - u_2(t)\|_{L^2(\Gamma)} \\ &\leq ce^{c't}(\|u_1(0) - u_2(0)\|_{H^{-1}(\Omega)} + \|u_1(0) - u_2(0)\|_{L^2(\Gamma)}),\ \forall t \geq 0, \end{aligned} \tag{2.90}$$

where the positive constants c and c' are independent of t, N and u_1 and u_2.

In particular, it follows from (2.88) that, at least for a subsequence which we do not relabel, u_N converges (at least weakly) to some function u in the corresponding spaces. Now, recalling that one expects nonexistence of classical solutions in general, u cannot be a (classical) solution to the Cahn–Hilliard system (2.83) with the original singular nonlinearity f_0 in general. However, we will see below that u_N converges to the solution to some variational inequality, derived from (2.83).

Our aim now is to pass rigorously to the limit in the regularized problems associated with (2.83). As already mentioned, as the limit is not a classical solution to (2.83) in general, we first need to define a proper weak formulation of the problem. More precisely, this will be done by considering a variational inequality (see also [197, 209] for a very close approach, but at an abstract level, i.e., based on duality arguments).

To do so, we first introduce the bilinear form

$$B(w, z) = ((\nabla w, \nabla z))_\Omega + \lambda((w, z))_\Omega + L(((-\Delta)^{-1}\overline{w}, \overline{z}))_\Omega + ((\nabla_\Gamma w, \nabla_\Gamma z))_\Gamma,$$

$(w, z) \in H^1(\Omega) \otimes H^1(\Gamma) = \{q \in H^1(\Omega),\ q|_\Gamma \in H^1(\Gamma)\}$, where the positive constant L is chosen such that the following coercivity relation holds:

$$\|\nabla w\|^2_{L^2(\Omega)^n} + \lambda\|w\|^2_{L^2(\Omega)} + L\|w\|^2_{H^{-1}(\Omega)} \geq \frac{1}{2}\|w\|^2_{H^1(\Omega)},\ \forall w \in H^1(\Omega) \text{ such that } \langle w\rangle = 0.$$

Furthermore, $-\Delta$ denotes again the minus Laplace operator with Neumann boundary conditions and acting on functions with null average, $\overline{w} = w - \langle w\rangle$ and $((\cdot,\cdot))_\Omega$ and $((\cdot,\cdot))_\Gamma$ denote the scalar products in $L^2(\Omega)$ and $L^2(\Gamma)$, respectively.

We then rewrite the problem in the following equivalent form:

$$\begin{cases} (-\Delta)^{-1}\frac{\partial u}{\partial t} - \Delta u + f_0(u) + \lambda u - \langle\mu\rangle = 0, \\ \mu = -\Delta u + f_0(u) + \lambda u, \\ \frac{\partial v}{\partial t} - \Delta_\Gamma v + g(v) + \frac{\partial u}{\partial \nu} = 0,\ \text{on } \Gamma,\ v = u|_\Gamma, \\ u|_{t=0} = u_0,\ v|_{t=0} = v_0. \end{cases} \tag{2.91}$$

We multiply the first equation of (2.91) by $u - w$, where $w = w(x)$ is such that

$$\langle u(t) - w\rangle = 0,\ \forall t \geq 0,$$

and have, owing to the boundary conditions,

$$(((-\Delta)^{-1}\frac{\partial u}{\partial t}, u-w))_\Omega + ((\frac{\partial u}{\partial t}, u-w))_\Gamma + B(u, u-w) + ((f_0(u), u-w))_\Omega$$
$$= L((u, (-\Delta)^{-1}(u-w)))_\Omega - ((g(u), u-w))_\Gamma.$$

Noting that B is positive and f_0 is monotone increasing, we finally obtain the following variational inequality:

$$(((-\Delta)^{-1}\frac{\partial u}{\partial t}, u-w))_\Omega + ((\frac{\partial u}{\partial t}, u-w))_\Gamma + B(w, u-w) + ((f_0(w), u-w))_\Omega \tag{2.92}$$
$$\leq L((u, (-\Delta)^{-1}(u-w)))_\Omega - ((g(u), u-w))_\Gamma$$

(what is important here, when considering a singular nonlinear term f_0, is that this function acts on the test functions).

We now introduce the phase space

$$\Phi = \{(q, r) \in L^\infty(\Omega) \times L^\infty(\Gamma),\ \|q\|_{L^\infty(\Omega)} \leq 1,\ \|r\|_{L^\infty(\Gamma)} \leq 1\}$$

and are in a position to give the definition of a variational (weak) solution.

Definition 2.16. Let (u_0, v_0) belong to Φ. Then, a pair (u, v) is a variational solution to (2.91) if
(i) $(u, v) \in C([0, +\infty); H^{-1}(\Omega) \times L^2(\Gamma)) \cap L^2(0, T; H^1(\Omega) \times H^1(\Gamma)),\ \forall T > 0$.
(ii) $(\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}) \in L^2(\tau, T; H^{-1}(\Omega) \times L^2(\Gamma)), \forall 0 < \tau < T$.
(iii) $f(u) \in L^1((0, T) \times \Omega),\ \forall T > 0$.
(iv) $-1 < u(t, x) < 1,$ for almost every $(t, x) \in \mathbb{R}^+ \times \Omega$.
(v) $u(0) = u_0$, $v(0) = v_0$.
(vi) $\langle u(t)\rangle = \langle u_0\rangle,\ \forall t \geq 0$.
(vii) $u(t)|_\Gamma = v(t)$, for almost every $t > 0$.
(viii) The variational inequality (2.92) is satisfied for almost every $t > 0$ and for every test function $w = w(x)$ such that $w \in H^1(\Omega) \otimes H^1(\Gamma)$, $f(w) \in L^1(\Omega)$ and $\langle w\rangle = \langle u_0\rangle$.

Remark 2.17. Of course, a classical solution is a variational one. Furthermore, the notion of a variational solution also makes sense when f_0 is regular and, in that case, the two notions of solutions are equivalent (see also [130]).

Remark 2.18. We can note that, in the above definition, $u(t)|_\Gamma = v(t)$ only for $t > 0$ and this condition does not necessarily hold for the initial data. However, as soon as $t > 0$, v can be found, once u is known. This also justifies that we wrote the variational inequality (2.92) in terms of u only.

We can now write the variational inequality (2.92) with u replaced by (a proper subsequence of) the solutions u_N to the regularized problems (when u_0 is not strictly separated from ± 1 or $u_0|_\Gamma \neq v_0$, we approximate the initial data by a sequence of smooth functions which satisfy these conditions) and, passing to the limit (owing to (2.88); one can proceed as in the proof of Theorem 2.6 to prove the separation property here), we have the

Theorem 2.19. *For every pair of initial data* $(u_0, v_0) \in \Phi$, (2.91) *possesses a unique variational solution* (u, v) *which is the limit of a sequence of solutions to the regularized problems and which satisfies* (2.88)*, the regularization property* (2.89) *and the Lipschitz continuity property* (2.90).

This result allows to define a semigroup $S(t)$ acting on the phase space Φ and associated with the variational solutions to (2.91). Furthermore, this semigroup is Lipschitz continuous in the following sense:

$$\|S(t)(u_1, v_1) - S(t)(u_2, v_2)\|_{H^{-1}(\Omega)\times L^2(\Gamma)} \leq ce^{c't}\|(u_1, v_1) - (u_2, v_2)\|_{H^{-1}(\Omega)\times L^2(\Gamma)},\ t \geq 0, \tag{2.93}$$

$\forall (u_1, v_1),\ (u_2, v_2) \in \Phi$ such that $\langle u_1\rangle = \langle u_2\rangle$.

Remark 2.20. We refer the reader to [305] for the study of finite-dimensional attractors for the semigroup $S(t)$.

Now, as already mentioned several times, a variational solution does not necessarily solve the Cahn–Hilliard system in the usual sense (this would be true if we had an H^2-regularity on u, but, here, we a priori only have an interior H^2-regularity, together with an L^2-estimate on the gradient of the tangential derivatives). Actually, we can be more precise. Indeed, we can prove that a variational solution solves the bulk equation, namely,

$$(-\Delta)^{-1}\frac{\partial u}{\partial t} - \Delta u + f_0(u) + \lambda u - \langle\mu\rangle = 0,$$

in the sense of distributions (and almost everywhere). However, it does not necessarily satisfy the dynamic boundary condition.

Again, we can be more precise. First, it follows from (2.88) (for u) that the trace

$$[\frac{\partial u}{\partial \nu}]_{\text{int}} = \frac{\partial u}{\partial \nu}$$

exists, say, in $L^\infty(\tau, T; L^1(\Gamma))$, $0 < \tau < T$. Furthermore, the (proper subsequence of the) solutions to the regularized problems satisfy the dynamic boundary condition

$$\frac{\partial v_N}{\partial t} - \Delta_\Gamma v_N + g(v_N) + \frac{\partial u_N}{\partial \nu} = 0,\ \text{on } \Gamma,\ v_N = u_N|_\Gamma,$$

in $L^\infty(\tau, T; L^2(\Gamma))$, $0 < \tau < T$. We can then pass to the limit in this equation, which yields that the limit

$$[\frac{\partial u}{\partial \nu}]_{\text{ext}} = \lim_{k\to+\infty}\frac{\partial u_N}{\partial \nu}$$

exists in $L^\infty(\tau, T; L^2(\Omega))$ weak star, $0 < \tau < T$, whence, at the limit

$$\frac{\partial v}{\partial t} - \Delta_\Gamma v + g(v) + [\frac{\partial u}{\partial \nu}]_{\text{ext}} = 0,\ \text{on } \Gamma,$$

almost everywhere, where $[\frac{\partial u}{\partial \nu}]_{\text{ext}}$ and $[\frac{\partial u}{\partial \nu}]_{\text{int}}$ do not necessarily coincide; in particular, a variational solution is a classical one when these two quantities coincide almost everywhere on the boundary.

Remark 2.21. Coming back to the scalar ODE considered in the beginning of this subsection, namely,

$$y'' - f(y) = 0, \text{ in } (-1, 1), \; y'(\pm 1) = K > 0,$$

we can prove that there exists a critical value K_0 of K such that, if $K > K_0$, there is no classical solution. However, there exists a variational solution which is solution to the ODE

$$y'' - f(y) = 0, \text{ in } (-1, 1), \; y(\pm 1) = \pm 1,$$

and, in that case,

$$y'|_{x=\pm 1} \neq K.$$

It is now natural and important to find sufficient conditions which ensure that a variational solution is a classical one (one such sufficient condition being the sign conditions (2.79) mentioned above). We saw that this is the case when the two quantities $[\frac{\partial u}{\partial \nu}]_{\text{ext}}$ and $[\frac{\partial u}{\partial \nu}]_{\text{int}}$ coincide almost everywhere on Γ. In particular, this is the case when $u(t)$ belongs to $H^2(\Omega)$, for almost every $t > 0$, which, in turn, is related to the (strict) separation from the singularities. Indeed, we have the following result.

Theorem 2.22. *Let (u, v) be a variational solution and set, for $\delta \in (0, 1)$ and $T > 0$,*

$$\Omega_\delta(T) = \{x \in \Omega, \; |u(T, x)| < 1 - \delta\}.$$

Then, $u(T) \in H^2(\Omega_\delta(T))$ and

$$\|u(T)\|_{H^2(\Omega_\delta(T))} \leq c_{\delta,T},$$

where $c_{\delta,T}$ is independent of u.

In particular, it follows from Theorem 2.22 that $u(t)$ is H^2-regular in each subdomain in which it is strictly separated from ± 1. Furthermore, a consequence of this result is that, if

$$|u(t, x)| < 1, \text{ for almost every } (t, x) \in \mathbb{R}^+ \times \Gamma, \tag{2.94}$$

then

$$[\frac{\partial u}{\partial \nu}]_{\text{ext}} = [\frac{\partial u}{\partial \nu}]_{\text{int}}, \text{ for almost every } (t, x) \in \mathbb{R}^+ \times \Gamma,$$

and a variational solution is a classical one. We thus finally see that the existence of a classical solution is related to the separation of u from the singularities of the nonlinear term, now on the boundary Γ.

Such a separation holds when f_0 has sufficiently strong singularities (see also [78] for a similar result for the Caginalp phase-field system). More precisely, we have the

Theorem 2.23. *We assume that*

$$\lim_{s \to \pm 1} F_0(s) = +\infty,$$

where F_0 is any antiderivative of f_0. Then, (2.94) *holds and a variational solution is a classical one.*

In particular, this holds when f_0 has a growth of the form

$$\frac{\pm 1}{(1-s^2)^p},\ p>1,$$

close to the singular points ± 1. Unfortunately, for the relevant logarithmic potentials, F_0 is bounded, so that this theorem cannot be applied. In that case, we can have $|u(t,x)| = 1$ on a set with nonzero measure on the boundary, or even on the whole boundary (see also the scalar ODE considered above).

We finally recall the

Theorem 2.24. *We assume that*

$$\pm g(\pm 1) > 0.$$

Then, a variational solution is a classical one.

As already mentioned, the sign conditions force the order parameter to stay away from the pure states on the boundary.

2.5. Concluding remarks

We mention in this subsection several important generalizations of the Cahn–Hilliard equation.

A first one consists in studying systems of Cahn–Hilliard equations to describe phase separation in multicomponent alloys (see [51, 91, 107, 139, 140, 146, 178–180, 298]).

We also mention the stochastic Cahn–Hilliard equation (also called the Cahn–Hilliard–Cook equation) which takes into account thermal fluctuations (see [29, 30, 32, 33, 63, 110, 112, 117, 118, 133, 202, 203, 219]).

Then, an important generalization of the Cahn–Hilliard equation is the viscous Cahn–Hilliard equation which accounts for viscosity effects in the phase separation of polymer/polymer systems (see [16, 65, 93, 142, 310]). The viscous Cahn–Hilliard equation can also be seen as a particular case of the generalizations proposed by M. Gurtin in [221] (which, in particular, account for anisotropy) and which are based on a microforce balance, i.e., a new balance law for interactions at a microscopic level (see [36, 37, 39, 79, 130, 131, 196, 206, 284–286, 294, 295, 299, 304, 332–334, 364] for the mathematical analysis); we also refer the reader to yet another approach proposed by P. Podio–Guidugli in [325] and studied in, e.g., [97–100, 105].

Another important generalization of the Cahn–Hilliard equation is the hyperbolic relaxation of the equation, proposed in [174–177, 260] to model the early stages of spinodal decomposition in certain glasses (see also [38, 184, 185, 210–212, 340] for the mathematical analysis and [338, 339] for the hyperbolic relaxation of the Cahn–Hilliard–Oono equation in the whole space). Actually, the hyperbolic relaxation of the equation is a particular case of more general memory relaxations (for an exponentially decreasing memory kernel) which were studied, e.g., in [106, 109, 186, 187] (see also [327]).

We also mention the convective Cahn–Hilliard equation which describes the dynamics of driven systems such as faceting of growing thermodynamically unstable crystal surfaces (see [126–128, 198, 267, 361] for the mathematical analysis).

It is important to note that, in realistic physical systems, quenches are usually carried out over a finite period of time, so that phase separation can begin before the final quenching is reached. It is thus

important to consider nonisothermal Cahn–Hilliard models. Such models were derived and studied in [8, 9, 170, 171, 297, 346].

The Cahn–Hilliard equation can be coupled with the Allen–Cahn equation which describes the ordering of atoms during the phase separation process (see [7]). This problem was studied, e.g., in [28, 114, 269, 296, 312, 374].

It can also be coupled with the equations for elasticity or viscoelasticity, to account for mechanical effects (see, e.g., [12, 27, 34, 35, 64, 121, 179–181, 284, 285, 318–321, 331]).

We also mention the coupling of the Cahn–Hilliard equation with the Navier–Stokes equations in the context of two-phase (multiphase) flows (see, e.g., [1, 3, 43–47, 49, 61, 62, 86, 94, 158, 166, 167, 169, 195, 222, 236, 247, 250, 252, 268, 273, 372, 375]) and some related models such as the Cahn–Hilliard–Hele–Shaw and Cahn–Hilliard–Brinkman equations (see, e.g., [42, 108, 119, 120, 153, 193, 225, 359, 360, 365, 371]). Related models can also be used to model tumor growth (see, e.g., [96, 101–103, 113, 159, 182, 237, 272]).

We finally refer the reader to, e.g., [5, 6, 10, 11, 16–22, 40, 46–50, 52, 67, 69, 71, 87, 89, 111, 122, 135–137, 141, 147, 150–152, 154, 159, 161, 183, 199, 204, 205, 207, 220, 223, 224, 226, 230, 231, 242, 243, 246–252, 261–263, 265, 266, 270, 275, 280, 283, 306, 330, 336, 344, 345, 347, 351, 358, 362, 366, 369, 377, 378] for the numerical analysis and simulations of the Cahn–Hilliard equation (and several of its generalizations).

3. The Cahn–Hilliard equation with a proliferation term

We consider the following initial and boundary value problem in a bounded and regular domain Ω of $\mathbb{R}^n$, $n = 1$, 2 or 3, with boundary Γ:

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) + g(u) = 0, \tag{3.1}$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0, \text{ on } \Gamma, \tag{3.2}$$

$$u|_{t=0} = u_0, \tag{3.3}$$

where f is a smooth function defined on $\mathbb{R}$ (as already mentioned, typically, $f(s) = s^3 - s$) and

$$g(s) = \sum_{i=0}^{q} b_i s^i, \ b_q \neq 0, \ q \geq 2. \tag{3.4}$$

The main feature of (3.1)-(3.2), together with a function g as in (3.4), is that we can have blow up in finite time.

In order to exhibit blow up in finite time, we look for spatially homogeneous solutions, i.e., solutions of the form

$$u(t, x) = y(t),$$

whence the ODE

$$y' + \sum_{i=0}^{q} b_i y^i = 0. \tag{3.5}$$

Let $x_0 < x_1 < \cdots < x_k$ be the real roots of g (we assume in what follows that $k \geq 1$, though the case $k = 0$ is also contained in Theorem 3.1 below; see also Remark 3.2). We thus have

$$g(s) = b_q \prod_{i=1}^{k} (s - x_i)^{\alpha_i} \prod_{i=1}^{r} (s^2 + \lambda_i s + \beta_i)^{u_i} \tag{3.6}$$

and

$$\frac{1}{g(s)} = \frac{1}{b_q}[\sum_{i=1}^{k}(\sum_{j=1}^{\alpha_i} \frac{\alpha_{i,j}}{(s - x_i)^j}) + \sum_{i=1}^{r}(\sum_{j=1}^{u_i} \frac{b_{i,j}s + c_{i,j}}{(s^2 + \lambda_i s + \beta_i)^j})] \tag{3.7}$$

(here, we assume, without loss of generality, that $r \geq 1$).

Therefore, (3.5) is equivalent to

$$\psi(y) = b_q(-t + \kappa), \tag{3.8}$$

where ψ is an antiderivative of $\frac{b_q}{g(s)}$ and

$$\kappa = \frac{1}{b_q}\psi(y_0),\ y_0 = y(0). \tag{3.9}$$

Let $\tilde{k}$ be the number of roots of g which have an odd order (i.e., for which α_i is odd).

We have the

Theorem 3.1. *We assume that $\tilde{k}$ is even. Then, we have blow up in finite time.*

Proof. We first note that ψ has finite limits $\lambda_\pm$ as s tends to $\pm\infty$. Indeed, the only terms in (3.7) which yield infinite limits when taking an antiderivative can be written as

$$\sum_{i=1}^{k} \frac{\alpha_{i,1}}{s - x_i} + \sum_{i=1}^{r} \frac{b_{i,1}s + c_{i,1}}{s^2 + \lambda_i s + \beta_i}.$$

Then, when taking an antiderivative, we need to deal with the function

$$h(s) = \sum_{i=1}^{k} \alpha_{i,1} \ln|s - x_i| + \sum_{i=1}^{r} \frac{b_{i,1}}{2} \ln(s^2 + \lambda_i s + \beta_i).$$

Now, noting that it follows from (3.7) that, necessarily,

$$\sum_{i=1}^{k} \alpha_{i,1} + \sum_{i=1}^{r} b_{i,1} = 0 \tag{3.10}$$

and writing

$$h(s) = \ln(\prod_{i=1}^{k} |s - x_i|^{\alpha_{i,1}} \prod_{i=1}^{r} (s^2 + \lambda_i s + \beta_i)^{\frac{b_{i,1}}{2}}),$$

it follows from (3.10) that

$$\lim_{s \to \pm\infty} h(s) = 0. \tag{3.11}$$

Furthermore, since $\tilde{k}$ is even, then ψ is monotone increasing on $]-\infty, x_1[$ and $\psi(]-\infty, x_1[) =]\lambda_-, +\infty[$. Similarly, ψ is monotone increasing on $]x_k, +\infty[$ and $\psi(]x_k, +\infty[) =]-\infty, \lambda_+[$.

Let us first assume that $b_q > 0$. In that case, we take $y_0 < x_1$ and there is a local (in time) solution to (3.5), with initial datum y_0 (as long as $y(t) < x_1$). Moreover,

$$y(t) = \psi^{-1}(b_q(-t + \kappa)) \tag{3.12}$$

and this solution exists as long as

$$b_q(-t + \kappa) > \lambda_-,$$

i.e.,

$$t < \kappa - \frac{\lambda_-}{b_q}, \tag{3.13}$$

meaning that we have blow up in finite time.

Similarly, if $b_q < 0$, we take $y_0 > x_k$ and we have a solution as long as

$$b_q(-t + \kappa) < \lambda_+,$$

i.e.,

$$t < \kappa - \frac{\lambda_+}{b_q}, \tag{3.14}$$

whence again blow up in finite time.

□

Remark 3.2. If $k = 0$, then ψ maps increasingly $\mathbb{R}$ onto $]\lambda_-, \lambda_+[$ and we can easily conclude, proceeding as above and taking y_0 arbitrarily.

Now, when $\tilde{k}$ is odd, we have the

Theorem 3.3. *We assume that $\tilde{k}$ is odd and $b_q < 0$. Then, we have blow up in finite time.*

Proof. Proceeding as above, we see that ψ is monotone decreasing on $]-\infty, x_1[$ and $\psi(]-\infty, x_1[) =]-\infty, \lambda_-[$. Taking $y_0 < x_1$, we have a solution to (3.5) as long as

$$b_q(-t + \kappa) < \lambda_-,$$

i.e.,

$$t < \kappa - \frac{\lambda_-}{b_q}, \tag{3.15}$$

meaning once more that we have blow up in finite time.

□

Remark 3.4. More generally, we have blow up in finite time whenever g is continuous on an interval $]-\infty, x_0[$ (resp., $]x_0, +\infty[$) and maps increasingly (resp., decreasingly) $]-\infty, x_0[$ (resp, $]x_0, +\infty[$) onto $]\lambda, +\infty[$. Here, λ is finite.

Example 3.5. We take

$$g(s) = \lambda s(s-1),\ \lambda > 0. \tag{3.16}$$

In that case, (3.1) has applications in wound healing and tumor growth and λ is a proliferation coefficient (see [244]). Here, $\tilde{k} = k = 2$ and $b_q = \lambda > 0$. It thus follows from Theorem 3.1 that we can have blow up in finite time (see also [85]). Furthermore, here, $x_0 = 0$, so that blow up in finite time occurs when $y_0 < 0$. We can note that, in this example, the biologically relevant interval is $[0, 1]$, so that a natural question is whether we can have blow up in finite time in (3.1) for initial data u_0 such that $u_0(x) \in [0, 1]$, a.e. in Ω. Numerical simulations performed in [85] suggest that this can indeed happen. Actually, what is important here is the choice of the nonlinear term f and, more precisely, the minima of the double-well potential F (we recall that $f = F'$).

Example 3.6. We take

$$g(s) = \frac{\lambda_d}{2}(1+s) - \lambda_g(1+s)^2(1-s)^2,\ \lambda_d,\ \lambda_g > 0. \tag{3.17}$$

In that case, (3.1) has biological applications and λ_d and λ_g are death and growth coefficients, respectively (see [14]). Furthermore, a study of the function g shows that either $\tilde{k} = k = 4$ or $\tilde{k} = 2$ and $k = 2$ or 3. Noting that $b_q = -\lambda_g < 0$, it follows from Theorem 3.1 that we can have blow up in finite time which occurs when $y_0 > 1$ (indeed, it is easy to show that, in this example, $x_k > 1$). Here, the biologically relevant interval is $[-1, 1]$ and, again, a natural question is whether we can have blow up in finite time for (3.1) when $u_0(x) \in [-1, 1]$, a.e. in Ω. Numerical simulations suggest that, in that case, the solutions remain in the biologically relevant interval (see [14, 149]).

Remark 3.7. Another interesting question is whether the solutions to (3.1) remain in the biologically relevant interval, assuming that the initial condition u_0 also belongs to this interval. It was proved in [85] that, for (3.16), this may not be the case (see also [324] for the Cahn–Hilliard equation). Let us now consider the function g defined in (3.17) and assume that $f(s) = s^3 - s$. In one space dimension, (3.1) then reads, with obvious notation,

$$u_t + u_{xxxx} - (u^3 - u)_{xx} + g(u) = 0. \tag{3.18}$$

We take the initial datum u_0 such that $u_0(x) = 1 - \frac{1+\lambda_d}{24}x^4$ in a neighborhood of 0 and extend it to a smooth function defined on $(-1, 1)$ and taking values in $[-1, 1]$. We note that $u_{0,x}(0) = u_{0,xx}(0) = 0$, while $u_{0,xxxx} = -1 - \lambda_d$. It thus follows from (3.18) that $u_t(0, 0) = 1$, so that

$$u_{0,t} = 1 + t + o(t).$$

Therefore, $u(0,t) > 1$ for $t > 0$ small, meaning that u does not stay in the biologically relevant interval.

Remark 3.8. The above results also show that we can have blow up in finite time for the reaction-diffusion equation

$$\frac{\partial u}{\partial t} - \Delta u + g(u) = 0, \tag{3.19}$$

associated with the Neumann boundary condition $\frac{\partial u}{\partial \nu} = 0$, on Γ, which is also relevant in view of biological applications. Actually, we can say more here. Indeed, let $u_0 \in L^\infty(\Omega)$ be an initial datum for (3.19) and let $y_\pm$ be the solutions to the ODE's

$$y'_\pm + g(y_\pm) = 0,\ y_\pm(0) = y_{\pm,0}, \tag{3.20}$$

where $y_{-,0} \leq u_0(x) \leq y_{+,0}$, a.e. in Ω. Then, it follows from the comparison principle for second-order parabolic equations that

$$y_-(t) \leq u(t,x) \leq y_+(t), \tag{3.21}$$

meaning that, if u_0 is properly chosen, u blows up in finite time. Similarly, the comparison principle also shows that, when u_0 is properly chosen (i.e., when $y_{\pm,0}$ yield solutions to (3.5) which do not blow up in finite time), then we have global (in time) existence; in particular, in the two examples above, we have global (in time) existence when u_0 remains in the biologically relevant interval. Now, such a comparison principle does not hold for fourth-order parabolic equations, so that the results obtained in Theorems 3.1 and 3.3 do not say more on the qualitative behavior of the solutions to (3.1) in general. However, in [85], we were able to obtain a more complete picture in the particular case (3.16) by studying the evolution equation for the spatial average of u. More precisely, we proved that, if u is a solution to (3.1)-(3.2), then either u blows up in finite time or u exists globally in time and $0 \leq \langle u(t)\rangle \leq \langle u_0\rangle + 1$, $\forall t \geq 0$. Furthermore, if u is a nonvanishing solution to (3.1)-(3.2) such that $u(t) \in [0,1]$, $\forall t \geq 0$, then u tends to 1 in $H^1(\Omega)$ as $t \to +\infty$. For a more general source term g (even polynomial), this seems much more complicated and will be studied elsewhere (see also [148, 149]).

Remark 3.9. When $\tilde{k}$ is odd and $b_q > 0$, then we do not have blow up in finite time. Indeed, in that case, necessarily, q is odd, $q \geq 3$, so that (3.5) is dissipative. Indeed, multiplying (3.5) by y, we easily obtain

$$\frac{d}{dt}y^2 + b_q y^{q+1} \leq c, \tag{3.22}$$

whence, in particular,

$$\frac{d}{dt}y^2 + cy^2 \leq c',\ c > 0. \tag{3.23}$$

Here and below, the same letters c and c' (and also c'') denote constants which may change from line to line. We thus deduce from (3.23) that

$$y^2(t) \leq y_0^2 e^{-ct} + c', \ c > 0, \tag{3.24}$$

and the solutions to (3.5) are indeed global in time. Furthermore, multiplying (3.1) by u and integrating over Ω, we find, assuming that the standard dissipativity assumption

$$f' \geq -c_0, \ c_0 \geq 0, \tag{3.25}$$

holds, a differential inequality of the form

$$\frac{d}{dt}\|u\|^2_{L^2(\Omega)} + \|\Delta u\|^2_{L^2(\Omega)} + b_q\|u\|^{q+1}_{L^{q+1}(\Omega)} \leq 2c_0\|\nabla u\|^2_{L^2(\Omega)^n} + c. \tag{3.26}$$

It follows from (3.26) that

$$\frac{d}{dt}\|u\|^2_{L^2(\Omega)} + c\|u\|^2_{H^2(\Omega)} + \frac{b_q}{2}\|u\|^{q+1}_{L^{q+1}(\Omega)} \leq c'\|u\|^2_{H^1(\Omega)} + c'', \ c > 0. \tag{3.27}$$

Employing the interpolation inequality

$$\|u\|_{H^1(\Omega)} \leq c\|u\|^{\frac{1}{2}}_{L^2(\Omega)}\|u\|^{\frac{1}{2}}_{H^2(\Omega)},$$

we deduce that

$$\frac{d}{dt}\|u\|^2_{L^2(\Omega)} + c\|u\|^2_{H^2(\Omega)} + \frac{b_q}{2}\|u\|^{q+1}_{L^{q+1}(\Omega)} \leq c'\|u\|^2_{L^2(\Omega)} + c'',$$

whence, owing to Young's inequality,

$$\frac{d}{dt}\|u\|^2_{L^2(\Omega)} + c\|u\|^2_{H^2(\Omega)} + \frac{b_q}{4}\|u\|^{q+1}_{L^{q+1}(\Omega)} \leq c', \ c > 0. \tag{3.28}$$

This yields that a solution to (3.1)-(3.2) (when it exists) is global in time and is dissipative in $L^2(\Omega)$ (in the sense that it follows from Gronwall's lemma that $\|u(t)\|^2_{L^2(\Omega)}$ is bounded independently of time and bounded sets of initial data for t large).

4. The Cahn–Hilliard equation with a fidelity term

We consider in this section the following generalization of the Cahn–Hilliard equation introduced in [24] in view of applications in image inpainting:

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) + \chi_{\Omega\setminus D}(x)(u - h) = 0, \tag{4.1}$$

where f is, for simplicity, the cubic function $f(s) = s^3 - s$ and $h \in L^2(\Omega)$ (actually, we will take no image, $h \equiv 0$, for simplicity). Here, we have taken all parameters equal to 1. This equation is endowed with the usual Neumann boundary conditions and the initial condition $u|_{t=0} = u_0$. Furthermore, D is an open bounded subset of Ω such that $D \subset\subset \Omega$ and χ denotes the indicator function.

The first existence and uniqueness result was obtained in [25]. Then, to go further and, in particular, to prove the existence of finite-dimensional attractors, we need to derive a global in time and dissipative

estimate. Obtaining such an estimate is not straightforward, due to the fact that we no longer have the conservation of mass.

Indeed, integrating (4.1) over Ω, we have

$$\frac{d\langle u\rangle}{dt} + \frac{1}{\mathrm{Vol}(\Omega)}\int_{\Omega\setminus D} u\,dx = 0.$$

In order to deal with this equation, we write

$$u = \langle u\rangle + v,$$

where v satisfies the equation

$$\begin{aligned}\frac{\partial}{\partial t}(-\Delta)^{-1}v - \Delta v + f(\langle u\rangle + v) - \langle f(\langle u\rangle + v)\rangle \\ +(-\Delta)^{-1}(\chi_{\Omega\setminus D}(x)u - \langle\chi_{\Omega\setminus D}(x)u\rangle) = 0.\end{aligned}$$

Furthermore, we can see that

$$\frac{d\langle u\rangle}{dt} + c_0\langle u\rangle = -\frac{1}{\mathrm{Vol}(\Omega)}\int_{\Omega\setminus D} v\,dx,\ c_0 = \frac{\mathrm{Vol}(\Omega\setminus D)}{\mathrm{Vol}(\Omega)}.$$

The left-hand side of the above equation is the simplest dissipative ODE and there just remains to control the right-hand side in order to have a global in time and dissipative estimate. To do so, we multiply the equation for v by v and integrate over Ω and by parts. However, in order to absorb bad terms which appear, we need some coercivity on the nonlinear term. More precisely, we write that

$$\begin{aligned}((f(\langle u\rangle + v) - \langle f(\langle u\rangle + v)\rangle, v)) &= ((f(\langle u\rangle + v) - f(\langle u\rangle), v)) \\ &\geq \frac{c_0}{2}\int_\Omega (v^4 + v^2\langle u\rangle^2)\,dx - \|v\|^2,\end{aligned}$$

where $((\cdot,\cdot))$ again denotes the usual L^2-scalar product, with associated norm $\|\cdot\|$. This then allows to have a global in time and dissipative estimate on v, then on $\langle u\rangle$ and finally on u. Having this, we can go further and obtain further regularity results and the existence of finite-dimensional attractors. We refer the reader to [73] for more details.

Remark 4.1. The question of the convergence of single trajectories to steady states is an important open problem. This question is all the more important that the final inpainting result is expected to be a steady state of the equation.

Now, again, the case of the thermodynamically relevant logarithmic nonlinear terms is much more involved. Considering such nonlinear terms is relevant here. Indeed, numerical simulations performed in [74] suggest better inpainting results as far as the convergence time is concerned. Furthermore, the final inpainting result is much better, when the inpainting domain D is large.

In what follows, we again take $h \equiv 0$. However, for a nonvanishing image h, we would need a condition of the form

$$\int_{\Omega\setminus D} h\,dx = 0,$$

meaning that we need some kind of symmetry.

We have the

Theorem 4.2. *We assume that* $u_0 \in H^1(\Omega)$, $|\langle u_0\rangle| < 1$ *and* $-1 < u_0(x) < 1$, a.e. $x \in \Omega$. *Then, there exists* $T_0 = T_0(u_0)$ *and a solution to the problem on* $[0, T_0]$ *such that* $u \in C([0, T_0]; H^{-1}(\Omega)) \cap L^\infty(0, T_0; H^1(\Omega)) \cap L^2(0, T_0; H^2(\Omega))$ *and* $\frac{\partial u}{\partial t} \in L^2(0, T_0; H^{-1}(\Omega))$. *Furthermore,* $-1 < u(t, x) < 1$, *a.e.* $(t, x) \in (0, T_0) \times \Omega$.

Proof. The proof is similar to the one performed in Section 2. However, we need to approximate the logarithmic nonlinear term in a careful way, as we need a coercivity property which is similar to the one obtained for the usual cubic nonlinear term for the approximated functions; this coercivity also needs to be uniform with respect to the approximation parameter.

To do so, we write (see (1.3)) $F(s) = \frac{\theta_c}{2}(1 - s^2) + F_1(s)$ and $f_1 = F_1'$. We then introduce, following [158] and for $N \in \mathbb{N}$, the approximated functions $F_{1,N} \in C^4(\mathbb{R})$ defined by

$$F_{1,N}^{(4)}(s) = \begin{cases} F_1^{(4)}(1 - \frac{1}{N}), \ s > 1 - \frac{1}{N}, \\ F_1^{(4)}(s), \ |s| \le 1 - \frac{1}{N}, \\ F_1^{(4)}(-1 + \frac{1}{N}), \ s < -1 + \frac{1}{N}, \end{cases} \tag{4.2}$$

$$F_{1,N}^{(k)}(0) = F_1^{(k)}(0), \ k = 0, \ 1, \ 2, \ 3, \tag{4.3}$$

so that

$$F_{1,N}(s) = \begin{cases} \sum_{k=0}^4 \frac{1}{k!} F_1^{(k)}(1 - \frac{1}{N})(s - 1 + \frac{1}{N})^k, \ s > 1 - \frac{1}{N}, \\ F_1(s), \ |s| \le 1 - \frac{1}{N}, \\ \sum_{k=0}^4 \frac{1}{k!} F_1^{(k)}(-1 + \frac{1}{N})(s + 1 - \frac{1}{N})^k, \ s < -1 + \frac{1}{N}. \end{cases} \tag{4.4}$$

Setting $F_N(s) = \frac{\theta_c}{2}(1 - s^2) + F_{1,N}(s)$, $f_{1,N} = F_{1,N}'$ and $f_N = F_N'$, there holds

$$f_{1,N}' \ge 0, \ f_N' \ge -\theta_c, \tag{4.5}$$

$$F_N \ge -c_1, \ c_1 \ge 0, \tag{4.6}$$

and (see [158, 305])

$$f_N(s)s \ge c_2(F_N(s) + |f_N(s)|) - c_3, \ c_2 > 0, \ c_3 \ge 0, \ s \in \mathbb{R}, \tag{4.7}$$

where the constants c_i, $i = 1$, 2 and 3, are independent of N, for N large enough. Furthermore, there holds, for N large enough,

$$(f_N(s + a) - f_N(a))s \ge c_4(s^4 + a^2 s^2) - c_5, \ c_4 > 0, \ c_5 \ge 0, \ s, \ a \in \mathbb{R}, \tag{4.8}$$

where the constants c_4 and c_5 are independent of N, which is the required coercivity property (see [74]).

We consider, for $N \in \mathbb{N}$, the approximated problems

$$\frac{\partial u_N}{\partial t} + \Delta^2 u_N - \Delta f_N(u_N) + \chi_{\Omega\setminus D}(x) u_N = 0, \tag{4.9}$$

$$\frac{\partial u_N}{\partial \nu} = \frac{\partial \Delta u_N}{\partial \nu} = 0, \text{ on } \Gamma, \tag{4.10}$$

$$u_N|_{t=0} = u_0. \tag{4.11}$$

We first derive uniform (with respect to N) a priori estimates. In particular, a crucial step is to prove that, at least locally in time, the spatial average of u_N is strictly separated from the pure states ± 1 and $f_N(u_N)$ is bounded in L^2, which will allow to prove the (local in time) existence of a solution. All constants below are independent of N. Furthermore, the same letters denote constants which may vary from line to line.

First, integrating (4.9) over Ω, we have

$$\frac{d\langle u_N\rangle}{dt} + \frac{1}{\mathrm{Vol}(\Omega)} \int_{\Omega\setminus D} u_N \, dx = 0. \tag{4.12}$$

Setting $u_N = \langle u_N\rangle + v_N$ (so that $\langle v_N\rangle = 0$), we can rewrite (4.12) as

$$\frac{d\langle u_N\rangle}{dt} + c_0\langle u_N\rangle = -\frac{1}{\mathrm{Vol}(\Omega)} \int_{\Omega\setminus D} v_N \, dx, \tag{4.13}$$

where, as above, $c_0 = \frac{\mathrm{Vol}(\Omega\setminus D)}{\mathrm{Vol}(\Omega)}$ and v_N is solution to

$$\frac{\partial v_N}{\partial t} + \Delta^2 v_N - \Delta(f_N(u_N) - \langle f_N(u_N)\rangle) + \chi_{\Omega\setminus D}(x)u_N - \langle \chi_{\Omega\setminus D}(x)u_N\rangle = 0, \tag{4.14}$$

$$\frac{\partial v_N}{\partial \nu} = \frac{\partial \Delta v_N}{\partial \nu} = 0, \text{ on } \Gamma, \tag{4.15}$$

$$v_N|_{t=0} = v_0 = u_0 - \langle u_0\rangle. \tag{4.16}$$

We rewrite (4.14)-(4.15) in the equivalent form

$$\begin{aligned} &(-\Delta)^{-1}\frac{\partial v_N}{\partial t} - \Delta v_N + f_N(u_N) - \langle f_N(u_N)\rangle \\ &+(-\Delta)^{-1}(\chi_{\Omega\setminus D}(x)u_N - \langle \chi_{\Omega\setminus D}(x)u_N\rangle) = 0, \end{aligned} \tag{4.17}$$

$$\frac{\partial v_N}{\partial \nu} = 0, \text{ on } \Gamma. \tag{4.18}$$

We multiply (4.17) by v_N to obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|v_N\|_{-1}^2 + \|\nabla v_N\|^2 \\ &+((f_N(u_N) - \langle f_N(u_N)\rangle, v_N)) + ((\chi_{\Omega\setminus D}(x)u_N, (-\Delta)^{-1}v_N)) = 0. \end{aligned} \tag{4.19}$$

Noting that

$$((f_N(u_N) - \langle f_N(u_N)\rangle, v_N)) = ((f_N(u_N) - f_N(\langle u_N\rangle), v_N)),$$

it follows from (4.8) that

$$((f_N(u_N) - \langle f_N(u_N)\rangle, v_N)) \geq c_4(\|v_N\|^4_{L^4(\Omega)} + \langle u_N\rangle^2\|v_N\|^2) - c. \tag{4.20}$$

Furthermore,

$$\begin{aligned} |((\chi_{\Omega\setminus D}(x)u_N, (-\Delta)^{-1}v_N))| &\leq c(\|v_N\|^2 + |\langle u_N\rangle|\|v_N\|) \\ &\leq \frac{c_4}{2}(\|v_N\|^4_{L^4(\Omega)} + \langle u_N\rangle^2\|v_N\|^2) + c. \end{aligned} \tag{4.21}$$

We thus deduce from (4.19)-(4.21) that

$$\frac{d}{dt}\|v_N\|^2_{-1} + \|\nabla v_N\|^2 + c_4(\|v_N\|^4_{L^4(\Omega)} + \langle u_N\rangle^2\|v_N\|^2) \leq c. \tag{4.22}$$

Next, it follows from (4.13) that

$$\frac{d\langle u_N\rangle^2}{dt} + c_0\langle u_N\rangle^2 \leq c\|v_N\|^2,$$

whence

$$\frac{d\langle u_N\rangle^2}{dt} + c_0\langle u_N\rangle^2 \leq \frac{c_4}{2}(\|v_N\|^4_{L^4(\Omega)} + \langle u_N\rangle^2\|v_N\|^2) + c. \tag{4.23}$$

Summing (4.22) and (4.23), we find a differential inequality of the form

$$\frac{dE_{1,N}}{dt} + c(\|u_N\|^2_{H^1(\Omega)} + \|v_N\|^4_{L^4(\Omega)} + \langle u_N\rangle^2\|v_N\|^2) \leq c',\ c > 0, \tag{4.24}$$

where

$$E_{1,N} = \langle u_N\rangle^2 + \|v_N\|^2_{-1}$$

satisfies

$$E_{1,N} \geq c\|u_N\|^2_{H^{-1}(\Omega)},\ c > 0. \tag{4.25}$$

We then multiply (4.9) by u_N and have, owing to (4.5),

$$\frac{d}{dt}\|u_N\|^2 + \|\Delta u_N\|^2 \leq 2\theta_c\|\nabla u_N\|^2 + c\|u_N\|^2. \tag{4.26}$$

Summing (4.24) and (4.26) multiplied by δ_1, where $\delta_1 > 0$ is chosen small enough, we obtain a differential inequality of the form

$$\frac{dE_{2,N}}{dt} + c(\|u_N\|^2_{H^2(\Omega)} + \|v_N\|^4_{L^4(\Omega)} + \langle u_N\rangle^2\|v_N\|^2) \leq c',\ c > 0, \tag{4.27}$$

where

$$E_{2,N} = \delta_1\|u_N\|^2 + E_{1,N}$$

satisfies

$$E_{2,N} \geq c\|u_N\|^2,\ c > 0. \tag{4.28}$$

We now rewrite (4.9)-(4.10) in the equivalent form

$$\frac{\partial u_N}{\partial t} + \chi_{\Omega\setminus D}(x)u_N = \Delta\mu_N, \tag{4.29}$$

$$\mu_N = -\Delta u_N + f_N(u_N), \tag{4.30}$$

$$\frac{\partial u_N}{\partial \nu} = \frac{\partial \mu_N}{\partial \nu} = 0,\ \text{on}\ \Gamma, \tag{4.31}$$

where, by analogy with the original Cahn–Hilliard equation, μ_N is called chemical potential.

We multiply (4.29) by μ_N and (4.30) by $\frac{\partial u_N}{\partial t}$ to find

$$\frac{1}{2}\frac{d}{dt}(\|\nabla u_N\|^2 + 2\int_\Omega F_N(u_N)\,dx) + \|\nabla\mu_N\|^2 = -((u_N, \chi_{\Omega\setminus D}(x)\mu_N)). \tag{4.32}$$

Furthermore, multiplying (4.30) by $\chi_{\Omega\setminus D}(x)u_N$, we have

$$((u_N, \chi_{\Omega\setminus D}(x)\mu_N)) = -((\Delta u_N, \chi_{\Omega\setminus D}(x)u_N)) + \int_{\Omega\setminus D} f_N(u_N)u_N\,dx. \tag{4.33}$$

We deduce from (4.7) and (4.32)-(4.33) that

$$\frac{d}{dt}(\|\nabla u_N\|^2 + 2\int_\Omega F_N(u_N)\,dx) \tag{4.34}$$

$$+c(\|\nabla\mu_N\|^2 + \int_{\Omega\setminus D}|f_N(u_N)|\,dx + \int_{\Omega\setminus D} F_N(u_N)\,dx) \leq c'\|u_N\|^2_{H^2(\Omega)},\ c > 0.$$

Summing (4.27) and (4.34) multiplied by δ_2, where $\delta_2 > 0$ is chosen small enough, we obtain a differential inequality of the form

$$\frac{dE_{3,N}}{dt} + c(\|u_N\|^2_{H^2(\Omega)} + \|v_N\|^4_{L^4(\Omega)} + \langle u_N\rangle^2\|v_N\|^2 \tag{4.35}$$

$$+ \int_{\Omega\setminus D}|f_N(u_N)|\,dx + \int_{\Omega\setminus D} F_N(u_N)\,dx + \|\nabla\mu_N\|^2) \leq c',\ c > 0,$$

where

$$E_{3,N} = \delta_2(\|\nabla u_N\|^2 + 2\int_\Omega F_N(u_N)\,dx) + E_{2,N}$$

satisfies

$$E_{3,N} \geq c\|u_N\|^2_{H^1(\Omega)} - c',\ c > 0. \tag{4.36}$$

Rewriting (4.29)-(4.30) in the equivalent form

$$(-\Delta)^{-1}\frac{\partial v_N}{\partial t} + (-\Delta)^{-1}(\chi_{\Omega\setminus D}(x)u_N - \langle\chi_{\Omega\setminus D}(x)u_N\rangle) = -(\mu_N - \langle\mu_N\rangle), \tag{4.37}$$

$$\mu_N - \langle\mu_N\rangle = -\Delta v_N + f_N(u_N) - \langle f_N(u_N)\rangle, \tag{4.38}$$

we deduce from (4.37) that

$$\|\frac{\partial v_N}{\partial t}\|_{-1} \leq c(\|u_N\| + \|\nabla\mu_N\|),$$

whence, owing to (4.13),

$$\|\frac{\partial u_N}{\partial t}\|_{H^{-1}(\Omega)} \leq c(\|u_N\| + \|\nabla\mu_N\|). \tag{4.39}$$

Furthermore, (4.38) yields

$$\|f_N(u_N) - \langle f_N(u_N)\rangle\| \leq c(\|u_N\|_{H^2(\Omega)} + \|\nabla\mu_N\|). \tag{4.40}$$

It thus follows from (4.35) and (4.39)-(4.40) that

$$\begin{aligned}\frac{dE_{3,N}}{dt} &+ c(\|u_N\|^2_{H^2(\Omega)} + \|v_N\|^4_{L^4(\Omega)} + \langle u_N\rangle^2\|v_N\|^2 \\ &+\|\frac{\partial u_N}{\partial t}\|^2_{H^{-1}(\Omega)} + \|f_N(u_N) - \langle f_N(u_N)\rangle\|^2 \\ &+ \int_{\Omega\setminus D}|f_N(u_N)|\,dx + \int_{\Omega\setminus D}F_N(u_N)\,dx + \|\nabla\mu_N\|^2) \leq c',\ c > 0.\end{aligned} \tag{4.41}$$

We can note that (4.41) is not sufficient to pass to the limit in the nonlinear term $f_N(u_N)$ (say, in a variational formulation). To do so, we also need an estimate on $|\langle f_N(u_N)\rangle|$ (in order to have an estimate on $\|f_N(u_N)\|$). This could be done if we were able to prove that $|\langle u_N(t)\rangle| \leq 1-\delta$, $t \geq 0$, $\delta \in (0,1)$ (see [301]; see also below). Unfortunately, we are not able to prove such a result and, therefore, we will only be able to obtain a local (in time) result.

We now assume that $|\langle u_0\rangle| < 1$. Then, there exists $\delta \in (0,1)$ such that $|\langle u_0\rangle| \leq 1-2\delta$. Therefore, since the function $t \mapsto \langle u_N(t)\rangle$ is continuous, there exists $T_0 = T_0(\delta, N)$ such that, if $t \in [0, T_0]$, then $|\langle u_N(t)\rangle| \leq 1-\delta$.

Actually, we can note that it follows from (4.13) that

$$\langle u_N(t)\rangle = e^{-c_0t}\langle u_0\rangle - e^{-c_0t}\int_0^t e^{c_0s}\,ds\int_{\Omega\setminus D}v_N\,dx,$$

so that

$$\begin{aligned}|\langle u_N(t)\rangle| &\leq |\langle u_0\rangle| + ce^{-c_0t}\int_0^t e^{c_0s}\|u_N\|\,ds \\ &\leq 1-2\delta + c(1-e^{-c_0t}),\end{aligned} \tag{4.42}$$

where we emphasize that $c = c(u_0)$ is independent of N (note indeed that it follows from (4.27)-(4.28) and Gronwall's lemma that $\|u_N\|$ is bounded uniformly with respect to time and N). We can thus find $T_0 = T_0(\delta, u_0)$ independent of N such that, if $t \in [0, T_0]$, then $|\langle u_N(t)\rangle| \leq 1 - \delta$.

Then, noting that we have a similar result for f (see [301]), it is not difficult to prove that, for N large enough,

$$f_N(s+m)s \geq c'_m|f_N(s+m)| - c''_m,\ c'_m > 0,\ c''_m \geq 0,\ s \in \mathbb{R},\ m \in (-1,1), \tag{4.43}$$

where the constants c'_m and c''_m depend continuously on m (see also [305]).

Taking $s = v_N$ and $m = \langle u_N\rangle$ in (4.43), integrating over Ω, noting that $\langle v_N\rangle = 0$ and employing Hölder's inequality, it follows that, for $N \geq N_0 = N_0(\delta)$,

$$|\langle f_N(u_N)\rangle| \leq c_\delta \|v_N\| \|f_N(u_N) - \langle f_N(u_N)\rangle\| + c'_\delta,\ t \in [0, T_0],$$

whence

$$\int_0^{T_0} |\langle f_N(u_N)\rangle|^2\, ds \leq c_\delta \|v_N\|^2_{L^\infty(0,T_0;L^2(\Omega))} \|f_N(u_N) - \langle f_N(u_N)\rangle\|^2_{L^2((0,T_0)\times\Omega)} + c'_\delta. \tag{4.44}$$

Therefore, noting that $v \mapsto (|\langle v\rangle|^2 + \|v - \langle v\rangle\|^2)^{\frac{1}{2}}$ is a norm on $L^2(\Omega)$ which is equivalent to the usual L^2-norm, (4.40) and (4.44) yield that

$$\begin{aligned} &\|f_N(u_N)\|_{L^2((0,T_0)\times\Omega)} \\ &\leq c_\delta(\|u_N\|_{L^\infty(0,T_0;L^2(\Omega))} + 1)(\|u_N\|_{L^2(0,T_0;H^2(\Omega))} + \|\nabla\mu_N\|_{L^2((0,T_0)\times\Omega)^n}) + c'_\delta. \end{aligned} \tag{4.45}$$

Noting finally that $\langle\mu_N\rangle = \langle f_N(u_N)\rangle$, we deduce that

$$\begin{aligned} &\|\mu_N\|_{L^2(0,T_0;H^1(\Omega))} \\ &\leq c_\delta(\|u_N\|_{L^\infty(0,T_0;L^2(\Omega))} + 1)(\|u_N\|_{L^2(0,T_0;H^2(\Omega))} + \|\nabla\mu_N\|_{L^2((0,T_0)\times\Omega)^n}) + c'_\delta. \end{aligned} \tag{4.46}$$

Having this, we can now proceed exactly as in Section 2 to pass to the limit and prove the separation property.

□

Remark 4.3. Actually, a more careful treatment of the equation for the spatial average of the order parameter allows to prove the global in time existence of solutions (see [194]). Indeed, rewriting (4.1) as

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) + u - \chi_D(x)u = 0,$$

we have, integrating this equation over Ω,

$$\frac{d\langle u\rangle}{dt} + \langle u\rangle = \frac{1}{\mathrm{Vol}(\Omega)} \int_D u\, dx.$$

This yields that

$$\langle u(t)\rangle = e^{-t}\langle u_0\rangle + \frac{1}{\mathrm{Vol}(\Omega)}e^{-t}\int_0^t e^s\,ds\int_D u\,dx.$$

Then, as long as the solution exists, necessarily, $|u(t)| \leq 1$, so that

$$|\langle u(t)\rangle| \leq e^{-t}|\langle u_0\rangle| + \frac{\mathrm{Vol}(D)}{\mathrm{Vol}(\Omega)}e^{-t}\int_0^t e^s\,ds,$$

whence

$$|\langle u(t)\rangle| \leq e^{-t}|\langle u_0\rangle| + \frac{\mathrm{Vol}(D)}{\mathrm{Vol}(\Omega)}(1-e^{-t}).$$

We now consider the function

$$\varphi(t) = e^{-t}|\langle u_0\rangle| + \frac{\mathrm{Vol}(D)}{\mathrm{Vol}(\Omega)}(1-e^{-t}).$$

Then,

$$\varphi'(t) = (-|\langle u_0\rangle| + \frac{\mathrm{Vol}(D)}{\mathrm{Vol}(\Omega)})e^{-t}.$$

If $-|\langle u_0\rangle| + \frac{\mathrm{Vol}(D)}{\mathrm{Vol}(\Omega)} \geq 0$, then φ is monotone increasing and

$$\varphi(0) \leq \varphi(t) \leq \lim_{t\to+\infty}\varphi(t),$$

that is,

$$|\langle u_0\rangle| \leq \varphi(t) \leq \frac{\mathrm{Vol}(D)}{\mathrm{Vol}(\Omega)}.$$

Furthermore, if $-|\langle u_0\rangle| + \frac{\mathrm{Vol}(D)}{\mathrm{Vol}(\Omega)} \leq 0$, then φ is monotone decreasing and

$$\lim_{t\to+\infty}\varphi(t) \leq \varphi(t) \leq \varphi(0),$$

that is

$$\frac{\mathrm{Vol}(D)}{\mathrm{Vol}(\Omega)} \leq \varphi(t) \leq |\langle u_0\rangle|.$$

It thus follows that

$$|\langle u(t)\rangle| \leq \max(|\langle u_0\rangle|, \frac{\mathrm{Vol}(D)}{\mathrm{Vol}(\Omega)}),$$

whence

$$|\langle u(t)\rangle| \leq 1-\delta,$$

where $\delta = \delta(u_0) \in (0,1)$ is independent of time. Therefore, the solutions are indeed global in time.

Remark 4.4. The uniqueness of solutions, as well as further regularity results, are important open problems in the case of logarithmic nonlinear terms and will be addressed in [194].

Remark 4.5. The Cahn–Hilliard inpainting model studied in this section was extended to color images in [75], by considering systems of Cahn–Hilliard equations, and to grayscale images in [76], by considering a complex version of the Cahn–Hilliard inpainting model; see also [41], where systems of Cahn–Hilliard equations were used for grayscale images.

Remark 4.6. As far as the numerical simulations are concerned, the authors in [24, 25] proposed a dynamic two-steps algorithm based on the interface thickness ε. More precisely, one first takes a large value of ε in order to join the edges (indeed, when the inpainting domain is large, the inpainting may fail if the interface thickness is too small) and then switches to a smaller value of ε in order to obtain the final restored image. This algorithm is very efficient as far as the computation time and the quality of the restored images are concerned. In [73–75], we proposed instead a one-step algorithm with threshold. Namely, we take an intermediate value of ε and then threshold, i.e., when the order parameter is larger than some given value, we take it equal to 1 (say, black) and, when it is smaller, we take it equal to 0 (white); of course, such an algorithm does not make sense for grayscale images. We observed that we can obtain results which are comparable with those in [24, 25], when the inpainting domain is not too large, but with a smaller computation time. When the inpainting domain is large, this algorithm may fail, but, as already mentioned, taking logarithmic nonlinear terms instead of polynomial ones, improves the simulations.

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Conflict of Interest

The author declares no conflicts of interest in this paper.

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